Advanced Quantum Mechanics
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II. Non-Local Field Theories

Non-local Field Theory

Jan, 1953

Last time I discussed the S-matrix theory from a more general standpoint than usual according to Heisenberg. As I mentioned, however, Heisenberg's approach was ~~too~~ so ~~different~~ much different from the orthodox field theory that many of the important features of present field theory which ~~was successful~~ enabled us successfully to account for great variety of phenomena in terms of the virtual creation and annihilation of particles such as photons, mesons, pairs of electrons, pairs of nucleons and so on. Moreover, ~~only~~ the formal general considerations of the structure of S-matrix alone give us no hint to the solution of ~~the~~ second important problem of mass-spectrum of elementary particles and its possible relation to

the first problem of convergence.
In connection with ~~As for~~ the second problem,
I discussed the non-local inter. between local field. It is clear, however, that such a theory cannot claim to be a fundamental ~~field~~ theory because ~~it~~ there is no principle in it to ~~determine~~ choose ~~the~~ form function from all other except for the convergence criterion. Nor ~~does~~ it suggest any connection to the second problem of mass spectrum.
I seems to me that the only conceivable way ~~of dealing~~ of coping with the two fundamental problems as mutually related is to change the concept of field itself.

N. L. F. (3)

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Prob. of ~~Mass~~ Mass Spectrum as Eigenvalue Problem
~~Lagrangian formalism~~
~~in N. L. F. T.~~

~~Since the ordinary~~ A local field, say a scalar or pseudoscalar field, $\varphi(x_\mu)$ is ordinarily supposed to satisfy a field equation of Klein-Gordon type ~~in which~~

$$\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - m^2\right)\varphi(x_\mu) = 0$$

in which m is an arbitrary parameter, ~~and there is no rule which give~~ Such a field is too ~~simple~~ simple whatsoever to give any ~~information on~~ further restriction ~~to~~ the possible (discrete) values of the mass m. For that reason, ~~we~~ we had better start from a more complicated ~~f~~ classical field which depends on two sets of space-time ~~points~~ parameters x'_μ, x''_μ : $\varphi(x'_\mu, x''_\mu)$

or $\varphi(x'_\mu, x''_\mu) = \varphi(X_\mu, r_\mu)$

where $X_\mu = \frac{1}{2}(x'_\mu + x''_\mu)$, η

$r_\mu = x'_\mu - x''_\mu$

This field is supposed to satisfy the field equation:

$$F\left(x'_\mu, x''_\mu, \frac{\partial}{\partial x'_\mu}, \frac{\partial}{\partial x''_\mu}\right)\varphi(x'_\mu, x''_\mu) = 0$$

where F is an invariant function of $x'_\mu, x''_\mu, \frac{\partial}{\partial x'_\mu}, \frac{\partial}{\partial x''_\mu}$.

In order that the field eq. is invariant with respect to an arb. inhom. Lorentz transformation, F must be independent of X_μ, so that we may write

$$F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(X_\mu, r_\mu) = 0.$$

In general, one may also write ~~φ can~~ be

$$\varphi(X_\mu, r_\mu) = \int\!\!\cdots\!\!\int u(k_\mu, r_\mu) e^{ik_\mu X_\mu}(dk)^4$$

$$F\left(ik_\mu, r_\mu, \frac{\partial}{\partial r_\mu}\right) u(k_\mu, r_\mu) = 0$$

Now F is a function of

$k_\mu k_\mu$, $r_\mu r_\mu$, $\frac{\partial^2}{\partial r_\mu \partial r_\mu}$, $k_\mu r_\mu$, $k_\mu \frac{\partial}{\partial r_\mu}$; $r_\mu \frac{\partial}{\partial r_\mu}$

Let us take the simple case in which ~~suppose that~~ one ~~can~~ may write

$$F = k_\mu k_\mu + f\left(r_\mu r_\mu, \frac{\partial^2}{\partial x_\mu \partial x_\mu}, k_\mu x_\mu, k_\mu \frac{\partial}{\partial x_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right).$$

so that the eigenvalue problem

$$f\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, k_\mu x_\mu, k_\mu \frac{\partial}{\partial r_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right) \times u(k_\mu, r_\mu) = m^2 u(k_\mu, r_\mu)$$

for the operator f can be solved with a certain eigenvalue

$$m^2_{n, k_\mu}$$

and eigenfunction

$$u_n(k_\mu, r_\mu).$$

Then we have

$$(k_\mu k_\mu + m^2_{n, k_\mu}) u_n(k_\mu, r_\mu) = 0$$

Thus m_{n, k_μ} can be interpreted as the mass of the particle, ~~due to~~ the mass ~~being~~ determined by its internal motion, provided that m^2_{n, k_μ} is independent of the direction of k_μ. In other words, either if F has the form

$$F = k_\mu k_\mu + f\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right)$$

or m^2_{n,k_μ} is actually independent of ~~k_μ~~ the direction of k_μ due to some supplementary imposed on u_n.

The above argument suggests us that

$$F = F^{(ext)} + F^{(int)}$$

$$F^{(ext)}\left(\frac{\partial}{\partial X_\mu}\right) = F^{(ext)}\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu}\right)$$

$$F^{(in)}\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, \cancel{\frac{\partial^2}{\partial X_\mu \partial r_\mu}}, \cancel{r_\mu \frac{\partial}{\partial X_\mu}}, r_\mu \frac{\partial}{\partial r_\mu}\right)$$

$$\varphi(X_\mu, r_\mu) = \sum_n \varphi_n^{(ex)}(X_\mu)\, \varphi_n^{(in)}(r_\mu)$$

where

$$F^{(in)} \varphi_n^{(in)}(r_\mu) = m_n^2 \varphi_n^{(in)}(r_\mu) \quad (1)$$

$$(F^{(ex)} + m_n^2)\, \varphi_n^{(ex)}(X_\mu) = 0.$$

Thus, if we take

$$F^{(ex)} = -\frac{\partial^2}{\partial X_\mu \partial X_\mu},$$

m_n^{2} becomes the mass of the particle.
By solving the eigenvalue problem (11), we have the mass spectrum for scalar ~~or pseudoscalar~~ elementary particles with 0 or integer spin.

Example. 1. Oscillator Model

$$F^{(in)} = -\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu$$

$$H = -p^2 + \frac{1}{\lambda^4} q^2 = \frac{1}{\lambda^2}(-P^2 + Q^2)$$

$$[q, p] = -1. \qquad P = \lambda p, \quad Q = \frac{q}{\lambda}$$

$$[Q, P] = -1.$$

$$E_n = \frac{1}{\lambda^2}(2n+1)$$

$$\cancel{[P + Q, P - Q] =}$$

$$[\tfrac{1}{\sqrt{2}}(Q+P), \tfrac{1}{\sqrt{2}}(Q-P)] = -\tfrac{1}{2}[Q, P] + \tfrac{1}{2}[P, Q]$$
$$= 1.$$

$$b = \frac{1}{\sqrt{2}}(Q+P) \qquad b^* = \frac{1}{\sqrt{2}}(Q-P)$$

$$b^* b = \tfrac{1}{2}(Q^2 - P^2) + \tfrac{1}{2}[Q, P]$$

$$b\, b^* = \tfrac{1}{2}(Q^2 - P^2) - \tfrac{1}{2}[Q, P]$$

$$Q^2 - P^2 = b^* b + b\, b^* = 2n+1$$

$$\frac{\partial}{\partial x'_\mu} = \frac{1}{2}\frac{\partial}{\partial X_\mu} + \frac{\partial}{\partial r_\mu}$$

$$\frac{\partial}{\partial x''_\mu} = \frac{1}{2}\frac{\partial}{\partial X_\mu} - \frac{\partial}{\partial r_\mu}$$

$$\frac{\partial^2}{\partial x'_\mu \partial x'_\mu} = \frac{1}{4}\frac{\partial^2}{\partial X_\mu \partial X_\mu} + \frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{\partial}{\partial X_\mu \partial r_\mu}$$

$$\frac{\partial^2}{\partial x''_\mu \partial x''_\mu} = \frac{1}{4}\frac{\partial^2}{\partial X_\mu \partial X_\mu} + \frac{\partial^2}{\partial r_\mu \partial r_\mu} - \frac{\partial}{\partial X_\mu \partial r_\mu}$$

$$\frac{\partial^2}{\partial x'_\mu \partial x'_\mu} + \frac{\partial^2}{\partial x''_\mu \partial x''_\mu} = \frac{1}{2}\frac{\partial^2}{\partial X_\mu \partial X_\mu} + 2\frac{\partial^2}{\partial r_\mu \partial r_\mu}$$

$$F^{(ex)} + F^{(in)} = -\frac{\partial^2}{\partial x'_\mu \partial x'_\mu} - \frac{\partial^2}{\partial x''_\mu \partial x''_\mu} + \frac{2}{\lambda^4}(x'_\mu - x''_\mu)(x'_\mu - x''_\mu)$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial X_\mu \partial X_\mu} + 2\underbrace{\left(\frac{-\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)}$$

$$\frac{2}{\lambda^2}\{(2n_1+1)+(2n_2+1)+(2n_3+1)-(2n_0+1)\}$$

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N. W. F (5)

$$m^2_{n_1 n_2 n_3 n_0} = \frac{8}{\lambda^2}\{n_1 + n_2 + n_3 - n_0 + 1\}$$

$$n_1 = n_2 = n_3 = n_0 = 0$$

$$m_0^2 = \frac{8}{\lambda^2}$$

$$\frac{\mu c}{\hbar} = \frac{\sqrt{8}}{\lambda}$$

$$\lambda = \sqrt{8}\,\frac{\hbar}{\mu c}$$

$$\mu = 2\frac{\hbar c}{e^2} m_e$$

$$\lambda = \sqrt{8}\cdot\frac{1}{2}\,\frac{e^2}{m_e c^2} = \sqrt{2}\,\frac{e^2}{m_e c^2}$$

$$\lambda' \equiv \frac{\lambda}{\sqrt{2}} = \frac{e^2}{m_e c^2}.$$

$$\Gamma^{(ed)} + \Gamma^{(in)} = \frac{1}{2}\frac{\partial^2}{\partial x_\mu \partial x_\mu} + 2\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda'^4} r_\mu r_\mu$$

$$\lambda' = \frac{e^2}{m_e c^2}$$

$$\left.\begin{aligned} m_1 &= \sqrt{2}\, m_0 \\ m_2 &= \sqrt{3}\, m_0 \\ m_3 &= 2\, m_0 \\ &\vdots \end{aligned}\right\} \text{etc.}$$

The trouble with this ~~and any other~~ model ~~and~~ is that m^2_{nn} may not be ~~positive, but~~ restricted to positive eigenvalues, but can be negative, if $n_0 > n_1 + n_2 + n_3 + 1$.
It is obviously possible to restrict m^2_{nn} to positive values by taking for example:

$$F^{(m)} = \left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)^2,$$

but this does not remove the more serious difficulty of degeneracy. Namely, for any

$$n = n_1 + n_2 + n_3 - n_0 + 1$$

there are infinitely many combinations of n_1, n_2, n_3, n_0.
In order to avoid such an ~~compli~~ undesirable complication, one has to assume a certain supplement condition of the type:

$$\frac{\partial \varphi^2}{\partial X_\mu \partial r_\mu} \qquad \varphi(X_\mu, r_\mu) = 0$$

In particular, if we take a solution of the form

$$\varphi = e^{ik_\mu x_\mu} \varphi^{(in)}(r_\mu)$$

$$k_\mu \frac{\partial}{\partial r_\mu} \varphi^{(in)} = 0$$

or another suppl. cond.:

$$\left(\frac{\partial^2}{\partial x_\mu \partial r_\mu} + \frac{1}{\lambda^2} \frac{\partial}{\partial x_\mu} r_\mu \right) \varphi = 0$$

or $$\frac{\partial}{\partial x_\mu} \left(\frac{\partial}{\partial r_\mu} + \frac{1}{\lambda^2} r_\mu \right) \varphi = 0$$

$$\varphi = e^{ik_\mu x_\mu} \varphi^{(in)}(r_\mu)$$

$$k_\mu \left(\lambda \frac{\partial}{\partial r_\mu} + \frac{1}{\lambda} r_\mu \right) \varphi^{(in)} = 0$$

or $$k_\mu b_\mu \varphi^{(in)} = 0$$

If we carry out a ~~tran~~ coordinate transformation

$$b'_\mu = a_{\mu\nu} b_\nu$$

with the coefficients

$$(a_{\mu\nu}) = \begin{pmatrix} 1+\left(\frac{k_1}{K}\right)^2 & \frac{k_1 k_2}{K^2} & \frac{k_1 k_3}{K^2} & i k_1/\kappa \\ \frac{k_1 k_2}{K^2} & 1+\left(\frac{k_2}{K}\right)^2 & \frac{k_2 k_3}{K^2} & i k_2/\kappa \\ k_1 k_3/K^2 & k_2 k_3/K^2 & 1+\left(\frac{k_3}{K}\right)^2 & i k_3/\kappa \\ i k_1/\kappa & i k_2/\kappa & i k_3/\kappa & i k_4/\kappa \end{pmatrix}$$

$$K = \sqrt{\kappa(\kappa + i k_4)}$$

$$\left(1+\left(\frac{k_1}{K}\right)^2\right)\frac{k_1 k_2}{K^2} + \left(\frac{k_1 k_2}{K^2}\right)\left(1+\left(\frac{k_2}{K}\right)^2\right)$$

$$+ \frac{k_1 k_3}{K^2}\,\frac{k_2 k_3}{K^2} - \frac{k_1 k_2}{\kappa^2}$$

$$= \frac{k_1 k_2}{K^2}\left\{2 + \frac{k_1^2 + k_2^2 + k_3^2}{K^2}\right\}$$

$$- \frac{k_1 k_2}{\kappa^2}$$

$$= \frac{k_1 k_2 \left\{2\kappa^2 + 2\kappa\sqrt{(k_1^2+k_2^2+k_3^2+\kappa^2)} + k_1^2 + k_2^2 + k_3^2\right\}}{K^4} - \frac{k_1 k_2}{\kappa^2}$$

$$\simeq \frac{k_1 k_2 \left(\kappa + \sqrt{k_1^2+k_2^2+k_3^2+\kappa^2}\right)^2}{K^2}$$

$$- \frac{k_1 k_2}{\kappa^2}$$

$$= \frac{k_1 k_2}{\kappa^2} - \frac{k_1 k_2}{\kappa^2} = 0$$

$$\left(1-\left(\frac{k_1}{K}\right)^2\right)^2+\left(\frac{k_1k_2}{K^2}\right)^2+\left(\frac{k_1k_3}{K^2}\right)^2$$

$$-\frac{k_1^2}{\kappa^2}=1+2\frac{k_1^2}{K^2}+\frac{k_1^2(k_1^2+k_2^2+k_3^2)}{K^4}$$

$$-\frac{k_1^2}{\kappa^2}=\frac{1}{K^4}\Big\{K^4+\cancel{2K^2k_1^2}\ 2K^2k_1^2$$
$$+k_1^2(k_1^2+k_2^2+k_3^2)-k_1^2(\kappa+\sqrt{\quad})^2\Big\}$$

$$=\frac{1}{K^4}\Big\{K^4+2K^2k_1^2-k_1^2\kappa^2$$
$$-2k_1^2\kappa\sqrt{k_1^2+k_2^2+k_3^2+\kappa^2}-k_1^2\kappa^2\Big\}$$

$$=\frac{1}{K^4}\Big\{\kappa^2(\kappa+\sqrt{\quad})^2+2k_1^2\kappa(\kappa+\sqrt{\quad})$$
$$-2k_1^2\kappa^2-2k_1^2\kappa\sqrt{\quad}\Big\}$$

$$=1$$

One may alternatively subtract from ~~F the~~ $F^{(in)}$, the term

$$1+\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu}\right)^2 \frac{1}{\kappa^2}\left(\gamma_\mu\frac{\partial}{\partial x_\mu}\right)^2 \text{ in } F$$

after multiplying the ~~first terms~~ first pair second by $-\frac{\partial^2}{\partial x_\mu \partial x_\mu}$

Then, we have

$$F^{(in)} = \frac{1}{\lambda^2}(2 b'^*_\mu b'_\mu + 1)$$

$$\frac{1}{\lambda^2}(2 b'^*_\mu b'_\mu + 2)\varphi^{(in)} = 0$$

with the supplementary cond.

$$b'_4 \varphi^{(in)} = 0.$$

$$E_n = \frac{2}{\lambda^2} \cancel{2m} (n'_1 + n'_2 + n'_3 + 1)$$

$$= (137 m_e)^2 (n'_1 + n'_2 + n'_3 + 1)$$

$$m_n^2 = (274 m_e)^2 (n'_1 + n'_2 + n'_3 + 1)$$

Thus, in the rest system, $r'_4 = i r'_0$ the eigenfn $\varphi^{(in)}_{0(r_\mu)}$ has the form:

$$\varphi^{(in)}_{0,r_\mu} = \left(\frac{1}{\sqrt{\pi}\cdot\lambda}\right)^4 e^{-(q_1'^2 + q_2'^2 + q_3'^2 + q_0'^2)/2\lambda^2}$$

$$= \frac{1}{\pi\lambda^2} e^{-(q_1'^2 + q_2'^2 + q_3'^2 + q_0'^2)/2\lambda^2}$$

$$\because \quad \frac{d^2}{dq^2} e^{-aq^2} = \frac{d}{dq}\left(-2aq\, e^{-aq^2}\right)$$

$$= -2a + 4a^2 q^2 e^{-aq^2}$$

$$\left(-\frac{d^2}{dq^2} + 4a^2 q^2\right) e^{-aq^2} = 2a\, e^{-aq^2}$$

$$a = \frac{1}{2}. \quad \left(-\frac{d^2}{dq^2} + q^2\right) e^{-q^2/2} = e^{-q^2/2}$$

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N.L.F. ⑧

The corresponding total Schrödinger fn is

$$\varphi_0(x_\mu, r_\mu) = \int \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{(2\pi)^3}} a_k e^{ik_\mu x_\mu} \frac{1}{\pi\lambda^2} e^{-(r_1'^2+r_2'^2+r_3'^2+r_0'^2)/2\lambda^2} (dk)^3$$

$$= \iiint \frac{1}{\sqrt{2k_0(2\pi)^3}} (a_k e^{ik_\mu x_\mu} + a_k^* e^{-ik_\mu x_\mu}) \frac{1}{\pi\lambda^2} e^{-(r_1'^2+r_2'^2+r_3'^2+r_0'^2)/2\lambda^2} (dk)^3$$

$$k_4 = i k_0 \qquad k_0 = +\sqrt{k^2+m_0^2}$$

$$[a_k, a_{k'}^*] = \delta(k, k')$$

$$\int (\varphi_0(x_\mu, r_\mu))^2 \times (dX)^3 (dr)^4 = \int \varphi_0(x_\mu, r_\mu)\varphi_0(x_\mu, -r_\mu) \times (dX)^3 (dr)^4$$

$$= \int (a_k^* a_k + a_k a_k^*)(dk)^3 + \text{fluctuation}$$

$$\int_{-\infty}^{+\infty} e^{-2aq^2} dq = \sqrt{\frac{\pi}{2a}} = \sqrt{\frac{\pi \cdot 2\lambda^2}{2}} = \sqrt{\pi}\cdot\lambda$$

$$\iint e^{-2a(x^2+y^2)} dx\,dy = 2\pi \int_0^\infty e^{-2ar^2} r\,dr$$

$$= \pi \int_0^\infty e^{-2ax} dx = \frac{\pi}{2a}$$

There are other excited states which correspond to

$n = n_1 + n_2 + n_3 = 1 \qquad n_0 = 0$

$\qquad = 2 \qquad n_0 = 0$

$\qquad = 3 \qquad n_0 = 0$

etc.

all of them are degenerate

$n = 1$. triple ~~ts~~ (P)

$n = 2$. $(2,0,0)\ (0,2,0)\ (0,0,2)$
$(1,1,0)\ (1,0,1)\ (0,1,1)$
6-fold
(D, S)

etc.

$n = 3$.

$n = 4$

The total Schrödinger fn is

$\psi(X, r) = \sum_n \varphi_n(X, r)$,

the procedure of the quantization being similar to the one for $n = 0$.

for Many-particle
General Eigenvalue Problem

N.L.F. (8)'

$$F\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu}, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu r_\mu, \frac{\partial^2}{\partial x_\mu \partial r_\mu}, r_\mu \frac{\partial}{\partial x_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right)\varphi(x_\mu, r_\mu) = 0$$

among 6-operators in F, $\frac{\partial^2}{\partial x_\mu \partial x_\mu}$ commute with all others, so that one can ~~take the representation in which~~ $\frac{\partial^2}{\partial x_\mu \partial x_\mu}$ expand φ into series of eigenfunctions of satisfying

$$\frac{\partial^2 \varphi_n^{(ex)}(x_\mu)}{\partial x_\mu \partial x_\mu} = m_n^2 \varphi_n^{(ex)}(x_\mu),$$

then the coefficients $\varphi_n^{(in)}(r_\mu)$ of expansion

$$\varphi(x_\mu, r_\mu) = \sum_n \varphi^{(ex)}_{n,k_\mu}(x_\mu)\, \varphi^{(in)}_{n,k_\mu}(r_\mu)$$

~~where~~ must satisfy

$$F\left(m_n^2, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu r_\mu, k_\mu \frac{\partial}{\partial r_\mu}, k_\mu r_\mu, r_\mu \frac{\partial}{\partial r_\mu}\right)\varphi^{(in)}_{n,k_\mu}(r_\mu) = 0$$

Example:

$$m_n^2\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right) \pm \left(k_\mu \frac{\partial}{\partial r_\mu}\right)^2 + \frac{1}{\lambda^4}(k_\mu r_\mu)^2$$

$$F = m_n^4 -$$

$$= m_n^2\left\{ m_n^2 - \frac{4}{\lambda^2}\left(n_1^{(in)} + n_2^{(in)} + n_3^{(in)} + 1\right)\right\}$$

However, this method is unsatisfactory in that the $\varphi^{(in)}(r_\mu)$ is not sufficiently restricted in $k_\mu r_\mu$-direction.*

On the other hand, the supplementary condition

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial}{\partial r_\mu}+\frac{1}{\lambda^2}r_\mu\right)\varphi=0$$

is compatible with the field eq. because

$$\frac{\partial^2}{\partial r_\mu \partial r_\mu}\left(\frac{1}{\lambda^2}r_\mu\right)\varphi$$

$$=\frac{\partial}{\partial x_\mu}\left\{\frac{1}{\lambda^2}r_\nu\frac{\partial}{\partial x_\mu}\right\}\varphi+\frac{\partial}{\partial x_\mu}\frac{1}{\lambda^2}\delta_{\mu\nu}\varphi$$

$$=\frac{1}{\lambda^2}r_\nu\cdot\frac{\partial^2\varphi}{\partial x_\mu\partial x_\mu}+2\frac{1}{\lambda^2}\delta_{\mu\nu}\cdot\frac{\partial\varphi}{\partial x_\mu}$$

$$\frac{1}{\lambda^4}r_\mu r_\mu\left(\frac{\partial}{\partial r_\nu}\right)\varphi$$

$$=\frac{\partial}{\partial r_\nu}\frac{1}{\lambda^4}r_\mu r_\mu\varphi$$

$$= \frac{2}{\lambda^4}\delta_{\mu\nu}r_\nu\varphi$$

$$\left(-\frac{\partial^2}{\partial r_\mu\partial r_\mu}+\frac{1}{\lambda^4}r_\mu r_\mu\right)\left(\frac{\partial}{\partial r_\nu}+\frac{1}{\lambda^2}r_\nu\right)\varphi$$

$$=\left(\frac{\partial}{\partial r_\nu}+\frac{1}{\lambda^2}r_\nu\right)(\qquad)\varphi-2\frac{1}{\lambda^2}\delta_{\mu\nu}\left(\frac{\partial}{\partial r_\nu}+\frac{r_\nu}{\lambda^2}\right)\varphi$$

* In order to get rid of this difficulty one can further impose a S.C.

$$k_\mu r_\mu \varphi = 0.$$

or

$$r_\mu \frac{\partial}{\partial x_\mu} \varphi = 0$$

This condition is compatible with the field equation, because F can be written in terms of

$$r_1', r_2', r_3' \quad \frac{\partial}{\partial r_1'}, \frac{\partial}{\partial r_2'}, \frac{\partial}{\partial r_3'}$$

alone and independent of

$$r_0' = -\frac{k_\mu r_\mu}{m} \quad \text{and} \quad \frac{\partial}{\partial r_0'}$$

This gives rise to the factor

$$\varphi \propto \delta(k_\mu r_\mu)$$

Another possibility is to assume a S.C.

$$\left\{\left(\frac{\partial^2}{\partial x_\mu \partial r_\mu}\right)^2 - \frac{1}{\lambda^4}\left(r_\mu \frac{\partial}{\partial x_\mu}\right)^2\right\}\varphi = ;$$

$$\frac{2}{\lambda^2} \frac{\partial^2 \varphi}{\partial x_\mu \partial x_\mu} \quad ?$$

N.L.F. (9)

Lagrangian formalism

$$L = \frac{1}{2}\iint \left\{ \frac{\partial \varphi(x',x'')}{\partial x'_\mu}\frac{\partial \varphi(x',x'')}{\partial x'_\mu} + \frac{\partial \varphi(x',x'')}{\partial x''_\mu}\frac{\partial \varphi(x',x'')}{\partial x''_\mu} \right\}(dx')^4(dx'')^4$$

$$+ \iint \frac{1}{\lambda^4} (x'_\mu - x''_\mu)(x'_\mu - x''_\mu) \times \varphi^2(x',x'') \} (dx')^4 (dx'')^4$$

$$\delta L = 0$$

$$-\frac{\partial^2 \varphi}{\partial x'_\mu \partial x'_\mu} - \frac{\partial^2 \varphi}{\partial x''_\mu \partial x''_\mu} + \frac{1}{\lambda^4}(x'_\mu - x''_\mu)^2 \varphi = 0$$

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial X_\mu \partial X_\mu} - 2\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{2}{\lambda^4} r_\mu r_\mu\right)\varphi = 0$$

$$\left(-\frac{\partial^2}{\partial X_\mu \partial X_\mu} + m_n^2\right)\varphi_n^{ex}(X_\mu) = 0$$

$$4\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)\varphi_n^{in}(r_\mu) = m_n^2 \varphi_n^{in}$$

$$m_n^2 = \frac{8}{\lambda^2}\{n_1 + n_2 + n_3 - n_0 + 1\}$$

$$= \left(\frac{\mu c}{\hbar}\right)^2 \quad \text{for } n_1 = n_2 = n_3 = n_0 = 0$$

$$\mu = 274\, m_e$$

for $\lambda = \sqrt{2}\dfrac{e^2}{m_e c^2}$

If $\varphi(x', x'') = \varphi(x'', x')$
or $\varphi(X, r) = \varphi(X, -r)$,
one may write L as the trace of an non-local operator.
Namely, if we introduce the non-local operators

$$(x'|\varphi|x'') \equiv \varphi(x', x'')$$
$$(x'|x_\mu|x'') = x'_\mu \, \delta(x'_\mu - x'')$$
$$(x'|p_\mu|x'') = i\, \delta'_\mu(x' - x'')$$

$$(x'|[x_\mu, p_\nu]|x'') = \int x'_\mu \, \delta(x' - x''') \, i\delta'_\nu(x''' - x'') \, dx'''$$
$$- \int i\delta'_\nu(x' - x''') \, x'''_\mu \, \delta(x''' - x'') \, dx'''$$
$$= -i \int x'_\mu \frac{\partial}{\partial x'''_\nu} \delta(x' - x''') \, \delta(x''' - x'') \, dx'''$$
$$- i \int \delta_{\mu\nu} \, \delta(x' - x''') \, \delta(x''' - x'') \, dx'''$$
$$- i \int \delta(x' - x''') \, x'''_\mu \frac{\partial}{\partial x'''_\mu} \delta(x''' - x'') \, dx'''$$
$$= -i \, \delta_{\mu\nu} \, \delta(x' - x'')$$

$$\int f(x') \, \delta'_\mu(x' - x'') \, dx' = -\frac{\partial f(x'')}{\partial x_\mu}$$

$$\int \delta'_\mu(x'-x'') f(x'') dx''$$
$$= \frac{\partial f(x')}{\partial x'_\mu}$$

$$(x'|[p_\mu, \varphi]|x'') = \int i\,\delta'_\mu(x'-x''')\varphi(x''',x'')\,dx'''$$
$$- \int i\,\varphi(x',x''')\,\delta'_\mu(x'''-x'')\,dx'''$$
$$= i\frac{\partial\varphi(x',x'')}{\partial x'_\mu} + i\frac{\partial\varphi(x',x'')}{\partial x''_\mu}$$
$$= i\frac{\partial\varphi(X,r)}{\partial X_\mu}$$

$$(x'|[p_\mu, \varphi]_+|x'') = i\frac{\partial\varphi(x',x'')}{\partial x'_\mu} - i\frac{\partial\varphi(x',x'')}{\partial x''_\mu}$$
$$= 2i\frac{\partial\varphi(X,r)}{\partial r_\mu}$$

$$\mathrm{Trace}\{[p_\mu,\varphi]\cancel{}[p_\mu,\varphi]\} = -\int\!\!\cdots\!\!\int \frac{\partial\varphi(X,r)}{\partial X_\mu}\frac{\partial\varphi(X,-r)}{\partial X_\mu}(dX)(dr)$$

$$\left.\begin{aligned} x'_\mu &= X_\mu + \tfrac{1}{2}r_\mu \\ x''_\mu &= X_\mu - \tfrac{1}{2}r_\mu \end{aligned}\right\}$$

$$\frac{\partial}{\partial X_\mu} = \frac{\partial}{\partial x'_\mu} + \frac{\partial}{\partial x''_\mu}, \quad \frac{\partial}{\partial r_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x'_\mu} - \frac{\partial}{\partial x''_\mu}\right)$$

$$\mathrm{Trace}\ [p_\mu, \varphi]_+ [p_\mu, \varphi]_+$$

$$= -4 \int\cdots\int \frac{\partial \varphi(X, r)}{\partial r_\mu} \frac{\partial \varphi(X, -r)}{\partial r_\mu}\, dX\, dr$$

$$\mathrm{Trace}\ [x_\mu, \varphi][x_\mu, \varphi]$$

$$= \int\cdots\int x_\mu r_\mu\, \varphi(X, r)\, \varphi(X, -r)\, dX\, dr$$

$$\bar{L} = \mathrm{Trace}\ \{[p_\mu, \varphi][p_\mu, \varphi]$$

$$+ [p_\mu, \varphi]_+ [p_\mu, \varphi]_+ \frac{4}{\lambda^4}[x_\mu, \varphi][x_\mu, \varphi]\}$$

$$\lambda' = \frac{\lambda}{\sqrt{2}} = \frac{e^2}{m_e c^2}$$

$$\lambda'^4 = \frac{\lambda^4}{4}$$

$$\bar{L} = \mathrm{Trace}\ \{[p_\mu, \varphi][p_\mu, \varphi]$$

$$+ [p_\mu, \varphi]_+ [p_\mu, \varphi]_+ + \frac{1}{\lambda'^4}[x_\mu, \varphi][x_\mu, \varphi]\}$$

If one adopt the unit in which the length is ~~measured in~~ expressed in units of $\lambda' = \frac{e^2}{m_e c^2}$, one obtains simply

$$\bar{L} = \mathrm{Trace}\{[\gamma_\mu, \varphi][\gamma_\mu, \varphi] + [\gamma'_\mu, \varphi]_+ [\gamma_\mu, \varphi]_+ + [x_\mu, \varphi][x_\mu, \varphi]\}$$

by multiplying the original $\bar{L}$ by a factor λ'^2 and replacing

$$\lambda' \gamma_\mu \to \gamma_\mu$$

$$\frac{x_\mu}{\lambda'} \to x_\mu$$

The supplementary condition will be

$$[\gamma_\mu, \{\tfrac{1}{2i}[\gamma_\mu, \varphi]_+ + [x_\mu, \varphi]\}] = 0$$

~~is commutative with φ~~

The notion of non-local field is modified a little bit.

Interaction of Non-local Fields

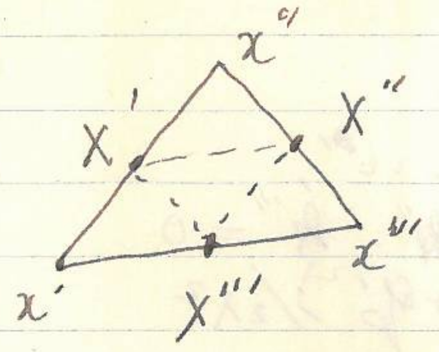
Suppose that two kinds of non-local fields φ and ψ interact with each other. Let us take a simple example of interaction between two scalar fields., ~~because~~ since non-local spinor fields are more difficult to deal with, we postpone the discussion of a more important case of interaction of a spinor and scalar fields.

Suppose that the interaction is represented by an additional term

$$\bar{L}_{int} = \mathrm{Trace}\,(g.\,\psi^*\varphi\psi$$

in the Lagrangian, where φ is a real scalar field and ψ is a complex scalar field.

$$\mathrm{Trace}\, g\,\psi^*\varphi\psi = g\int\cdots\int (x'|\psi^*|x'') \times (x''|\varphi|x''')(x'''|\psi|x')\,dx'dx''dx'''$$



$$\left.\begin{aligned} X' &= \tfrac{1}{2}(x'+x'') \\ X'' &= \tfrac{1}{2}(x''+x''') \\ X''' &= \tfrac{1}{2}(x'''+x') \end{aligned}\right\}$$

$$\left.\begin{aligned} r' &= x'-x'' = 2(X'''-X'') \\ r'' &= x''-x''' = 2(X'-X''') \\ r''' &= x'''-x' = 2(X''-X') \end{aligned}\right\}$$

$$\varphi(x,r) = \sum_{n,k_\mu} \varphi^{ex}_{n,k_\mu}(X)\, \varphi^{in}_{n,k_\mu}(r)$$

$$\sum_{\substack{n',n'',n''' \\ k',k'',k'''}} \int\cdots\int \underbrace{\psi^{*ex}_{n',k'}(X')\,\varphi_{n'',k''}(X'')\,\psi_{n''',k'''}(X''')}_{\Phi_{n',k';\,n'',k'';\,n''',k'''}(X'''-X'',\; X'-X''',\; X''-X')} dX'\,dX''\,dX'''$$

$$\Phi = G\, \psi^{*(in)}_{n',k'}(2(X'''-X''))\, \varphi^{(in)}_{n'',k''}(2(X'-X'''))\, \psi^{(in)}_{n''',k'''}(2(X''-X'))$$

$$= G\, \psi^{*in}_{n',k'}(\xi')\, \varphi^{in}_{n'',k''}(-\xi'-\xi'')\, \psi^{(in)}_{n''',k'''}(\xi''')$$

If all the or some of Φ's reduce to a certain multiple of $\delta(X'''-X'')\,\delta(X''-X')$, we have the ordinary local field theory.

Actually, for $x' = x'' = x'''$,

we have $p' = p'' = p''' = 0$

$$e^{-(q_1'^2 + q_2'^2 + q_3'^2 + q_0'^2)/2\lambda^2}$$

$$\times e^{-(q_1''^2 + q_2''^2 + q_3''^2 + q_0''^2)/2\lambda^2}$$

$$\times e^{-(q_1'''^2 + q_2'''^2 + q_3'''^2 + q_0'''^2)/2\lambda^2}$$

$$= e^{-(r_1'^2 + \cdots)/2\lambda^2}$$

$$\times e^{-((r_1' + r_3''')^2 + \cdots)/2\lambda^2}$$

$$\times e^{-(r_3''^2 + \cdots)/2\lambda^2}$$

$$= e^{-r_1'^2/\lambda^2} e^{-r_2'''^2/\lambda^2} e^{-\cancel{2} r_1' r_1'''/\lambda^2}$$

$$\times \cdots$$

$$= e^{-3(r_1' + r_1''')^2/4\lambda^2} e^{-(r_1' - r_1''')^2/4\lambda^2}$$

$$\times \cdots$$

$$\times \cdots$$

$$\times \cdots$$

$$\int_{-\infty}^{+\infty} e^{-2aq^2} dq = \sqrt{\frac{\pi}{2a}}$$

$$a = \frac{3}{\lambda^2}$$

$$\int e^{-3X^2/\lambda^2} dX = \sqrt{\frac{\pi}{6}} \cdot \lambda$$

$$a = 1/4\lambda^2$$

$$\int e^{-r^2/4\lambda^2} dr = \sqrt{\frac{\pi}{8}} \cdot \lambda$$

$$\left(\frac{\pi}{4\sqrt{3}} \lambda^2 \right)^4 \quad \delta\left(\tfrac{1}{2}(r'_\mu + r'''_\mu)\right) \delta(r'_\mu - r'''_\mu)$$
$$\delta(X''' - X') \, \delta\left(\tfrac{1}{2}(X' + X''' - 2X'')\right)$$

Normalization factor.

$$\left(\frac{1}{\pi \lambda^2} \right)^3$$

Thus the coupling G_F must be proportional to

$$G_F \sim \text{const.} \, g/\lambda^2$$

More generally, if some of k', k'', k'' are not zero, ~~we must investigate the Fourier transform of~~ we have

$$q_1'^2 + q_2'^2 + q_3'^2 + q_0'^2 = f_{\mu\nu} r_\mu' r_\nu'$$

$$q_1''^2 + q_2''^2 + q_3''^2 + q_0''^2 = g_{\mu\nu} r_\mu'' r_\nu''$$

$$q_1'''^2 + q_2'''^2 + q_3'''^2 + q_0'''^2 = h_{\mu\nu} r_\mu''' r_\nu'''$$

where $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are different symmetric coefficients.

$$\sum_\mu (q_\mu'^2 + q_\mu''^2 + q_\mu'''^2) = f_{\mu\nu} r_\mu' r_\nu' + h_{\mu\nu} r_\mu''' r_\nu''' + g_{\mu\nu}(r_\mu' + r_\mu''')(r_\nu' + r_\nu''')$$

$$= f_{\mu\nu} r_\mu' r_\nu' + h_{\mu\nu} r_\mu''' r_\nu''' + g_{\mu\nu}(r_\mu' r_\nu''' + r_\mu''' r_\nu')$$

$$e^{-\sum_\mu (q_\mu'^2 + q_\mu''^2 + q_\mu'''^2)/2\lambda^2} \equiv f(r_\mu', r_\mu''')$$

converges to a multiple of $\delta(r_\mu')\,\delta(r_\mu''')$ in the limit ~~with~~ $\lambda \to a$,

Only question is whether the ~~normalization~~ numerical factor is independent of k's or not.

$$q_1'^2 + q_2'^2 + q_3'^2 + q_0'^2$$
$$= q_1'^2 + q_2'^2 + q_3'^2 - q_0'^2 + 2q_0'^2$$

$$= r_1'^2 + r_2'^2 + r_3'^2 - r_0'^2 + 2\left(\frac{k_\mu' r_\mu'}{m}\right)^2$$

$$= r_1'^2 + r_2'^2 + r_3'^2 - r_0'^2 + \frac{2}{m^2}\left(k_1'^2 r_1'^2 + k_2'^2 r_2'^2\right.$$

$$+ k_3'^2 r_3'^2 + k_0'^2 r_0'^2 + 2 k_1' k_2' r_1' r_2' + \cdots$$

$$\left. + \cdots - 2 k_1' k_0' r_1' r_0' - \cdots \right)$$

$$= \left(1 + \frac{2k_1'^2}{m^2}\right) r_1'^2 + \left(1 + \frac{2k_2'^2}{m^2}\right) r_2'^2 + (\quad) r_3'^2$$

$$+ \left(-1 + \frac{2k_0'^2}{m^2}\right) r_0'^2 + \frac{4 k_1' k_2'}{m^2} r_1' r_2' + \cdots$$

$$- \frac{4 k_1' k_0'}{m^2} r_1' r_0' - \cdots$$

$$(q_1''')^2 + (q_2''')^2 + (q_3''')^2 + (q_0''')^2$$

$$= \left(1 + \frac{2k_1'''^2}{m^2}\right) r_1'''^2 + \left(1 + \frac{2k_2'''^2}{m^2}\right) r_2'''^2 + \cdots$$

$$+ \left(-1 + \frac{2k_0'''^2}{m^2}\right) r_0'''^2 + \frac{4 k_1''' k_2'''}{m^2} r_1''' r_2''' + \cdots$$

$$- \frac{4 k_1''' k_0'''}{m^2} r_1''' r_0''' - \cdots$$

$$(q_1'')^2 + (q_2'')^2 + (q_3'')^2 + (q_0'')^2$$

$$= \left(1 + \frac{2k_1''^2}{\mu^2}\right)(r_1' + r_1''')^2 + \cdots \quad (r_2' + r_2''')$$

$$+ \left(-1 + \frac{2k_0''^2}{\mu^2}\right)(r_0' + r_0''')^2 + \frac{4 k_1'' k_2''}{\mu^2}(r_1' + r_1''')$$

$$+ \cdots - \frac{4 k_1'' k_0''}{\mu^2}(r_1' + r_1''')(r_0' + r_0''') - \cdots$$

Interaction between Non-local fields

Let us take a simpler example of interaction between a non-local scalar (or pseudoscalar) field and a local scalar or spinor field. The interaction hamiltonian will be

$$L_{int} = g \,\mathrm{Trace}\, \rho\varphi = G\,\mathrm{Trace}\, \varphi\rho$$

$$= G\int\cdots\int (x'|\varphi|x'')\,\psi^{+}(x'')\,\psi(x')\,dx'dx''$$

or

$$= G\int\cdots\int \psi^{+}(x')\,(x'|\varphi|x'')\,\psi(x'')\,dx'dx''$$

provided that

$$(x'|\varphi|x'') = (x''|\varphi|x')$$

or $$\varphi(X, r) = \varphi(X, -r)$$

$$L_{int} = G\int\cdots\int dX\,dr\,\varphi(X,r)\,\psi^{+}(X-\tfrac{1}{2}r)\,\psi(X+\tfrac{r}{2})$$

$$\left.\begin{aligned}
\varphi(X,r) &= \sum_n \int v_{l'',n}\, e^{il''X}\, dl''\, \varphi_{l'',n}(r)\\
\psi(X+\tfrac{r}{2}) &= \int u_{l'''}\, e^{il''X}\, e^{i\frac{l'''}{2}r}\, dl'''\\
\psi(X-\tfrac{r}{2}) &= \int u^{+}_{-l'}\, e^{il'X}\, e^{-i\frac{l'}{2}r}\, dl'
\end{aligned}\right\}$$

$$\overline{L}_{int} = G \sum_n \int \cdots \int (2\pi)^{-4} \delta(l' + l'' + l''') dl' dl'' dl'''$$

$$u^{\dagger}_{-l'} v_{l'',n} u_{l'''} \int \varphi_{l'',n}(r) e^{-i\frac{l'-l'''}{2}r} dr$$

$$l' + l'' + l''' = K = 0$$

$$\frac{l'-l'''}{2} = l \qquad -l'' = L$$

$$l' + l''' = L$$

$$\overline{L}_{int} = G \underbrace{\sum_n}_{wait.} \int \cdots \int (2\pi)^{-4} \cancel{dl} \, dl \, dL$$

$$\times\, u^{\dagger}_{-\frac{L}{2}+l} v_{-L,n} u_{\frac{L}{2}-l} \Phi(\quad)$$

$$\Phi(\quad) = \int e^{-\frac{1}{2\lambda^2}(r_1'^2 + r_2'^2 + r_3'^2 + r_0'^2)}$$

$$e^{-i\frac{(l')'-(l''')'}{2}r'} dr'$$

$$= \text{const} \cdot \exp\left[-\text{const}\left\{ \left(\frac{(l_1')'-(l_1''')'}{2}\right)^2 + \left(\frac{(l_2')'-(l_2''')'}{2}\right)^2 + \left(\frac{(l_3')'-(l_3''')'}{2}\right)^2 + \left(\frac{(l_0')'-(l_0''')'}{2}\right)^2 \right\}\right]$$

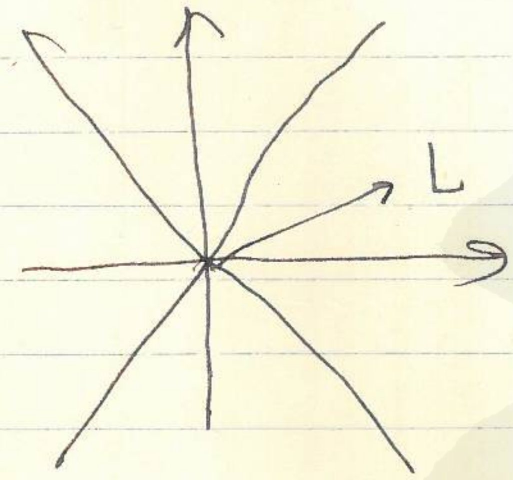
For a time-like L, we have

$$= \text{const} \exp\left[-\text{const}\left\{ l^2 - \frac{2(lL)^2}{L^2} \right\}\right]$$

See; Katayama, Soryu. Ken. 4 (1952), Nr. 9, 97

~~If~~ the above formula is true for a ~~space~~ time-like L, but, if L is space-like,

$\cancel{l^2 + 2l_0^{res}}$



$\cancel{\vec{l}_1^{\,2}} +$

$$\vec{l}'^2 + l_0'^2$$

$$= 2\vec{l}'^2 + (l_0'^2 - \vec{l}'^2)$$

$$= 2\frac{(lL)^2}{L^2} - l^2 > 0$$

$$l' = \frac{l_1L_1 + l_2L_2 + l_3L_3 - l_0L_0}{L_1^2 + L_2^2 + L_3^2 - L_0^2}$$

$(L_1 = l_1,\ L_2 = l_2,\ L_3 = l_3,\ L_0 = 0)$ $\quad l_1^2 + l_2^2 + l_3^2 + l_0^2$

Anyway, ~~independ~~ the Fourier transform of the form function decreases exponentially in any direction for large values of any one of $(l'_\mu - l''_\mu)^2$. $(\mu = 1, 2, 3, 0)$ or

$\cancel{(l'_\mu - l''_\mu)(l'_\mu - l''_\mu)}$

$l^2 - \frac{2(lL)^2}{L^2}$ for any time-like L and

$$2\frac{(lL)^2}{L^2} - l^2$$

for any space-like L.

Field Equations　　N.L.F. (19)

$$L = \mathrm{Trace}\,\tfrac{1}{2}\{-[p_\mu,\varphi][p_\mu,\varphi] + \{p_\mu,\varphi\}_+\{p_\mu,\varphi\}_+ \tfrac{4}{\lambda^4}[x_\mu,\varphi][x_\mu,\varphi] + [\gamma_\mu p_\mu + M, \rho]_{+} + \underset{\text{Trace}}{g}\,\varphi\,\rho\}\ ^{*}$$

$$(x'|\rho_{\alpha\beta}|x'') = \bar{\psi}_\alpha(x')\psi_\beta(x'')$$

for scalar meson.

$g\varphi\rho$ is to be replaced by

$g\varphi\tilde{\gamma}_5\rho$ where

$$(\tilde{\gamma}_5)_{\alpha\beta} = (\gamma_5)_{\beta\alpha}$$

more precisely,

$$(x'|(\tilde{\gamma}_5)_{\alpha\beta}|x'') = (\gamma_5)_{\beta\alpha}\,\delta(x'-x'')$$

$$L = \tfrac{1}{2}\int\cdots\int\Big\{\frac{\partial\varphi(X,r)}{\partial X_\mu}\frac{\partial\varphi(X,-r)}{\partial X_\mu} + 4\,\frac{\partial\varphi(X,r)}{\partial r_\mu}\frac{\partial\varphi(X,-r)}{\partial r_\mu} + \frac{4}{\lambda^4}\,r_\mu r_\mu\,\varphi(X,r)\varphi(X,-r)$$

$$+ \Big[-\gamma_\mu\frac{\partial\bar{\psi}(x')}{\partial x'_\mu} + M\bar{\psi}(x')\Big]\psi(x'')\delta(x'-x'') + \bar{\psi}(x')\Big[\gamma_\mu\frac{\partial\psi(x'')}{\partial x''_\mu} + M\psi(x'')\Big)\delta(x'-x'')\Big\}\,dX\,dr$$

$$^{*}\ (x'|[p_\mu,\varphi]^{|x''}_{(x''|}\{p_\mu,\varphi\}_+|x') = -\Big(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x''}\Big)\varphi(x',x'')$$

$$\times\Big(\frac{\partial}{\partial x''} - \frac{\partial}{\partial x'}\Big)\varphi(x'',x') = 4\,\frac{\partial\varphi(X,r)}{\partial r}\frac{\partial\varphi(X,-r)}{\partial r}$$

$\varphi_n(X)\ \varphi_n^{in}(X)$

$$-\iint g\,(x'|\varphi|x'')\,\bar\psi_\alpha(x'')\,\psi_\alpha(x')\ dx'\,dx''$$

or

$$-\iint g\ \varphi(X,r)\,\bar\psi_\alpha(X-\tfrac{1}{2}r)\,\psi_\alpha(X+\tfrac{1}{2}r)\ dX\,dr$$

Field equations:

$$-\frac{\partial^2\varphi(X,r)}{\partial X_\mu\partial X_\mu}-4\frac{\partial^2\varphi(X,r)}{\partial r_\mu\partial r_\mu}+\frac{4}{\lambda^4}r_\mu r_\mu\,\varphi(X,r)-g\,\bar\psi_a(X+\tfrac{1}{2}r)\times\psi_a(X-\tfrac{1}{2}r)=0$$

or

$$\frac{\partial^2\varphi(X,r)}{\partial X_\mu\partial X_\mu}+4\frac{\partial^2\varphi(X,r)}{\partial r_\mu\partial r_\mu}-\frac{4}{\lambda^4}r_\mu r_\mu\,\varphi(X,r)=-g\,\bar\psi_a(X+\tfrac{1}{2}r)\,\psi_a(X-\tfrac{1}{2}r)$$

$$\left(\gamma_\mu\frac{\partial\psi(x')}{\partial x'_\mu}+M\psi(x')\right)=g\int\cancel{x'|\varphi|x'}\,\psi(x'')(x''|\varphi|x')\,dx''$$

$$\left(\gamma_\mu\frac{\partial\bar\psi(x')}{\partial x'_\mu}-M\bar\psi(x')\right)=-g\int(x'|\varphi|x'')\,\bar\psi(x'')\,dx''$$

$\varphi^{in}_n(r)$: complete set of orthogonal eigenfunctions of satisfying

$$-4\frac{\partial^2\varphi_n(X,r)}{\partial r_\mu\partial r_\mu}+\frac{4}{\lambda^4}r_\mu r_\mu\varphi_n(X,r)$$

$$=m_n^2\varphi_n(X,r).$$

$$\varphi(X,r)=\sum_n\varphi^{ex}_n(X)\varphi^{in}_n(r)$$

$$\frac{\partial^2\varphi^{ex}_n(X)}{\partial X_\mu\partial X_\mu}-m^2=-g\int\varphi^{in}_n(r)\bar{\psi}_\alpha(X+\tfrac{1}{2}r)$$

$$\times\psi_\alpha(X-\tfrac{1}{2}r)\,dr.$$

$$=-g\int\varphi^{in}_n(x'-x'')\bar{\psi}_\alpha(x')\psi_\alpha(x'')\,dx'''\delta(\tfrac{x'+x''}{2}-X)\,dx'$$

provided that φ_n is normalized:

$$\int(\varphi^{in}_n(r))^2\,dr=1,$$

$$\left(\gamma_\mu\frac{\partial\psi(x')}{\partial x'_\mu}+M\psi(x')\right)=g$$

$$\times\sum_n\int\psi(x'')\varphi^{ex}_n\left(\frac{x'+x''}{2}\right)\varphi^{in}_n(x''-x')\,dx''$$

$$=g\sum_n\int\psi(x'')\varphi^{ex}_n(x''')\varphi^{in}_n(x''-x')$$

$$\times\delta\left(\frac{x'+x''}{2}-x'''\right)dx''dx'''$$

$$\frac{x'+x''}{2}=x'''$$

$$\left(\frac{\partial^2 \varphi_n^{ex}(x'')}{\partial x_\mu \partial x_\mu} - m_n^2 \varphi_n^{ex}(x'')\right)$$

$$= -g \int \Phi_n(x', x'', x''') \bar{\psi}_\alpha(x') \psi_\alpha(x''') dx' dx'''$$

with

$$\Phi_n(x', x', x'') = \varphi_n^{in}(x' - x''') \delta\left(\frac{x' + x'''}{2} - x''\right)$$

$$\left(\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')\right)$$

$$= g \sum_n \int \Phi'_n(x', x'', x''') \varphi_n^{ex}(x'') \psi(x''') dx'' dx'''$$

$$\Phi'_n(x', x'', x''') = \varphi_n^{in}(x''' - x')$$
$$\times \delta\left(\frac{x' + x'''}{2} - x''\right)$$

$\Phi'_n = \pm \Phi_n$
accordings as n is even or odd.

N. L. F (19)

$$\gamma_\mu \frac{\partial \bar{\Psi}(x')}{\partial x'_\mu} - M\bar{\Psi}(x')$$

$$= g\sum_n \int \bar{\Psi}(x'')\varphi_n^{\ell l}\left(\frac{x'+x''}{2}\right)\varphi_n^{in}(x'-x'')\,dx''$$

$$= g\sum_n \bar{\Psi}(x''')\varphi_n^{\ell l}(x'')\,\Phi_n^{\ell l}(x',x'',x''')\,dx''$$
$$\times\, dx'''$$

$$\Phi_n^{\ell l}(x',x'',x''') = \varphi_n^{in}(x'-x''')\,\delta\left(\frac{x'+x'''}{2}-x''\right)$$

If we confine our attention to <u>any one</u> of the even state (~~with~~ a set of n_1, n_2, n_3, n_0 with $n = n_1+n_2+n_3-n_0 = \text{even}$) one can process just as done in Moller paper,

$$\Phi(x',x'',x''') = (2\pi)^{-8}\int G(l',l''')$$

$$\cancel{\exp i\left\{k\left(\frac{x'+x'''}{2}-x''\right)+l(x'-x''')\right\}\,dk\,dl}$$

$$k = \quad \exp i\{l'x'+l'''x'''-(l'+l''')x''\}\,dl'\,dl'''$$

$$= (2\pi)^{-8} \int G(l', l''') \exp i\left\{ L\left(\frac{x'+x''}{2} - x''\right) + l(x'-x'')\right\} \times dL\, dl$$

with

$$L = l' + l''' \qquad l = \frac{l' - l'''}{2}$$

$$G(l, L) = G(l)$$

in our case.

In particular for

$$n_1 = n_2 = n_3 = n_0 = 0,$$

we ~~have~~ may take for ~~our~~ coord. sys.

$$G(l) = \exp -\frac{\lambda^2}{2}(l_1^2 + l_2^2 + l_3^2 + l_0^2)$$

independent ~~of~~ of the external motion.

The form factors appearing in the self-energies are

$$|G(K, -K-P')|^2 \delta(K^2+M^2)$$

$$l = K + P/2$$

$$|G(P', k-P')|^2 \delta(k^2+m^2)$$

$$l = P' - k/2$$

$$|G(K, -P')|^2 \delta(K^2+M^2)$$

$$l = K + P'$$

Thus, as long as one can take up a definite ~~internal~~ set of ~~even~~ intermediate state, the convergence of self-energies are secured.
However, the situation is more ~~so~~ complicated due to the ~~fa~~ supplement condition.
Let us assume that, even in the presence of interaction, the S.C.

$$\frac{\partial}{\partial X_\mu}\left(\frac{\partial}{\partial r_\mu} + \frac{1}{\lambda^2} r_\mu\right)\varphi = 0$$

is true. (I shall discuss ~~the~~ whether this is legitimate or not later on.)
~~On a~~ Once we assume S.C., the form factor for

$$n = n_1 = n_2 = n_3 = 0$$

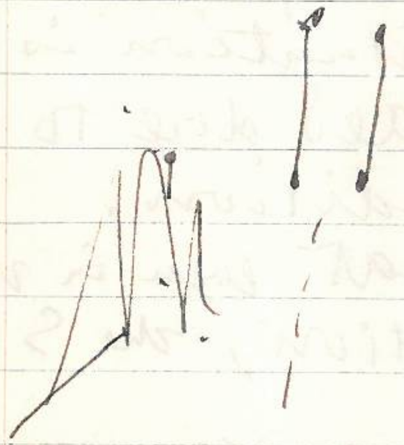
will be

$$G(l) = \exp -\frac{\lambda^2}{2}(l_1^2 + l_2^2 + l_3^2 + l_0^2)$$

in the rest system for the meson in question, or in the original system

$$G(l) = \exp -\frac{\lambda^2}{2}\left(l^2 - \frac{2(l\,k)^2}{k^2}\right)$$

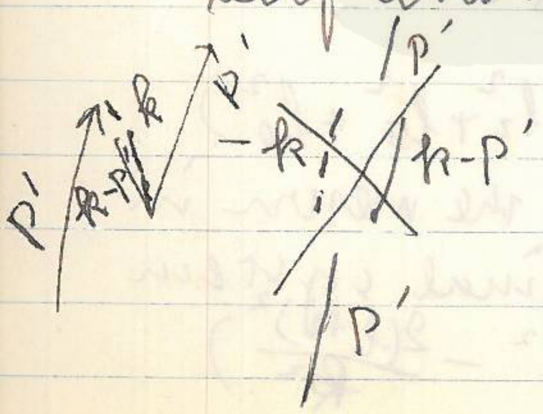
Now, ~~the~~ k for the meson in the calculation of self-energy of the meson is that of the meson in the initial and final states.



So that it is simply $(k = -L = p')$
$k_1 = k_2 = k_3$ $k_0 = m$
for a meson at rest.

In other word, the form factor is $G(l) = \exp\frac{\lambda^2}{2}\left(\vec{k}^2 + (k_0 + \frac{m}{2})^2\right)$
i.e. ~~independent~~ with l_1, l_2, l_3, l_0 for the original system. (rest)

The ~~cor~~ k for the nucleon self-energy is for the first term:



$-k = L$

$(p' - k/2)^2 - \frac{2((p' - \frac{k}{2})k)^2}{k^2}$

For the nucleon at rest
$p_1' = p_2' = p_3' = 0$ $p_0' = M$

$$k^2 = -m^2$$

$$\left(\frac{\vec{k}}{2}\right)^2 - \left(M - \frac{k_0}{2}\right)^2 + \frac{2\left(-\left(\frac{\vec{k}}{2}\right)^2 \left(M - \frac{k_0}{2}\right)k_0\right)^2}{m^2}$$

$$= \left(\frac{\vec{k}}{2}\right)^2 - M^2 + Mk_0 - \left(\frac{k_0}{2}\right)^2$$

$$+ \frac{2(m^2 - Mk_0)^2}{m^2}$$

$$= -m^2 - M^2 + Mk_0 + \frac{2(m^4 - 2m^2Mk_0 + M^2k_0^2)}{m^2}$$

$$= +m^2 - M^2 - 3Mk_0 + \frac{2M^2k_0^2}{m^2} > 0$$

$$\cancel{= 2(M^2k_0}$$

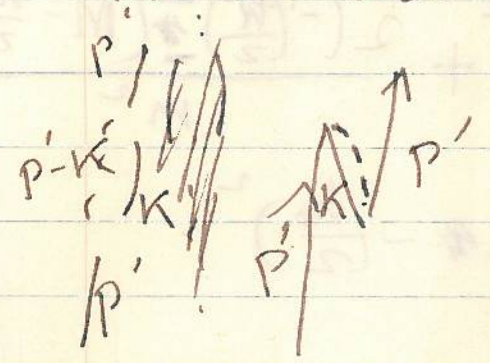
$$\geq \frac{2M^2}{\cancel{m^2}} - 3Mm - M^2 + m^2 > 0$$

In any case, for a very large value of

$$k_0 \gg M,$$

the argument in the Gauss fn is always as large as k_0^2.

For the second term, the meson in the intermediate state



$k = -L = p' - K$, which is ~~space~~-like

If the supplementary condition is assumed to the virtual meson in the ~~virtual~~ intermediate state

$k_\mu b_\mu \varphi = 0$,

we have the restriction on the internal motion of the meson:

$n_1' = 0$

if we take ~~the int the~~ as new x-axis the direction of k_μ. In that case, n_0' is no longer restricted to 0 and the infinite degeneracy reappears.

However, ~~if we~~ this came from the (incoming) free meson field for which k_μ is always time-like. The factor G in

$$|G(K,-P')|^2 \delta(K^2+M^2)$$

becomes thus

$$\exp -\frac{\lambda^2}{2}\left((K-P')_1^2+(K-P')_2^2\right.$$
$$\left.+(K-P')_3^2+(K-P')_0^2\right)$$

in the rest system for the meson

$$\vec{k} = K-P'$$
$$k_0 = \sqrt{(K-P')^2+m^2}$$

$$(\vec{K}-\vec{P}')^2-(K_0-P_0')^2$$
$$+\frac{2\{(K-P')(K-P')-k_0(K_0-P_0')\}^2}{m^2}$$

rest system:

$$\cancel{(\vec{K}-\vec{P}')^2-(K_0-P_0')^2} \qquad k_0=\sqrt{K^2+m^2}$$

$$\vec{K}^2-(K_0-M)^2$$
$$+\frac{2\{\vec{K}^2-k_0(K_0-M)\}^2}{m^2} \qquad 2M(K_0-M)$$

$$=\vec{K}^2-K_0^2+2K_0M-M^2$$
$$+\frac{2}{m^2}\left\{\vec{K}^2-\sqrt{\vec{K}^2+m^2}(K_0-M)\right\}^2>0$$

$$\shortparallel$$

$$(\vec{K}^2)^2-2\sqrt{\vec{K}^2+m^2}(K_0-M)\vec{K}^2$$
$$+(\vec{K}^2+m^2)(K_0-M)^2$$

$$\varphi(x, r) = \int w_{l', l''} e^{i(l'x' + l''x'')} (dl')(dl'')$$

$$= \int v(L, l) e^{i(LX + lr)} (dL)(dl)$$

$$\left. \begin{array}{l} L = l' + l'' \\ l = \frac{l' - l''}{2} \end{array} \right\} \quad \left. \begin{array}{l} l' = \frac{L}{2} + l \\ l'' = \frac{L}{2} - l \end{array} \right]$$

$$\int \left\{ -L^2 - 4\left(l^2 - \frac{1}{\lambda^4} \frac{\partial^2}{\partial l_\mu \partial l_\mu}\right) \right\} v(L, l) e^{i(LX + lr)} \times (dL)(dl)$$

$$= -g \int \bar{u}_\alpha\left(-\tfrac{L}{2} - l\right) u_\alpha\left(\tfrac{L}{2} - l\right) e^{i(LX + lr)} (dL)(dl)$$

$$\psi_\alpha(x'') = \int u_\alpha(l'') e^{il''x''} (dl'')$$

$$\bar{\psi}_\alpha(x') = \int \bar{u}_\alpha(l') e^{-il'x'} (dl')$$

$$= \int \bar{u}_\alpha(-l') e^{il'x} (dl')$$

$$\bar{\psi}_\alpha\left(X + \tfrac{1}{2} r\right) \psi_\alpha\left(X - \tfrac{1}{2} r\right)$$

$$= \int \bar{u}_\alpha(-l') u_\alpha(l'') e^{i(l'x' + l''x'')} (dl')(dl'')$$

$$= \int \bar{u}_\alpha\left(-\tfrac{L}{2} - l\right) u_\alpha\left(\tfrac{L}{2} - l\right) e^{i(LX + lr)} (dL)(dl)$$

$$\left\{-L^2-4\left(l^2-\frac{1}{\lambda^4}\frac{\partial^2}{\partial l_\mu \partial l_\mu}\right)\right\} v(L,l)$$
$$= -g\,\bar{u}_\alpha\left(-\frac{L}{2}-l\right)u_\alpha\left(\frac{L}{2}-l\right)$$

$$\int (i\gamma_\mu l'_\mu + M)_{\alpha\beta} u_\beta(l')\, e^{il'x'}(dl')$$
$$= g\int u_\alpha(l'')e^{il''x''}\, w(l''',l')\, e^{i(l'''x''+l'x')}\, dx''\, dl''\, dl'''\, dl'$$
$$= g\int u_\alpha(-l''')w(l''',l')\cdot(2\pi)^4\, dl'''\, e^{il'x'}\, dl'$$
$$(i\gamma_\mu l'_\mu + M)_{\alpha} u(l') = (2\pi)^4 g\int u(-l''')w(l''',l') \times dl'''$$

Particular solution:
~~$v(L,l)=$~~
$$v_{0,L}(l) = N_0\, e^{-\frac{\lambda^2}{2}(l_1^2+l_2^2+l_3^2+l_0^2)}$$
$$v(L,l) = b_L\, v_{0L} \qquad \int (v_{0,L}(l))^2\, dl = 1$$
$$(-L^2 \mp m_0^2)\quad b_L v_{0L}(l) = -g\,\bar{u}_\alpha\left(-\frac{L}{2}-l\right)u_\alpha\left(\frac{L}{2}-l\right)$$
$$(-L^2 \mp m_0^2)\, b_L = -(2\pi)^4 g\int v_{0L}(l)\,\bar{u}_\alpha\left(-\frac{L}{2}-l\right) u_\alpha\left(\frac{L}{2}-l\right) dl$$

$$\int(-L^2-m_0^2)\,b_L\,e^{iLX}\,dL$$

$$= -g(2\pi)^4\int v_{0L}(l)\,\bar{u}_\alpha\left(-\tfrac{L}{2}-l\right)e^{i(\frac{L}{2}+l)(X+\frac{1}{2}r)}$$

$$\cdot u_\alpha\left(\tfrac{L}{2}-l\right)e^{i(\frac{L}{2}-l)(X-\frac{1}{2}r)}\,(dL)(dl)\,e^{-ilr}$$

$$= -g(2\pi)^4\iint \bar{u}_\alpha(-l')\,e^{il'x'}$$

$$u_\alpha(l'')\,e^{il''x''}\cdot\cdot\, v_{0L}(l)\,e^{-ilr}\,dL\,dl$$

$$= -g(2\pi)^4\int\cdot\cdot\int \bar{u}_\alpha(-l')\,e^{il'x'}\,u_\alpha(l'')\,e^{il''x''}$$

$$v_0(l',l'')\,e^{-i\frac{l'-l''}{2}(x'-x'')}\,dl'\,dl''$$

$$\cancel{= -g(2\pi)^4\int\cdot\cdot\int \bar{u}_\alpha(-l')\,e^{il'x'}\,u_\alpha(l'')\,e^{il''x''}}$$

$$\cancel{v_0(l''',l^{IV})\,e^{-i\frac{l'''-l^{IV}}{2}(x'-x'')}\,dl'\,dl''\,dl'''\,dl^{IV}\,\delta(l'-l''')\,\delta(l''-l^{IV})}$$

$$\cancel{= -g(2\pi)^4\,\bar{u}_\alpha(x')\,u_\alpha(x'')}$$

$$= -g(2\pi)^4\int\cdot\cdot\int \Phi(x',x'')\,\bar{u}_\alpha(x')\,u_\alpha(x'')$$

Interaction between local spinor field and N.L. scalar field　　N.L.F. (25)

$$\bar{L}_{\cancel{total}} = \bar{L}_{sc} + \bar{L}_{sp} + \bar{L}_{int}$$

$$\bar{L}_{sc} = \frac{1}{2}\,\mathrm{Trace}\{[p_\mu,\varphi][p_\mu,\varphi] - [p_\mu,\varphi]_+[p_\mu,\varphi]_+ + \frac{4}{\lambda^4}[x_\mu,\varphi][x_\mu,\varphi]\}$$

$$\bar{L}_{sp} = \frac{1}{2}\,\mathrm{Trace}\{[+i\,\gamma_\mu p_\mu, \rho_\mu]_+ - [M,\rho]_+\}$$

$$\bar{L}_{int} = g\,\mathrm{Trace}\;\varphi\rho$$

$(x'|\rho_\mu|x'') = \bar{\psi}(x')\gamma_\mu\psi(x'')$

$(x'|\rho|x'') = \bar{\psi}(x')\psi(x'')$

$$(x'|\rho_{\alpha\beta}|x'') = \bar{\psi}_\alpha(x')\psi_\beta(x'')$$

$$(x''|\rho_{\beta\alpha}|x') = \bar{\psi}_\beta(x'')\psi_\alpha(x')$$

$$= (x'|\rho^*_{\alpha\beta}|x'') = \bar{\psi}_\beta(x'')\psi_\alpha(x')$$

hermitian

$$(x'|\tilde{\varphi}|x'') = (x''|\varphi|x')$$

$$\tilde{\varphi}(X,r) = \varphi(X,-r)$$

$$\bar{L}_{sc} = -\frac{1}{2}\int\!\!\cdots\!\!\int\Big\{\frac{\partial\tilde{\varphi}(X,r)}{\partial X_\mu}\frac{\partial\varphi(X,r)}{\partial X_\mu} + 4\frac{\partial\tilde{\varphi}(X,r)}{\partial r_\mu}\frac{\partial\varphi(X,r)}{\partial r_\mu} + \frac{4}{\lambda^4}r_\mu r_\mu\tilde{\varphi}(X,r)\varphi(X,r)\Big\}dX\,dr$$

$$\bar{L}_{sp} = +\frac{1}{2}\Big\{\int \Big\{ \frac{\partial \bar{\psi}(x')}{\partial x'_\mu}\gamma_\mu - M\bar{\psi}(x')\Big\}\psi(x')dx'$$

$$-\int \bar{\psi}(x')\Big\{\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')\Big\}dx'$$

$$\bar{L}_{int} = \begin{cases} g\iint \varphi(X,r)\bar{\psi}_a(X-\tfrac{1}{2}r)\psi_a(X+\tfrac{1}{2}r) \\ \qquad \times dX\,dr \\ g\iint (x'|\varphi|x'')\bar{\psi}_a(x'')\psi_a(x')dx'dx'' \end{cases}$$

Field equations:

$$\frac{\partial^2\varphi(X,r)}{\partial X_\mu \partial X_\mu} + 4\frac{\partial^2 \varphi(X,r)}{\partial r_\mu \partial r_\mu} - \frac{4}{\lambda^4} r_\mu r_\mu \varphi(X,r)$$

$$+ g\,\bar{\psi}_a(X+\tfrac{1}{2}r)\psi_a(X-\tfrac{1}{2}r) = 0$$

$$\left.\begin{array}{l} \gamma_\mu \dfrac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x') \\ \qquad - g\int (x''|\varphi|x')\psi_a(x'')dx'' = 0 \\ \dfrac{\partial \bar{\psi}(x')}{\partial x'_\mu}\gamma_\mu - M\bar{\psi}(x') \\ \qquad + g\int (x'|\varphi|x'')\bar{\psi}(x'')dx'' = 0 \end{array}\right\}$$

First, ignore all about supplementary condition and expand $\varphi(X, r)$ into series of orthogonal function $\chi_n^{(in)}$ defined by

$$\left\{4\frac{\partial^2\varphi}{\partial r_\mu \partial r_\mu} + \frac{4}{\lambda^4} r_\mu r_\mu \varphi + m_n^2 \varphi\right\} \chi_n^{(in)}(r) = \left\{\begin{array}{l} \cancel{\varphi_n^{(in)}(r)} \\ 0 \end{array}\right.$$

$$m_n^2 \gtrless 0$$

$\varphi_n^{(in)}(r)$: even — real

$\varphi_n^{(in)}(r)$: odd — pure imag.

$\tilde{\varphi}(r) = -\varphi(-r)$

$$\varphi(X, r) = \varphi_n^{(ex)}(X)\,\chi_n^{(in)}(r)$$

normalization

$$\int \tilde{\chi}_n(r)\,\chi_n(r)\,dr = 1.$$

$$\frac{\partial^2 \varphi_n^{(ex)}(X)}{\partial X_\mu \partial X_\mu} - m_n^2 \varphi_n^{(ex)}(X) \qquad x' - X - X + x'' = 0$$

$$= -g \int \underbrace{\tilde{\chi}_n^{(in)}(X, r)\,\overline{\psi}_\alpha(\overset{x'}{X + \tfrac{1}{2}r})\,\psi_\alpha(\overset{x''}{X - \tfrac{1}{2}r})}\;dr$$

$$r = 2(x' - X) = 2(X - x'')$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g \sum_n \int \underbrace{(x'|\tilde{\varphi}_n|x'')\,\varphi_n^{(ex)}(X)}_{\tilde{\varphi}^{(in)}}\,\psi(x'')\,dx'' \qquad X = \frac{x' + x''}{2}$$

$$\frac{\partial \overline{\psi}(x')}{\partial x'_\mu}\gamma_\mu - M\overline{\psi}(x')$$

$$= -g \sum_n \int \overline{\psi}(x'')\,\overbrace{(x''|\tilde{\varphi}|x')}^{\varphi_n^{(ex)}(X)\,\chi_n(-r)}\,dx''$$

$$\frac{\partial^2 \varphi^{(ex)}_{n\mu}(x'')}{\partial x''_\mu \partial x''_\mu} - m^2_{n\mu} \varphi^{(ex)}_{n\mu}(x'')$$

$$= -g \int \Phi_{n\mu}(x', x'', x''') \bar{\psi}(x') \psi(x''') \, dx' dx'''$$

$$\Phi_{n\mu}(x', x'', x''') = \tilde{\varphi}^{in}_{n\mu}(x' - x''') \delta\left(\frac{x' + x'''}{2} - x''\right)$$

$$\tilde{\chi}_{n\mu}$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g \sum_n \int \Phi_{n\mu}(x', x'', x''') \varphi^{ex}_{n\mu}(x'') \psi(x''') \, dx'' dx'''$$

etc.

Thus, if a form function, for example

$$\Phi_0(x', x'', x''')$$

was ~~so chosen as to give converg~~ appropriately chosen, one can expect the same convergence as in Møller's theory.

Unfortunately, ~~the~~ ~~this~~ such a function does not seem to exist for the following reasons:
An eigenfunction of the type

$$e^{-\frac{1}{2\lambda^2}(x_1^2+x_2^2+x_3^2+x_0^2)}$$

is not invariant. One might weight

$$e^{-\frac{1}{2\lambda^2}\left(x_\mu x_\mu-\frac{2(k_\mu x_\mu)^2}{k_\mu k_\mu}\right)}$$

which is formally invariant, where k_μ is an arbitrary fixed time-like vector. ~~Then~~ However, one will be asked how the direction of k_μ is determined. Only if k_μ is determined in a reasonable way, above considerations really make sense.

This difficulty is intimately connected with the fact that hermitian invariant operators are, in general, not positive definite. Even, if one makes it positive definite by taking the square, for instance, the infinite degeneracy remains

(Born,

Schrödinger)

In order to ~~suppress the~~ remove the degeneracy, one need further restriction on the internal motion. In the case of oscillator model, one may assume a S.C, however, the S.C. was so chosen as to connect the external and internal motion. ~~that~~ For that reason, the ~~further develop. of~~ field equations does not have the simple form which is ~~equivalent to those~~. reduced to those of n.l. interaction theory.

In other words, the internal motion is affected by the change in the external motion, which makes our picture really very complicated.

Supplementary Conditions, I, N.L.F. (28)

For the reasons above mentioned, the coupling between external and internal motion is to be considered more seriously:

$$\frac{\partial}{\partial X_\mu}\left(-\frac{\partial}{\partial r_\mu}+\frac{1}{\lambda^2}r_\mu\right)\varphi=0$$

or

$$[\overset{\partial}{p}_\mu,\{-i[p_\mu,\varphi]_+ +\frac{1}{\lambda^2}[x_\mu,\varphi]\}]=0$$

$$\sum_{j=1,2,3}\frac{\partial}{\partial X_j}\left(-\frac{\partial}{\partial r_j}+\frac{1}{\lambda^2}r_j\right)\varphi(X,r)$$

$$-(i)\frac{\partial}{\partial X_0}\left((i)\frac{\partial}{\partial r_0}+\frac{1}{\lambda^2}(i)r_0\right)\varphi=0$$

If φ is independ of X_j and only depends on X_0

$$\left(\frac{\partial}{\partial r_0}+\frac{1}{\lambda^2}r_0\right)\varphi(X_0,r)=0$$

$$\varphi\propto e^{-\frac{1}{2\lambda^2}r_0^2}$$

Further if φ is a function of $k_\mu X_\mu$, where k_μ is a time-like vector, then φ must ~~be of the form~~ leave the factor

$$e^{-\frac{1}{2\lambda^2}(r_0')^2}$$

$$r_0' = \frac{-k_\mu r_\mu}{\sqrt{-k_\mu k_\mu}}.$$

$$\left[\lambda\frac{\partial}{\partial r_0} + \frac{1}{\lambda}r_0,\ -\lambda\frac{\partial}{\partial r_0} + \frac{1}{\lambda}r_0\right]\varphi$$

$$= 2\varphi$$

$$b_0 = \frac{1}{\sqrt{2}}\left(\lambda\frac{\partial}{\partial r_0} + \frac{1}{\lambda}r_0\right) \qquad [b_0, b_0^*] = 1$$

$$b_0^* = \frac{1}{\sqrt{2}}\left(-\lambda\frac{\partial}{\partial r_0} + \frac{1}{\lambda}r_0\right)$$

Similarly,

$$\left.\begin{aligned} b_j &= \frac{1}{\sqrt{2}}\left(\lambda\frac{\partial}{\partial r_j} + \frac{1}{\lambda}r_j\right) \\ b_j^* &= \frac{1}{\sqrt{2}}\left(-\lambda\frac{\partial}{\partial r_j} + \frac{1}{\lambda}r_j\right) \end{aligned}\right\}$$

$$\left[\frac{\partial}{\partial X_j}\cdot b_j^* + \frac{\partial}{\partial X_0}b_0\right]\varphi = 0$$

$$\left.\begin{aligned} b_j^*\chi_n &= \sqrt{n+1}\,\chi_{n+1} \\ b_j\chi_n &= \sqrt{n}\,\chi_{n-1} \end{aligned}\right\}$$

$$(n+1|b_j^*|n) = \sqrt{n+1}, \quad (n-1|b_j|n) = \sqrt{n}.$$

031

N.L.F. 29

χ_n : normalized osc. eigen fn

$$\chi_n = N_n H_n(x) e^{-\frac{x^2}{2}} \qquad x = \frac{r}{\lambda}$$

$$\varphi(X, r) = \sum_n \varphi^{(ex)}_{n_1 n_2 n_3 n_0}(X)\, \chi^{(r)}_{n_1 n_2 n_3 n_0}$$

$$\left\{ \frac{\partial}{\partial X_j} b_j^* + \frac{\partial}{\partial X_0} b_0 \right\} \varphi$$

$$= \left\{ \sum_j \frac{\partial \varphi^{ex}_{n_1 n_2 n_3 n_0}}{\partial X_j} \sqrt{n_j+1}\, \chi^{(r)}_{n_1, n_j+1, n_3, n_0} + \frac{\partial \varphi^{ex}_{n_1 n_2 n_3 n_0}}{\partial X_0} \sqrt{n_0}\, \chi^{(r)}_{n_1, n_2, n_3, n_0-1} \right\} = 0$$

$$= \sum \chi_{n_1 n_2 n_3 n_0} \left\{ \sqrt{n_1} \frac{\partial \varphi^{(ex)}_{n_1-1, n_2, n_3, n_0}(X)}{\partial X_1} + \sqrt{n_2} \frac{\partial \varphi^{(ex)}_{n_1, n_2-1, n_3, n_0}(X)}{\partial X_2} + \sqrt{n_3} \frac{\partial \varphi^{(ex)}_{n_1, n_2, n_3-1, n_0}(X)}{\partial X_3} + \sqrt{n_0+1} \frac{\partial \varphi_{n_1, n_2, n_3, n_0+1}(X)}{\partial X_0} \right\} = 0$$

$n_1 = n_2 = n_3 = 0$:

$$\frac{\partial \varphi_{0,0,0,n_0+1}}{\partial X_0} = 0$$

$n_1 = 1,\ n_2 = n_3 = 0$

$$\frac{\partial \varphi_{0,0,0,n_0}}{\partial X_1} + \sqrt{n_0+1} \frac{\partial \varphi_{1 0 0 0 n_0+1}}{\partial X_0} = 0$$

$$\varphi^{ex}_{n_1 n_2 n_3 n_0}(X) = \int \varphi^{ex}_{n_1 n_2 n_3 n_0}(k) e^{i(kX - k_0 X_0)} d^4k$$

$$\sqrt{n_1}\, k_1 \varphi^{(ex)}_{n_1-1, n_2, n_3, n_0}(k)$$

$$+ \sqrt{n_2}\, k_2 \varphi^{(ex)}_{n_1, n_2-1, n_3, n_0}(k) + \sqrt{n_3}\, k_3 \varphi^{(ex)}_{n_1, n_2, n_3-1, n_0}(k)$$

$$- \sqrt{n_0+1}\; k_0 \varphi^{(ex)}_{n_1, n_2, n_3, n_0+1}(k) = 0$$

$n_1 = n_2 = n_3 = 0$

$$\sqrt{n_0+1}\; k_0 \varphi^{(ex)}_{0,0,0,n_0+1}(k) = 0$$

$$\varphi^{(ex)}_{0,0,0,1}(k) = \varphi^{(ex)}_{0,0,0,2}(k) = \cdots = 0$$

if $k_0 \neq 0$.

$n_1 = 1, \quad n_2 = n_3 = 0$.

$$k_1 \varphi^{(ex)}_{0,0,0,n_0}(k) = \sqrt{n_0+1}\; k_0 \varphi^{(ex)}_{1,0,0,n_0+1}(k)$$

$$k_1 \varphi^{(ex)}_{0,0,0,0}(k) = k_0 \varphi^{(ex)}_{1,0,0,1}(k).$$

$$\varphi^{(ex)}_{1,0,0,2}(k) = \varphi^{(ex)}_{1,0,0,3}(k) = \cdots = 0$$

if $k_1 \neq 0$.

If further $k_2 = k_3 = 0$.

$$\sqrt{n_1}\, k_1 \varphi^{(ex)}_{n_1-1, n_2, n_3, n_0}$$

$$= \sqrt{n_0+1}\; k_0 \varphi^{(ex)}_{n_1, n_2, n_3, n_0+1}$$

$$(0,0,0,0) \to (1,0,0,1) \to (2,0,0,2)$$

$$\to \cdots\cdots \to (n,0,0,n)$$

$$\sqrt{n+1}\, k_1 \varphi_{n,0,0,n} = \sqrt{n+1}\, k_0 \varphi_{n+1,0,0,n+1}$$

$$\frac{\varphi_{n+1,0,0,n+1}}{\varphi_{n,0,0,n}} = \frac{k_1}{k_0}$$

Thus, if $|k_1/k_0| < 1$

$$\sum_n \int \varphi^{(ex)}_{n,0,0,n}(k) \chi_{n,0,0,n}(r) \cdot e^{i(kx - k_0 x_0)} dk$$

$$= \int f(k, r)\, e^{i(kx - k_0 x_0)} dk$$

$$f(k, r) = \sum_n \varphi^{(ex)}_{n,0,0,n}(k) \chi_{n,0,0,n}(r)$$

$$= \sum_n \left(\frac{k_1}{k_0}\right)^n \chi_{n,0,0,n}(r) \quad \cancel{e^{i(k_1 x_1 - k_0 x_0)} dk}$$

$$\sum_n \int \left(\frac{k_1}{k_0}\right)^{2n} \chi^2_{n,0,0,n}(r)\, dr$$

$$= \frac{1}{1 - \left(\frac{k_1}{k_0}\right)^2} = \frac{k_0^2}{k_0^2 - k_1^2}$$

For any ~~sp~~ time-like direction k $f(k, r)$ is quadratically integral fn in r-space.

On the contrary, for any space-like k, i.e. $\left|\frac{k}{k_0}\right| > 1$, $f(k, v)$ is not a ~~bounded~~ quadratically integral fn of v. Thus, in order that $\varphi(x, v)$ it is a quadratically integral fn of ~~x and~~ v, φ must be the superposition of plane waves with ~~space~~-like k. This is certainly true for the free particle with a real mass, but ~~for~~ the field quantity φ, which ~~is~~ satisfies the field equation with the interaction, contains, in general, the superposition of plane waves with space-like k as well as time-like k. For instance, the field equation with the static source ~~thus~~ is the superposition of wave functions ~~space waves~~ with $k_0 = 0$ ~~the space-like k~~ and of ~~those with~~ the free plane-waves.

~~In any perturbation calculation,~~

Møller has suggested a form function which decreases for $|\pi^2| > M$ for both signs of $\pi^2 = l^2 - \frac{(lL)^2}{L^2}$,

, which may be negative for a space-like L. (For instance, for $L_1 = l_1$, $L_2 = l_2$, $L_3 = l_3$, $L_0 = 0$, $\Pi^2 = l_1^2 + l_2^2 + l_3^2 - l_0^2 - \frac{(l_1^2+l_2^2+l_3^2)^2}{l_1^2+l_2^2+l_3^2} = -l_0^2 < 0$)

Another ~~One~~ suggestion ~~may be~~ is to ~~start fr~~ introduce a system of four unit vectors $e_\mu^{(\nu)}$ $(\nu = 1, 2, 3, 4)$ orthog to each other and assume

$$F^{in} = \sum_\nu \left\{ -\left(e_\mu^{(\nu)} \frac{\partial}{\partial r_\mu}\right)^2 + \frac{1}{\lambda^4}\left(e_\mu^{(\nu)} r_\mu\right)^2 \right\}$$

which is formally invariant.* However, the ~~diffi~~ objection immediately arises as for the arbitraryness of $e_\mu^{(\nu)}$. Only if all the ~~cons~~ physically meaningful results are independent of the choice of $e_\mu^{(\nu)}$ (or the ~~specific~~ arbitrary internal coordinate system), the above idea makes sense. Unfortunately, there is no guarantee for that, ~~because~~ For example, if ~~the~~ $e_4^{(\nu)}$ coincides

* In this case, S.C. is no longer necessary, because the eigenvalues are of the form $\frac{2}{\lambda^2}(n_1 + n_2 + n_3 + n_0 + 2)$

with ~~the~~ the direction wave vector k_n, the external motion with this k_n may behave in such ~~particular~~ a way ~~property which may~~ as to ~~well~~ violate the relativisty principle.

Thus, ~~we are~~ it is very likely that we have to give up the method of direction integration of field equations. We had better invent a method of ~~solving~~ expressing the outgoing field $\varphi^{(in)}$ in terms of the incoming field $\varphi^{(out)}$. This amount to the same thing as to construct directly the S-matrix without taking recourse to Y. F. K.'s method of approach. The supplementary condition may or may not be satisfied by the actual fields φ ~~, which~~.

In other words, ~~during~~ the interaction ~~existing~~ due to the relation between the external and internal motion is so different from that for free field that S.C. may not work at all.

Supplementary Conditions II, N. L. F. (32)

In the presence of the interaction, the S.C. ~~cannot be~~

$$\frac{\partial}{\partial x_\mu}\left(-\frac{\partial}{\partial r_\mu}+\frac{1}{\lambda^2}r_\mu\right)\varphi=0$$

is not adequate, because we have to take into account both the time-like k and the space-like k. Let us assume the new supplementary conditions:

$$k_\mu\left(-\frac{\partial}{\partial r_\mu}+\frac{1}{\lambda^2}r_\mu\right)\varphi=0 \quad \text{for } k_\mu k_\mu<0$$

$$k_\mu\left(\frac{\partial}{\partial r_\mu}+\frac{1}{\lambda^2}r_\mu\right)\varphi=0 \quad \text{for } k_\mu k_\mu>0$$

In the limit $k_\mu k_\mu \to 0$, both of these conditions can be satisfied, because ~~they do~~ the two operators commute with each other for $k_\mu k_\mu=0$.

The ~~solution~~ orthogonal eigenfunctions of the 8-dimensional eigenvalue problem

$$\frac{\partial^2\varphi}{\partial x_\mu\partial x_\mu}+4\frac{\partial^2\varphi}{\partial r_\mu\partial r_\mu}+\frac{4}{\lambda^4}r_\mu r_\mu\varphi=\mu^2\varphi$$

which satisfies the above S.C., has, in general, the form

$$e^{ik_\mu x_\mu}\chi_{n,k_\mu}(r)$$

where, ~~$n_\mu=(n_1,n_2,n_3,n_0)$~~ ~~$n=n_0+n_1+n_2+n_3$~~ $\chi_{n,k_\mu}(r)$ is normalized:

$$\int \chi_{n,k_\mu/2}(r)\,\chi_{n,k_\mu}(-r)(dr)^4 = 1$$

k_μ and m_n is connected by the relation

$$-k_\mu k_\mu - m_n^2 = \mu$$

or $\quad k_\mu k_\mu + m_n^2 + \mu = 0,$

$$m_n^2 = \frac{8}{\lambda^2}(n_1' + n_2' + n_3' - n_0'),$$

~~So~~ k_μ can be time-like or space-like according as

$$m_n^2 + \mu \gtrless 0.$$

For a time-like k_μ, one can chose a coordinate system, in which $n_0' = 0$. ~~For~~ a space-like k_μ, ~~n may not be simply a linear combination of n_μ''s, but one can define because of the degeneracy~~ $m_n^2 < 0$ is always degenerate, but n can still be ~~a~~.

$$n = n_1' + n_2' + n_3' - n_0'$$

in an appropriate coordinate system, in which one of ~~the~~ the space axes coincide with k_μ.

 035

Suppose, then, that

$$\varphi(X,r) = \sum_n \int v_{n,k_\mu} e^{ik_\mu X_\mu} \chi_{n,k_\mu}(r)(dk)^4,$$

where k_μ can ~~take~~ be an arbitrary vector. The field equations

$$\frac{\partial^2 \varphi(X,r)}{\partial X_\mu \partial X_\mu} + 4\frac{\partial^2\varphi}{\partial r_\mu \partial r_\mu} + \frac{4}{\lambda^4} r_\mu r_\mu \varphi$$

$$+ g\,\overline{\psi}(X+\tfrac{1}{2}r)\psi(X-\tfrac{1}{2}r) = 0$$

becomes

$$-k_\mu k_\mu v_{n,k_\mu} + m_n^2 v_{n,k_\mu}$$

$$+ \frac{g}{(4\pi)^4}\int e^{-ik_\mu X_\mu} \underbrace{\quad}_{\varphi_{n,k_\mu}(-r)} \overline{\psi}(X+\tfrac{1}{2}r)\psi(X-\tfrac{1}{2}r)\, dX\, dr = 0$$

Then, let us introduce an equivalent local scalar field

$$V_n^{(\cancel{equiv})}(x'') = \int v_{n,k_\mu} e^{ik_\mu x''_\mu}(dk)^4$$

$$\frac{\partial^2 V_n(x'')}{\partial x''_\mu \partial x''_\mu} - m_n^2 V_n(x'')$$

$$+ \frac{g}{\cancel{(4\pi)^4}} \int\!\!\int \Phi_n(\cancel{x''-X}\ X-x'', \cancel{-r}) \overline{\psi}(X+\tfrac{1}{2}r)\psi(X-\tfrac{1}{2}r)\, dX\, dr$$

where

$$\Phi_n(\cancel{x''-X}\, X-x'', +r) = \frac{1}{(2\pi)^4}\int e^{-ik_\mu(X'-x'')} \varphi(-r)(dk)^4$$

$$g\int \Phi_n(x''-X, +r)\,\bar{\psi}(X+\tfrac{1}{2}r)\psi(X-\tfrac{1}{2}r)\, dX\, dr$$

$$= g\int \Phi_n\left(\cancel{x''-\frac{x'+x'''}{2}}, \frac{x'+x''}{2}-x'', x'-x''\right) \bar{\psi}(x')\psi(x''')\, dx'\, dx'''$$

$$\left.\begin{aligned} x' &= X+\tfrac{1}{2}r \\ x''' &= X-\tfrac{1}{2}r \end{aligned}\right\}$$

$$\frac{\partial^2 V_n(x'')}{\partial x''_\mu \partial x''_\mu} - m_n^2 V_n(x'')$$

$$= -g\int \Phi_n\left(\frac{x'+x''}{2}-x'', x'-x''\right) \bar{\psi}(x')\psi(x''')\, dx'\, dx'''.$$

This is ~~equation~~ equivalent to the field equation with the form function Φ_n in the non-local interaction theory.

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g\int (x''|\varphi|x')\,\psi(x'')\,dx''$$

$$= g\sum_n \int v_{n,k_\mu}\, e^{ik_\mu \cancel{X_\mu}}\, \chi_{n,k_\mu}(-r)\,\psi(\cancel{X}''-\tfrac{1}{2}r) \times dx''$$

$$\left.\begin{array}{l} x' = X+\tfrac{1}{2}r \\ x'' = X-\tfrac{1}{2}r \end{array}\right\}$$

$$x''' = X-\tfrac{1}{2}r$$

Then, the general form for $\varphi(X, r)$ is

$$\varphi(X,r) = \cancel{v_{n_\mu, k_\mu}, C_{n_\mu, k_\mu} e^{ik_\mu}} \sum_{n_\mu} \varphi^{(ex)}_{n_\mu}(X)\, \chi_{n_\mu}(r)$$

$$= \sum_{n_\mu k_\mu} \int e^{ik_\mu X_\mu} C_{n_\mu k_\mu} \chi_{n_\mu}(r) (dk)^4$$

$$= \sum_{n, j} \int v^{(j)}_{n, k_\mu} e^{ik_\mu X_\mu} \chi^{(j)}_{n, k_\mu}(r) (dk)^4$$

where j refers to different orthogonal states with the same $n = n_1 + n_2 + n_3 - n_0$.

$$\sum_{n, j} v^{(j)}_{n, k_\mu} \chi^{(j)}_{n, k_\mu}(r) = \sum_{n_\mu} C_{n_\mu, k_\mu} \chi_{n_\mu}(r)$$

$$v^{(j)}_{n, k_\mu} \chi^{(j)}_{n, k_\mu}(r) = \sum_{n_\mu} C_{n_\mu, k_\mu} \chi_{n_\mu}(r)$$

$$v^{(j)}_{n, k_\mu} = \sum_{n_\mu} C_{n_\mu, k_\mu} f^{(j)}_{n; n_\mu} \qquad n_1+n_2+n_3-n_0 = n, \quad f(n_\mu) = j$$

$$\frac{\partial^2 \varphi^{(ex)}_{n_\mu}(x'')}{\partial x''_\mu \partial x''_\mu} - m_n^2 \varphi^{(ex)}_{n_\mu}(x'')$$

$$= -g \int \Phi_{n_\mu}(x', x'', x''')\, \bar{\psi}(x')\, \psi(x''')\, dx' dx'''$$

$$\varphi^{(ex)}_{n_\mu}(X) = \int e^{ik_\mu X_\mu} C_{n_\mu k_\mu} (dk)^4$$

$$(-k_\mu k_\mu + m_n^2)\, \varphi^{(ex)}_{n_\mu} C_{n_\mu k_\mu}$$

$$= \frac{-g}{(2\pi)^4}\int e^{-ik_\mu x''_\mu}\,\Phi_{n_\mu}(x', x'', x''')\bar{\psi}(x')\psi(x''')\,dx'dx''dx'''$$

$$(-k_\mu k_\mu - m_n^2)\, v^{(j)}_{n,k_\mu}$$

$$= \frac{-g}{(2\pi)^4}\sum_{n_\mu}\int f^{(j)}_{n,n_\mu,k_\mu} e^{-ik_\mu x''_\mu}\,\Phi_{n_\mu}(x', x'', x''') \times \bar{\psi}(x')\psi(x''')\,dx'dx''dx'''.$$

$$\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - m_n^2\right)\varphi_n^{(ex)(j)}(x''_\mu)$$

$$= -g\sum_{n_\mu}\int f^{(j)}_{n,n_\mu,k_\mu}\Phi_{n_\mu}(x', x'', x''')\, e^{ik_\mu(x_\mu'' - x_\mu''')}\,dk\,dx''' \;\bar{\psi}(x')\psi(x''')\,dx'dx'''$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g\sum_{n_\mu}\int \Phi_{n_\mu}(x', x'', x''')\, C_{n_\mu,k_\mu} e^{ik_\mu x''_\mu}\,dk_\mu \;\cancel{\psi(x')}\psi(x''')\,dx'dx'''$$

$$v^{(j)}_{n,k_\mu}\chi^{(j)}_{n,k_\mu}(r) = \sum_{\substack{n = n_1+n_2+n_3-n_0 \\ f(n_\mu) = j}} C_{n_\mu,k_\mu}\chi_{n_\mu}(r)$$

$$f^{(j)}_{n,n_\mu,k_\mu} = \int \tilde{\chi}^{(j)}_{n,k_\mu}(r)\chi_{n_\mu}(r)\,dr$$

$$C_{n_\mu,k_\mu} = v^{(j)}_{n,k_\mu}\cdot \tilde{f}^{(j)}_{n,n_\mu,k_\mu}$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g \sum_{n_\mu} \int \Phi_{n_\mu}(x', x'', x''') \tilde{f}^{(j)}_{n_j, n_\mu, k_\mu} v^{(j)}_{n, k_\mu} dk$$

$$e^{i k_\mu x''_\mu} \cancel{\Phi} \psi(x''') dx'' dx'''$$

$$= g$$

Direct Integration of Field Equations

N.L.F. (37)

$$\frac{\partial^2 \varphi_{n\mu}(x'')}{\partial x''_\mu \partial x''_\mu} - m^2_{n\mu}\varphi_{n\mu}(x'')$$

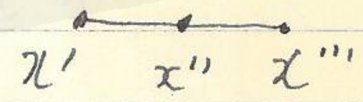
$$= -g \int \Phi_{n\mu}(x', x'', x''')\bar{\psi}(x')\psi(x''') \times dx'\,dx'''$$

$$\Phi_{n\mu}(x', x'', x''') = \tilde{\chi}_{n\mu}(x' - x''') \times \delta\left(\frac{x'+x'''}{2} - x''\right)$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x')$$

$$= g \sum_{n\mu} \int \Phi_{n\mu}(x', x'', x''')\varphi_{n\mu}(x'')\psi(x''') \times dx''\,dx'''$$

etc



Since all functions $\Phi_{n\mu}(x', x'', x''')$ decreasing rapidly as the distance between x'' and x''' (and x'' and x') increases, in any direction, one can formally integrate the field equations as follows, according to Y.F.K. method:

$$\varphi_{n_\mu}(x) = \varphi^{(in)}_{n_\mu}(x) + g\int \Delta^{(ret)}_{n_\mu}(x-x'')$$

$$\times\ \Phi_{n_\mu}(x',x'',x''')\,\bar\psi(x')\,\psi(x''')\,dx'dx''dx'''$$

$$\psi(x) = \psi^{(in)}(x) - g\sum_{n_\mu}\int S^{(ret)}_M(x-x')$$

$$\times\ \Phi_{n_\mu}(x',x'',x''')\,\varphi_{n_\mu}(x'')\,\psi(x''')\,dx'dx''dx'''$$

$$\varphi_{n_\mu}(x) = \varphi^{(out)}_{n_\mu}(x) + g\int \Delta^{(adv)}_{n_\mu}(x-x'')$$

$$\times\ \Phi_{n_\mu}(x',x'',x''')\,\bar\psi(x')\,\psi(x''')\,dx'dx''dx'''$$

$$\psi(x) = \psi^{(out)}(x) - g\sum_{n_\mu}\int S^{(adv)}_M(x-x')$$

$$\times\ \Phi_{n_\mu}(x',x'',x''')\,\varphi_{n_\mu}(x'')\,\psi(x''')$$
$$dx'dx''dx'''$$

$$(\Box - m^2_{n_\mu})\begin{pmatrix}\Delta^{(ret)}_{n_\mu}(x-x')\\ \Delta^{(adv)}_{n_\mu}(x-x')\end{pmatrix} = -\delta(x-x')$$

$$\left.\begin{array}{l}\lim\limits_{x_0\to\mp\infty} \Delta^{\substack{ret\\adv}}_{n_\mu}(x-x') = 0\\[2ex] \lim\limits_{x_0\to\mp\infty} \dfrac{\partial\Delta^{\substack{ret\\adv}}_{n_\mu}(x-x')}{\partial x_0'} = 0\end{array}\right\}$$

For $m_{n\mu}^2 \geq 0$, the properties of $\Delta_\kappa^{ret, adv}$ are well known

$$\Delta_\kappa^{ret} = \bar{\Delta}_\kappa - \frac{1}{2}\Delta_\kappa$$

$$\Delta_\kappa^{adv} = \bar{\Delta}_\kappa + \frac{1}{2}\Delta_\kappa$$

with

$$\begin{cases} \bar{\Delta}_\kappa = \frac{1}{(2\pi)^4}\int \frac{1}{k^2+\kappa^2} e^{ikx} dk \\ \Delta_\kappa = \frac{1}{(2\pi)^3}\frac{1}{i}\int \varepsilon(k)\,\delta(k^2+\kappa^2)\, e^{ikx} dk \end{cases}$$

where $\kappa = m_{n\mu}$.

For $m_{n\mu}^2 < 0$, $\kappa = +\sqrt{-m_{n\mu}^2}$,

$$\bar{\Delta}_{i\kappa} = \frac{1}{(2\pi)^4}\int \frac{1}{k^2-\kappa^2} e^{ikx} dk$$

still satisfies formally the equation. However, ~~if~~ we have ~~tak~~ first to carry out the integral over angular variables of $\vec{k}$, and then take the principal value for the integration over $|\vec{k}|$, because the integrand has poles at $\pm\sqrt{k_0^2+\kappa^2}$. Finally one integrates over k_0. Similarly, for $\Delta_{i\kappa}$, one has to start from the space interactions. $\Delta_{in}^{(x)}$ is no long zero outside the light-cone. These and other things will give rise to further complications.

For that reason we had better restrict the eigenvalues of $m_{n\mu}$ to non-negative values by choosing a suitable eigenvalue problem for the mass. For instance, if $F^{(in)}$ has the form

$$\cancel{\left(r_\mu \frac{\partial}{\partial r_\mu}\right)^2 - (r_\mu r_\mu)^2}$$

$$\left(\frac{\partial^2}{\partial r_\mu \partial r_\mu} - \frac{1}{\lambda^4} r_\mu r_\mu\right)^2 \chi = \frac{m^2}{\lambda^2} \chi$$

or

$$\left(\frac{\partial^2}{\partial r_\mu \partial r_\mu} - \frac{1}{\lambda^4} r_\mu r_\mu\right) \chi \overset{\mu = \pm \frac{m}{\lambda}}{=} \mu\, \theta$$

$$\left(\frac{\partial}{\partial r_\mu \partial r_\mu} - \frac{1}{\lambda^4} r_\mu r_\mu\right) \theta = \mu\, \chi$$

$$(\qquad)(\chi + \theta) = \mu(\chi + \theta)$$

$$(\qquad)(\chi - \theta) = -\mu(\chi - \theta)$$

Thus, if we take a complete set of eigenfunctions for the original oscillator problem, it is also the eigenfunctions for the new eigenvalue problem, which has, however, only positive eigenvalues.
(The infinite degeneracy still remains)
Now

$$m_{n\mu} = I\,\frac{8}{\lambda}(n_1+n_2+n_3-n_0+1)$$

If we restrict, in this way, the eigenvalues m_n to real values, we have further

$$\left.\begin{aligned}\varphi^{(out)}_{n_\mu}(x) &= \varphi^{(in)}_{n_\mu}(x) - g\int \Phi_{n_\mu}(x', x'', x''')\,\bar{\psi}(x')\\ &\quad \Delta_{n_\mu}(x-x'')\,\psi(x''')\,dx'dx''dx'''\\ \psi^{(out)}(x) &= \psi^{(in)}(x) + g\sum_{n_\mu}\int \Phi_{n_\mu}(x', x'', x''')\\ &\quad S_M(x-x')\,\varphi_{n_\mu}(x'')\,\psi(x''')\,dx'dx''dx'''\end{aligned}\right\}$$

~~We~~ As shown by Bloch, the incoming and outgoing field are connected by a unitary ~~real~~ S-matrix by the relation, ~~if~~

$$\varphi_0^{(out)}(x) = S^{-1}\varphi_0^{in}(x) S,$$

if ~~φ^{in}~~ we restrict our attention to $n_\mu = 0$ and ignore all other states and ~~assume~~ quantize the field $\varphi_0^{(out)}(x)$ as usual:

$$[\varphi_0^{(in)}(x), \varphi_0^{(in)}(x')] = i\Delta_{m_0}(x-x')$$

However, such a restriction to $n_\mu = 0$ will inevitably violate the relativistic invariance. We have at least to take into account all the states belonging to the same lowest eigenvalue $m = m_0$ in order to ~~be in~~ conform to relativity principle.

Then, the question of convergence could not ~~will~~ again be ~~very hard~~ easily ~~to be~~ answered. ~~Perhaps~~ From here on, we have to take up the restriction due to the S.C. again seriously.

(with S. C.)

Quantization of N. L. S. F. N. L. F. (40)

Now our problem is as follows:
Take an incoming field $\varphi^{in}_{n\mu}(x)$ which is so chosen as to satisfy the S. C. ~~Is it possible that~~ Is the relation between $\varphi^{(out)}$ outgoing field $\varphi^{(out)}_{n\mu}(x)$, which would be obtained by solving the integral equations, related to ~~the~~ $\varphi^{(in)}_{n\mu}(x)$ by a unitary matrix

$$\varphi^{(out)}_{n\mu}(x) = S^{-1}\varphi^{(in)}_{n\mu}(x)\,S,$$

so that $\varphi^{(out)}_{n\mu}(x)$ satisfies not only the same field equation as $\varphi^{(in)}_{n\mu}(x)$, but also it satisfies the same commutation relations and the same supplementary condition?

To answer this question is ~~as~~ not easy, because the commutation relations for $\varphi^{(in)}_{n\mu}(x)$ is to be modified ~~to~~ due to S. C.

In any case, we have to investigate the C. R. for $\varphi^{(out)}_{n\mu}(x)$ in detail.

First of all the general form for the field $\varphi(X, r)$ which satisfies both the supplementary condition and the field equation is:

$$\varphi(X,r)=\sum_{n,j}\int v_{k_\mu}^{(n,j)} e^{ik_\mu X_\mu} \chi_{k_\mu}^{(n,j)}(r) \times \delta(k_\mu k_\mu + m_n^2)(dk)^4$$

where n is the quantum which characterizes the mass m_n and j is that what discriminate orthogonal ~~states~~ internal eigenfunction belong to the same n or same m_n, $v_{k_n}^{(n,j)}$ ~~are~~ is an arbitrary coefficient.

Although $\chi_{k_n}^{(n,j)}(r)$ can be normalized in r-space, two such functions with the same n, j and different k_n are not orthogonal to each other: ~~Namely~~ Let us take the case $n_1 = n_2 = n_3 = 0$ $(n_0 = 0)$ ~~without k_μ~~ In this case, the extra quantum number j is superfluous.

$$\chi^{(0)}_{k_\mu}(r) = N_{k_\mu} e^{-\frac{1}{2\lambda^2}\left\{r_\mu r_\mu + \frac{2(k_\mu r_\mu)^2}{m_0^2}\right\}}$$

where N_{k_μ} is the normalization factor.

$$\int |\chi^{(0)}_{k_\mu}(r)|^2 (dr)^4 = |N_{k_\mu}|^2 \int e^{-\frac{1}{\lambda^2}\left\{r_\mu r_\mu + \frac{2(k_\mu r_\mu)^2}{m_0^2}\right\}} \times (dr)^4$$

$$= |N_{k_\mu}|^2 \int e^{-\frac{1}{\lambda^2}(r_1'^2 + r_2'^2 + r_3'^2 + r_0'^2)} \times dr_1' dr_2' dr_3' dr_0'$$

$$= |N_{k_\mu}|^2 \left(\sqrt{\frac{\pi\lambda^2}{\cancel{}}}\right)^4 = |N_{k_\mu}|^2 \left(\frac{\pi\lambda^2}{\cancel{}}\right)^2$$

$$= 1$$

or $$N_{k_\mu} = \frac{1}{\pi\lambda^2}.$$

$$\int \chi^{(0)}_{k_\mu}(r) \chi^{(0)}_{k'_\mu}(r) (dr)^4 = \left(\frac{1}{\pi\lambda^2}\right)^2 \int e^{-\frac{1}{\lambda^2}\left\{r_\mu r_\mu + \frac{(k_\mu r_\mu)^2 + (k'_\mu r_\mu)^2}{m_0^2}\right\}} (dr)^4$$

If we go over to the centre of mass

system, in which

$$k_1' = -l_1', \quad k_0' = l_0'$$

$$\frac{(k_1'r_1' + k_0'r_0')^2 + (-k_1'r_1' - k_0'r_0')^2}{m_0^2}$$

$$= 2(k_1'^2 r_1'^2 + k_0'^2 r_0'^2)/m_0^2$$

$$r_\mu' r_\mu' + \frac{2}{m_0^2}(k_1'^2 r_1'^2 + k_0'^2 r_0'^2)$$

$$= \left(1 + \frac{2k_1'^2}{m_0^2}\right) r_1'^2 + r_2'^2 + r_3'^2 + \left(1 + \frac{2k_0'^2}{m_0^2}\right) r_0'^2$$

Thus

$$\int \chi^{(0)}_{k_\mu}(r') \chi^{(0)}_{l_\mu}(r') (dr')^4.$$

$$= \frac{1}{\sqrt{\left(1 + \frac{2k_1'^2}{m_0^2}\right)\left(1 + \frac{2k_0'^2}{m_0^2}\right)}}$$

$$= \frac{1}{\sqrt{\left(1 + \frac{2k_1'^2}{m_0^2}\right)\left(1 + \frac{2k_1'^2}{m_0^2}\right)}} = \frac{1}{1 + \frac{2k_1'^2}{m_0^2}}$$

$$= \frac{1}{\frac{2k_0'^2}{m_0^2} - 1}. = \frac{m_0^2}{k_0 l_0 - k_j l_j} = \frac{m_0^2}{-k_\mu l_\mu}$$

Proof:
In the centre of mass system

$$(k_1+l_1)^2+(k_2+l_2)^2+(\ \ )^2-(k_0+l_0)^2 \doteq \varkappa^2$$

$$k_0' = -\frac{k_1+l_1}{\varkappa}\,k_1 - \frac{k_2+l_2}{\varkappa}\,k_2 - \frac{k_3+l_3}{\varkappa}\,k_3 + \frac{k_0+l_0}{\varkappa}\,k_0$$

$$= \frac{m_0^2 - (k_1l_1+k_2l_2+k_3l_3-k_0l_0)}{\varkappa}$$

$$= l_0' \quad \#$$

$$(k_1+l_1)^2+(k_2+l_2)^2+(k_3+l_3)^2-(k_0+l_0)^2$$

$$\doteq 2m_0^2 + 2(k_1l_1+k_2l_2+k_3l_3-k_0l_0)$$

$$\doteq -\varkappa^2$$

$$\varkappa^2 = 2m_0^2 - 2(k_1l_1+k_2l_2+k_3l_3-k_0l_0)$$

$$k_0' = \frac{1}{\sqrt{2}}\sqrt{m_0^2+(k_0l_0-k_jl_j)}$$

$k_\mu = l_\mu$: $\quad \underline{k_0' = m_0}$

$$\frac{2k_0'^2}{m_0^2} - 1 = \frac{m_0^2+(k_0l_0-k_jl_j)}{m_0^2} - 1$$

$$= (k_0l_0-k_jl_j)/m_0^2$$

On the other hand, φ can be expanded in the form

$$\varphi(X,r) = \sum_{n_\mu} \varphi_{n_\mu}(X)\chi_{n_\mu}(r)$$

whence

$$\varphi_{n_\mu}(X) = \sum_{n,j}\int v_{k_\mu}^{(n,j)} e^{ik_\mu x_\mu} \overbrace{\delta(k_\mu k_\mu + m_n^2)(dk)^4}$$

$$\times \sum_{n,j}\int \tilde{\chi}_{n_\mu}(r)\chi_{k_\mu}^{(n,j)}(r)(dr)^4$$

Now ~~the last~~ $\chi_{n_\mu}(r)$ and $\chi_{k_\mu}^{(n,j)}$ are orthogonal to each other, if

$$n \neq n_1+n_2+n_3-n_0,$$

because they ~~belong~~ are eigenfunctions of the same oscillator problem with the different eigenvalues.

$$C_{n_\mu,k_\mu}^{(n,j)} = \int \tilde{\chi}_{n_\mu}(r)\chi_{k_\mu}^{(n,j)}(r)(dr)^4$$

~~Let us take again the case $n=0$. Then, for the eigenfunction χ_{n_μ} which gives non-vanishing~~

For $C^{(n,j)}_{n_\mu,k_\mu}$, we have

$$\sum_{n,j,n_\mu} |C^{(n,j)}_{n_\mu,k_\mu}|^2 = 1$$

provided that χ_{n_μ} is normalized, because

$$\chi^{(n,j)}_{k_\mu}(r) : \chi^{(n,j)}_{k_\mu}(r) = \sum_{n_\mu} C^{(n,j)}_{n_\mu,k_\mu} \chi_{n_\mu}(r)$$

$$1 = \sum_{n_\mu} C^{(n,j)}_{n_\mu,k_\mu} \int \tilde{\chi}^{(n,j)}_{k_\mu}(r) \chi_{n_\mu}(r)(dr)^4$$

$$\varphi_{n_\mu}(X) = \sum_{n,j} \int v^{(n,j)}_{k_\mu} e^{ik_\mu x_\mu} C^{(n,j)}_{n_\mu,k_\mu} \times \delta(k_\mu k_\mu + m_n^2)(dk)^4$$

also we have

$$\sum_{n,j} |C^{(n,j)}_{n_\mu,k_\mu}|^2 \leqq 1$$

because $\{\chi^{(n,j)}_{k_\mu}(r)\}$ ~~is~~ may not be a complete ~~$\chi_{n_\mu}(r)$~~ ~~$\chi^{(n,j)}_{k_\mu}(r)$~~ set of orthogonal functions for a fixed k_μ, because we have chosen only those functions which satisfy S.C.

$$\chi_{n_\mu}(r) = \sum_{(n,j)} c^{(n,j)}_{n_\mu, k_\mu} \chi^{(n,j)}_{k_\mu}(r) + \chi'_{n_\mu, k_\mu}(r)$$

where $\chi'_{n_\mu,k_\mu}(r)$ is the residual which does not ~~sats~~ satisfy S.C. and which is orthogonal to all of $\chi^{(n,j)}_{k_\mu}$.

In this sense, ~~$\varphi_{n_\mu}(X)$~~ if $v^{(n,j)}_{k_\mu}$ satisfies the usual com. R. of the type

$$[\underset{\delta(k_\mu k_\mu + m_n^2)}{v^{(n,j)}_{k_\mu}}, \underset{\delta(k'_\mu k'_\mu + m_{n'}^2)}{v^{(n';j')}_{k'_\mu}}] = \text{const.}\ \delta_{nn'}\delta_{jj'}\delta(k_\mu + k'_\mu) \times \varepsilon(k_\mu)\delta(k_\mu k_\mu + m_n^2)$$

$$[\varphi_{n_\mu}(X), \varphi_{n_\mu}(X')] = \int e^{ik_\mu X} \cdot e^{ik'_\mu X'_\mu} dk\, dk' \sum_{n,j} [\underset{\delta(k_\mu k_\mu + m^2_{n_\mu})}{v^{(n,j)}_{k_\mu}}, \underset{\delta(k'_\mu k'_\mu + m^2_{n_\mu})}{v^{(n',j')}_{k'_\mu}}] (c^{n,j}_{n_\mu, k_\mu})^2$$

$$= \underset{\text{const.}}{} \int e^{ik_\mu (X_\mu - X'_\mu)} \underbrace{\sum_{n,j} (C^{n,j}_{n_\mu} k_\mu)^2 \, dk}_{a(k_\mu)\,\delta(k_\mu k_\mu + m_n^2)}$$

Thus, the commutation relations for $\varphi_{n_\mu}(X)$, $\varphi_{n_\mu}(X')$ will take the usual form only if

$$\sum_{n,j} (C^{(n,j)}_{n_\mu, k_\mu})^2 = 1$$

independent of k_μ.

In other words, $\varphi_{n_\mu}(X)$ and $\varphi_{n_\mu}(X')$ may not be commutative for ~~two~~ X, X' which are space-like to each other.

Also, $\varphi_{n_\mu}(X)$, $\varphi_{n'_\mu}(X')$ are not in general commutative even for different n_μ and n'_μ.

On the contrary, if we assume ~~usual~~ usual commutation relations for $\varphi_{n_\mu}(X)$ in order to obtain a unitary S-matrix, then the C. R. for $v_{k_\mu}^{(n,j)}$ will be different:

$$\varphi_{n_\mu}(X) = \sum_{n,j} \int v_{k_\mu}^{(n,j)} c_{n_\mu, k_\mu}^{(n,j)} e^{i k_\mu x_\mu} \delta(k_\mu k_\mu + m_n^2)(dk)^4$$

$$-\vec{k}^2 + k_0^2 - m_n^2 = x$$

$$k_0 = \pm\sqrt{x + \vec{k}^2 + m_n^2}$$

$$2 k_0 dk_0 = dx$$

$$\int_{-\infty}^{+\infty} f(k_0)\delta(k_\mu k_\mu + m_n^2)\, dk_0 = \int_0^\infty f(+\sqrt{\ })\frac{1}{2\sqrt{\ }}\delta(x)\,dx \qquad \cancel{\frac{1}{2k_0}} \cancel{(dk)^3}$$

$$+ \int_\infty^0 f(-\sqrt{\ })\frac{1}{(-)2\sqrt{x+\cdots}}\delta(x)\,dx$$

$$+ \int_0^{+\infty} f(-\sqrt{\ })\frac{1}{2\sqrt{x+\cdots}}\delta(x)\,dx$$

$$\varphi_{n_\mu}(X) = \sum_{n,j} \int \Big[v_{\vec{k}, +k_0}^{(n,j)} c_{n_\mu, \vec{k}, +k_0}^{(n,j)} e^{i k_\mu x_\mu} + v_{\vec{k}, -k_0}^{(n,j)} c_{n_\mu, \vec{k}, -k_0}^{(n,j)} e^{i k_\mu x_\mu} \Big] \frac{(d\vec{k})^3}{2 k_0}$$

$$k_0 = +\sqrt{\vec{k}^2 + m_n^2}$$

From the commutation relation

$$[\varphi_{n_\mu}(X), \varphi_{n_\mu}(X')] = i \Delta_{m_n}(X - X'),$$

we obtain

$$\Big[\sum_{(n,j)} v^{(n,j)}_{\mathbf{k}, +k_0} c^{(n,j)}_{n_\mu, \mathbf{k}, +k_0} ,$$

$$\sum_{(n',j')} v^{(n',j')}_{\mathbf{k}', -k_0'} c^{(n',j')}_{n_\mu, -\mathbf{k}', -k_0'} \Big]$$

$$= \text{const.}\ \delta(\mathbf{k}, \mathbf{k}')$$

If, $\varphi_{n_\mu}(X)$ is real,

$$\tilde{c}^{(n,j)}_{n_\mu, \mathbf{k}, +k_0} = c^{(n,j)}_{n_\mu, -\mathbf{k}, -k_0}$$

$$\cancel{\int \tilde{c}^{(n',j')}_{n_\mu, \mathbf{k}, +k_0} v^{(n,j)}_{n_\mu, \mathbf{k}+k_0} c^{(n,j)}_{n_\mu, \mathbf{k}, +k_0}}$$

$$\sum_{\substack{n, j, \\ n', j'}} c^{(n,j)}_{n_\mu, \mathbf{k}, +k_0} c^{(n',j')}_{n_\mu, -\mathbf{k}', -k_0'} \left[v^{(n,j)}_{\mathbf{k}, +k_0}, v^{(n',j')}_{\mathbf{k}', -k_0'} \right]$$

$$= \text{const.}\ \delta(\mathbf{k}, \mathbf{k}')$$

$$\sum_{\substack{n, j \\ n', j'}} f_{n,j,n',j'}(n_\mu, \mathbf{k}, \mathbf{k}') \left[v^{(n,j)}_{\mathbf{k}, +k_0}, v^{(n',j')}_{-\mathbf{k}', -k_0'} \right]$$

$$= \text{const}\ \delta(\mathbf{k}, \mathbf{k}')$$

$$\cancel{\left[v_{\mathbf{k}, +k_0}, v^{n,\cdot}_{-\mathbf{k}', -k_0'} \right.}$$

Thus, the C.R. for $V^{(n,j)}_{\mathbf{k},+k_0}$ ~~will be singular,~~ does not ~~not~~ have such a form as to allow the usual interpretation:

V_{+k_0} : annih. operator
V_{-k_0} : creat. operator

~~The real annihilation~~

In any case, if we started from the integral equations for the fields with the initial incoming field satisfying the S.C., the outgoing field, which would be obtained ~~by~~ as the result of ~~sup~~ successive integration, would ~~not~~ no longer satisfy the S.C.. In other words, the unitary S-matrix which connect the in- and out- fields must be obtained by some other way other than the direct integration of field equations.

It might be thought possible to modify the S.C. However, the mere ~~change~~ in S.C. does not help addition of an interaction term much, unless the spinor field is restricted at the same time by another S.C.

Or else, one can say that the field equations themselves are already so much modified by S.C. that there remains ~~an~~ little similarity between them and the field equations for N.L.I. ~~to~~

As for the method of constructing S.M., ~~independent~~ by ~~giving up~~ deducing the field equations from variation principle was suggested ~~by~~ in my earlier papers.
One may also go over first to the moment space ~~an~~ repr. of Lagrangian action operator.

S. 1

Summary of the course of Q. Q. M. for Winter Semester Sept. 1952 ~ Jun. 1953

A. General formulation of Relativistic Field Theories of Local fields
(a) Variation principle.
(b) Direct Integration of Field Equations
(c) Structure of S-matrix
(d) Divergence – Renormalization
(e) Q. E. D. (Supp. Cond.)
(f) General Rules of constructing S-matrix

B. Non-local ~~Fiel~~ ~~Interaction~~ Field Theories
(a) N. L. Interaction
(b) N. L. Fields
mass spectrum versus form factor.

There are a number of problems which were not discussed in detail: Among other things
(i) Computation of (for collision process) physically observable quantities
(ii) Problems of bound states
However, I think I could cover the most fundamental problems of the present field theories and, in particular, their most serious difficulties.

Program of Spring Semester
Application of Group Theory to ~~Phys.~~ quantum mechanics
(i) permutation group
(ii) rotation group
(iii) Lorentz group
(iv) ~~Possible~~ other groups of possible interest.

In the meantime, I hope I shall be able to discuss further developments in field theories.
In any case, I think I better change the subject

Another possibility
More elementary approach to Meson-Nucleon Problem without bothering too much about relativistic formulation of the theory.

{ Weyl
van der Waerden
Wigner

Carson, Perturbation Methods in the Q. M. of n-Electron Systems, 1951
Murnagan, Theory of G. Repr.

Variation Principle with Constraint

$$F\left(\frac{\partial}{\partial x_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(x, r) = 0$$

$$C\left(\frac{\partial}{\partial x_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(x, r) = 0$$

F, C; self-adjoint operators

$$\delta\int \varphi \cdot F\varphi \, dx\, dr = 0$$

$$\delta\int \varphi \cdot C\varphi \, dx\, dr = 0$$

μ: Lagrange multiplier

$$\delta\left[\int \varphi F \varphi + \mu \int \varphi \cdot C\varphi\right] = \delta I = 0$$

for arbitrary change of φ and μ.

φ must satisfy the eq.

$$F\varphi + \mu C\varphi = 0$$

μ: certain positive constant,

~~where μ is to be so chosen as to~~ ,

~~give φ~~ where μ is to be so chosen as to give

$$\int \underbrace{\varphi \cdot C\varphi}_{\text{positive definite operator for timelike } k_\mu} = 0$$

$$\varphi(x, r) = \sum_{n_\mu}\int C_{k_\mu, n_\mu} e^{ik_\mu x_\mu} \cdot \chi_{n_\mu}(x)(dk)^4$$

$$(-k_\mu k_\mu - m_n^2)C_{k_\mu, n_\mu} - \frac{n_\mu(k_\mu b_\mu^* + k_\mu b_\mu)}{\sqrt{n_\kappa n_\lambda}\, k_\kappa k_\lambda} C_{k_\mu, n_\mu - \delta_{\mu\kappa} + \delta_{\mu\lambda}} = 0$$

For $k_1 = k_2 = k_3 = 0$:

$$(k_0^2 - m_n^2)\, C_{0,0,0,k_0;\, n_1 n_2 n_3 n_0}$$

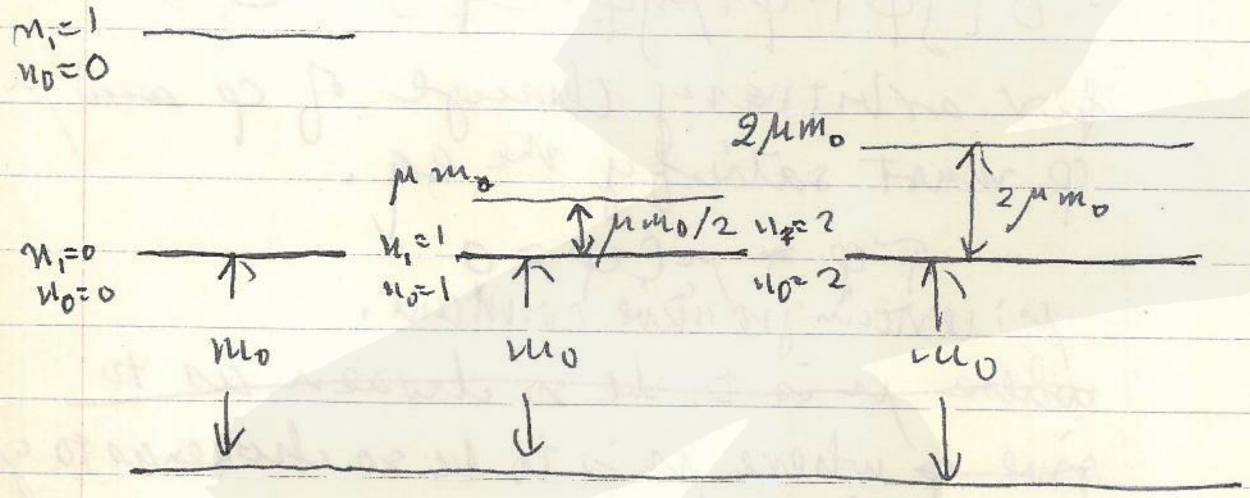
$$\pm \mu m_n \underbrace{n_i}_{n_0} C_{0,0,0,k_0,\, n_1 n_2 n_3 n_0} = 0$$

If $k_0^2 = m_n^2$: $n_0 = 0$

or $k_0^2 =$ otherwise

$$(k_0^2 - m_n^2 \pm \mu m_n^2 n_0)\, C_{0,0,0,k_0,\, n_1 n_2 n_3 n_0} = 0$$

$$k_0 = \pm\sqrt{m_n^2 \pm \mu m_n^2 n_0} = m_0\sqrt{1+\mu n_0}$$



$$k_0 = +\sqrt{m_0^2 + \mu m_0^2 n_0}$$

$$= m_0\sqrt{1 + \mu n_0} \simeq m_0 + m_0 \frac{\mu n_0}{2}$$

If μ is positive, we have a non-degenerate lowest mass ~~of~~ state $n_0 = n_1 = n_2 = n_3$ for the particle at rest.

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N.L.F. (48)

$$\cancel{=}\ a(n_1+n_2+n_3-n_0)^2$$
$$+\, b\,(\underbrace{n_1+n_2+n_3}_{})\,n_0 \quad (-n_0)$$

$$\cancel{= a(n_1+n_2+n_3)^2 - 2a(}$$

$$= an^2 - 2ann_0 + an_0^2$$
$$+\, b\,nn_0 - bn_0^2$$

$a=1.$ $\quad \underline{b=\mu}$

$$n^2 - (2-b)\,nn_0 + n_0^2$$
$$= \{n - (1-\tfrac{b}{2})\,n_0\}^2$$
$$+\{1-(1-\tfrac{b}{2})^2\}\,n_0^2$$

$$-1 \le 1-\frac{b}{2} \le 1$$

$$\frac{b}{2} \le 2 \qquad b \le 4$$
$$\frac{b}{2} \ge 0 \qquad b \ge 0$$

$$0 \le \frac{b}{2} \le 2 \qquad \boxed{0 \le b \le 4}$$
$$0 \le \frac{b}{4} \le 1 \qquad b = \mu$$

$\mu = 4$, $(n + n_0)^2 = (n_1 + n_2 + n_3 + n_0)^2$

$n_? = 2$, $n_1 = 1, n_2 = 1$

$n_1 = 1$,

$n_\mu = 0$.

$$(n_1 + n_2 + n_3 - n_0)^2$$
$$+ \mu (n_1 + n_2 + n_3 - n_0) n_0$$
$$= (n - n_0)^2 + \mu (n n_0 - n_0^2)$$
$$= n^2 - (2 - \mu) n n_0 + (1 - \mu) n_0^2$$
$$= (n - (1 - \tfrac{\mu}{2}) n_0)^2 + \{(1+\mu) - (1 - \tfrac{\mu}{2})^2\} n_0^2$$

$$1 + \mu - (1 - \tfrac{\mu}{2})^2 > 0$$
$$(\tfrac{\mu}{2})^2 < 0 \qquad \times$$

$$-\left(\frac{\partial^2}{\partial x_\mu \partial r_\mu}+\frac{1}{\lambda^2}\frac{\partial}{\partial x_\mu}r_\mu\right)\left(-\frac{\partial^2}{\partial x_\mu \partial r_\mu}+\frac{1}{\lambda^2}\frac{\partial}{\partial x_\mu}r_\mu\right)\varphi=0$$

$$\varphi = e^{ik_\mu x_\mu}\chi_{k_\mu}(r)$$

$$-ik_\nu\left(\frac{\partial}{\partial r_\nu}+\frac{1}{\lambda^2}r_\nu\right)\cdot ik_\mu\left(\underbrace{\frac{\partial}{\partial r_\mu}}_{ik_0}+\frac{1}{\lambda^2}\underbrace{r_\mu}_{ir_0}\right)$$

k_μ: time-like

$$-ik_0' b_0'^{*}\cdot ik_0' b_0' = k_0'^2 b_0'^{*}b_0' \quad \text{(pos. op.)}$$

k_μ: space-like

$$-ik_j' b_j'\cdot ik_j' b_j^{*} = k_j'^2 b_j' b_j^{*} \quad \text{(pos. op.)}$$

Quantization:
Interaction:

Free field can be expanded into series of eigenfunctions of the e.v.p.

$$(F-\mu C)\varphi = \kappa\varphi$$

Thus, we have ~~an~~ eigenvalue problem ~~of~~ ~~wt~~ in general, with two numerical

~~for~~ constants in addition to the fundamental length λ,

$$\varphi(X, r) = \sum_{n, n_0, j} \int c^{n_0, j}_{k_\mu, n} e^{ik_\mu x_\mu} \chi^{n_0, j}_{k_\mu, n}(r) \, (dk_\mu)^4$$

For the free field, for which

$$(F - \mu C)\varphi = 0$$

is satisfied, all k_μ must be ~~tue~~ time-like with the factor

$$\delta(k_\mu k_\mu + m^2_{n, n_0}).$$

If there is the interaction with ~~another~~ field, k_μ can no longer be restricted to the ~~wave~~ of time-like vector. For a space-like vector k_μ, $\chi^{n_0, j}_{k_\mu, n}(r)$ still is a function which decreases with increasing $r_1^2, r_2^2, r_3^2, r_0^2$.

$$k_\mu k_\mu + m_n^2 + \mu m_n (n_j + \tfrac{1}{2}) = \kappa.$$

$$(n-n_0)^2 + \mu(n-n_0)n_0$$

$$= n^2 - 2nn_0 + \mu n n_0 - \mu n_0^2 + n_0^2$$

$$= n^2 - (2-\mu) n n_0 + (1-\mu) n_0^2$$

$$= \left(n - \left(1-\frac{\mu}{2}\right)n_0\right)^2 + \left\{(1-\mu) - \left(1-\frac{\mu}{2}\right)^2\right\} n_0^2$$

$$1-\mu-1 = \mu - \left(\frac{\mu}{2}\right)^2$$

$$(n_1+n_2+n_3-2n_0)^2 \begin{cases} (1+\mu n_0) \\ (1+\mu(n_j+\frac{1}{2})) \end{cases}$$

Direct Integration of F. E.

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$$\left\{ \frac{\partial^2}{\partial x_\mu \partial x_\mu} - F\left(\frac{\partial}{\partial x_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right) \right\} \varphi(X, r)$$

$$= -g \bar{\psi}(X + \tfrac{1}{2} r) \psi(X - \tfrac{1}{2} r)$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M \psi(x')$$

$$= g \int (x'' | \varphi | x') \psi(x'') dx''$$

$$\left\{ \frac{\partial^2}{\partial x_\mu \partial x_\mu} - F(\ , \ , \ ) \right\} \chi^{(n'_\mu)}_{\cancel{n} k_\mu}(X, r)$$

$$= \gamma \chi^{(n'_\mu)}_{\cancel{n} k_\mu}(X, r)$$

$$\varphi(X, r) = \underbrace{\qquad}_{\varphi^{(0)}(X, r) +} g \int G(X, r; X', r') \, dX' dr' \, \bar{\psi}(X' + \tfrac{1}{2} r') \psi(X' - \tfrac{1}{2} r')$$

$$\{ \quad \} G(X, r; X', r') = -\delta(X - X') \delta(r - r')$$

$$G(X, r; X', r') = \sum_{n'_\mu} P \int \frac{\chi^{(n'_\mu)}_{k_\mu}(X, r) \tilde{\chi}^{(n'_\mu)}_{k_\mu}(X', r')}{\gamma} dk$$

$$+ \cancel{\mu} \sum_{(p)} \int \chi^{n'_\mu}(X, r) \tilde{\chi}^{n'_\mu}_{k_\mu}(X', r') \delta(\gamma) dk$$

$$\gamma = \gamma_1 k_\mu k_\mu + \gamma_2$$

$\gamma_2 \geq 0$. $\gamma_1 \lessgtr 0$.

$(\gamma_1 - 1) k_\mu k_\mu$ ~~according as~~ $\geq$ ~~all~~ always.

or $\gamma_1 < 1$ for $k_\mu k_\mu < 0$

$\gamma_1 > 1$ for $k_\mu k_\mu > 0$

$$\chi^{(n_\mu')}_{k_\mu}(X, r) = \int e^{i k_\mu X_\mu} w^{(n_\mu')}_{k_\mu}(r) \times dk_\mu$$

$$G(X, r; X', r') = \sum_P \int dk \frac{\chi^{(n_\mu')}_{k_\mu}(X, r)\, \tilde{\chi}^{(n_\mu')}_{k_\mu}(X', r')}{\gamma_1 k_\mu k_\mu + \gamma_2}$$

$$= \sum \int \frac{e^{-ik_0(X_0 - X_0')}}{\gamma_1(k^2 - k_0^2) + \gamma_2} dk_0\, e^{ik(X - X')} \times w^{(n_\mu')}_{k_\mu}(r)\, \tilde{w}^{(n_\mu')}_{k_\mu}(r') \times dk$$

$$\left(\frac{1}{\gamma_1(k^2 - k_0^2) + \gamma_2} + \mu\, \delta(\gamma_1(k^2 - k_0^2) + \gamma_2)\right)$$

Quantum Mechanical V. P. (1)

Variation Principle

$$I = \iint L(\varphi(x',x'')) \, dx' dx'' \cancel{dx'''}.$$

$$\varphi(x',x'') = \sum_n a_n \cancel{c_n} \chi_n(x',x'')$$

χ_n: 8-dimensional eigenfunctions of a certain eigenvalue problem.

$$\delta I = 0$$

$$\frac{\partial}{\partial a_n} I = \iint \frac{\partial}{\partial a_n} L(c_n) \, dx' dx''$$

$$= 0$$

$$[b_n, I] = 0 \quad \left(b_n = -\frac{\partial}{\partial a_n} \right.$$

$$[a_n, b_{n'}] = \delta_{nn'}$$

classical variation principle is equivalent to the operator relation

$$[b_n, I] = 0,$$

if c_n in I are all commutative with each other.

Generalization to q.m. would be as follows:

$$\left.\begin{aligned} [b_n, I(a_n, b_n)] &= 0 \\ [a_n, I(a_n, b_n)] &= 0 \end{aligned}\right\}$$

where a_n, b_n would be identified with annihilation and creation

operators ~~prov~~ if we choose
$a_n^* = b_n$.

Then we obtain, for m

$$L = \sum_{n,n'} G_{nn'} a_n^* a_{n'} \qquad \tilde{G}_{nn'} = G_{n'n}$$

$$[a_n, L] = \sum_{n'} G_{nn'} a_{n'} = 0$$

$$[a_n^*, L] = -\sum_{n'} a_{n'}^* G_{n'n} = 0$$

$$G_{nn'} = \int\!\!\int \tilde{c}_n c_{n'} \tilde{\chi}_n(x',x'') \underbrace{\chi_{n'}(x',x'')}_{F\left[\frac{\partial}{\partial x'}, \frac{\partial}{\partial x''}, x'-x''\right]} dx'dx''$$

$$F\chi_{n'} = f_{n'}\chi_{n'}$$

$$G_{nn'} = |c_n|^2 \cdot f_n \delta_{nn'}$$

$$\left.\begin{aligned} |c_n|^2 f_n a_n &= 0 \\ a_n^* |c_n|^2 f_n &= 0 \end{aligned}\right\}$$

These equations are satisfied only for such c_n:

$$\left.\begin{aligned} c_n &= 0 \quad \text{for } f_n \neq 0 \\ c_n &\neq 0 \quad \text{for } f_n = 0 \end{aligned}\right\}$$

$$c_n = c_n' \delta_{f_n,0}$$

V. P. ②

There is an essential difference between the field equations and commutation relations as operator relations ~~in~~ the ordinary formulation as the above field eq. ~~and C. R. for~~ the f-eq. and ~~at~~ C. R. for a_n, a_n^*, ~~which~~ ~~because~~ are to be regarded as the condition satisfied by the Schrödinger functional Ψ

$$[a_n, L]\Psi = |C_n|^2 f_n a_n \cdot \Psi = 0$$

$$[a_n^*, L]\Psi = -a_n^* |C_n|^2 f_n \cdot \Psi = 0$$

while the C. R.

$$[a_n, a_{n'}^*] = \delta_{nn'}$$

are regarded as operator relations.

~~For that reason, the relations like~~

$$\cancel{[a_n^*[a_{n'}^*, L]] = 0}$$

~~is not true~~

Interaction

$$L = \sum_{n,n'} G_{n,n'} a_n^* a_{n'} + \sum_{m,m'} H_{mm'} b_m^* b_{m'}$$

$$+ \sum I_{n,m,m'} a_n b_m^* b_{m'} + \sum \tilde{J}_{n,m,m'} a_n^* b_m^* b_{m'}$$

$$\sum_{n'} G_{n,n'} a_{n'} + \sum_{m',m} \tilde{J}_{n,m';m} b_m^* b_{m'}$$

$$= 0$$

$$\sum_{n'} G_{n',n} a_{n'}^* + \sum_{m,m'} J_{n,m,m'} b_m^* b_{m'}$$

$$= 0$$

$$[b_m^*, b_{m'}^*]_+ = \delta_{m,m'}$$

$$[\cancel{b_m, b_{m'}^*}]$$

$$[b_{m''}, b_m^* b_{m'}] = [b_{m''}, b_m^*]_+ b_{m'}$$

$$- b_m^* [b_{m''}, b_{m'}]_+ = \delta_{m'',m} b_{m'}.$$

$$\sum_{m'} H_{mm'} b_{m'} + \sum_{n,m'} J_{n,m,m'} a_n b_{m'}$$

$$+ \sum_{n,m'} \tilde{J}_{n,m';m} a_n^* b_{m'} = 0$$

Green's functions:

$$\sum_{n'} K_{n'',n'} G_{n,n'} = \delta_{n'',n'}$$

$$a_{n''} + \sum_{m',m} K_{n'',n} \tilde{J}_{n,m';m} b_m^* b_{m'} = 0$$

$$|C_n|^2 f_n a_n + \sum_{m',m} \tilde{J}_{n,m',m}\, b_m^* b_{m'} = 0$$

$$|C_{n'}|^2 f_{n'} a_{n'}^* + \sum_{m'',m'''} J_{n',m'',m'''}\, b_{m''}^* b_{m'''} = 0$$

$$[b_m^* b_{m'}, b_{m''}^* b_{m'''}]$$

$$= [b_m^* b_{m'}, b_{m''}^*]_- b_{m'''}$$

$$+ b_{m''}^* [b_m^* b_{m'}, b_{m'''}]_-$$

$$= \{-[b_m^*, b_{m''}^*]_+ b_{m'}$$

$$+ b_m^* [b_{m'}, b_{m''}^*]_+\} b_{m'''}$$

$$+ b_{m''}^* \{ b_m^* [b_{m'}, b_{m'''}]_+$$

$$- [b_m^*, b_{m'''}] b_{m'}\}$$

$$= + b_m^* b_{m'''} \delta_{m'm''} - b_{m''}^* b_{m'} \delta_{m m'''}$$

$$(|C_n|^2 f_n)^2 \delta_{nn'}$$

$$+ \sum_{\substack{m,m'\\ m'',m'''}} \tilde{J}_{n,m',m} J_{n',m'',m'''} (b_m^* b_{m'''} \delta_{m'm''}$$

$$- b_{m''}^* b_{m'} \delta_{m m'''}) \quad b_{m''}^* b_{m'''}$$

$$= \sum \{ \tilde{J}_{n,m',m} J_{n',m',m'''} b_m^* b_{m'''} - \tilde{J}_{n,m',m} J_{n',m'',m}$$

$$\sum_m L_{m'',m} H_{m,m'} = \delta_{m'',m'}$$

$$b_{m''} + \sum \mathcal{J}_{m_1} L_{m'',m'} J_{n,m,m'} a_n b_{m'}$$

$$+ \sum L_{m'',m'} \tilde{J}_{n,m',m} a_n^* b_{m'} = 0$$

$$K_{n'',n'} = \tilde{c}_n$$

$$G_{n,n'} = \tilde{c}_n c_{n'} \tilde{\chi}_n ($$

$$\sum \tilde{J}_{n,m',m} J_{n',m',m'''} b_m^* \, b_{m'''}$$

$$- \sum \tilde{J}_{n,m''',m} J_{n',m,m'} \, b_m^* \, b_{m''}$$

General Formulation

Free non-local field

$$\varphi(X, r) = (x'|\varphi|x'')$$

$$\bar{I} = \int \left(\frac{1}{2}\frac{\partial\tilde{\varphi}(X,r)}{\partial X_\mu}\frac{\partial\varphi(X,r)}{\partial X_\mu}\right)dXdr + \text{pos. definite}$$
quadratic form of φ

$$\underbrace{\frac{\partial}{\partial X_\mu}\left(-\lambda\frac{\partial}{\partial r_\mu} + \frac{1}{\lambda}r_\mu\right)\varphi}$$

$$F_2 = \frac{\partial}{\partial X_\mu}\left(-\lambda\frac{\partial}{\partial r_\mu} + \frac{1}{\lambda}r_\mu\right)\tilde{\varphi}$$

$$\underbrace{\left(-\lambda^2\frac{\partial^2}{\partial r_\mu \partial r_\mu} \quad + \frac{1}{\lambda^2}r_\mu r_\mu\right)\varphi(X,r)}$$

$$F_1 = \frac{1}{\lambda^2}\left(-\lambda^2\frac{\partial^2}{\partial r_\mu \partial r_\mu} \quad + \frac{1}{\lambda^2}r_\mu r_\mu\right)\tilde{\varphi}(X,r)$$

$$\cancel{\varphi = e^{ik_\mu X_\mu}}$$

$$\text{''}F_0\text{''}$$

$$\bar{I} = \iiint\left\{\frac{1}{2}\frac{\partial\tilde{\varphi}(X,r)}{\partial X_\mu}\frac{\partial\varphi(X,r)}{\partial X_\mu} + \alpha F_1 + \beta F_2\right\}dXdr$$

α, β: positive constant.

$$(F_0' + \alpha F_1' + \beta F_2')\varphi = \varkappa\varphi.$$

$$\varphi = e^{ik_\mu X_\mu}\chi_{k_\mu, n}(r)$$

F_1', F_2' have both discrete positive eigenvalues m_1^2, m_2^2

$$k_\mu k_\mu + \alpha \underbrace{m_1^2}_{(n_\mu)} + \beta m_2^2(n_\mu, k_\mu) = \gamma$$

where m_2^2 depends both on n_μ and k_μ, while m_1^2 depends only on n_μ. In general

$$\varphi(x, r) = \sum_{n_\mu} \int c_{k_\mu, n_\mu} e^{-ik_\mu x_\mu} \chi_{n_\mu}(r_\mu)(dk)^4$$

Free field: $\underline{\gamma = 0}$

$k_\mu k_\mu < 0$ ~~...~~, k_μ: time-like

General field:

$k_\mu k_\mu - \gamma < 0$

or $k_\mu k_\mu < \gamma - \alpha m_1^2 - \beta m_2^2$

$\gamma = \alpha m_1'^2 + \beta m_2^2$:

$k_\mu k_\mu = 0$.

$\gamma > \alpha m_1'^2 + \beta m_2^2$

$k_\mu k_\mu > 0$ k_μ: space-like

Feb 1953

061

(C, 1)

Coupling between External and Internal Motions

$$F \equiv -\frac{\partial^2}{\partial x_\mu \partial x_\mu} + \alpha^2\lambda^2\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)^2$$

$$-\beta^2\lambda^2\left\{-\left(\frac{\partial}{\partial x_\mu \partial r_\mu}\right)^2 + \frac{1}{\lambda^4}\left(r_\mu \frac{\partial}{\partial x_\mu}\right)^2\right\}$$

$$\varphi(x, r) = e^{ik_\mu x_\mu} \chi(k_\mu, r_\mu)$$

$$\Big[k_\mu k_\mu + \alpha^2\lambda^2\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)^2$$

$$+\beta^2\lambda^2\left\{-\left(k_\mu \frac{\partial}{\partial r_\mu}\right)^2 + \frac{1}{\lambda^4}(k_\mu r_\mu)^2\right\}\Big]$$

$$\times \chi(k_\mu, r_\mu) = 0$$

$k_\mu k_\mu > 0$: space-like

$k_1 = \kappa, \quad k_2 = k_3 = k_0 = 0$

$$\Big[\kappa^2 + \alpha^2\lambda^2\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)^2$$

$$+\beta^2\lambda^2\kappa^2\left\{-\frac{\partial^2}{\partial r_1^2} + \frac{1}{\lambda^4} r_1^2\right\}\Big]$$

$$\times \chi(\kappa, 0, 0, 0; r_\mu) = 0$$

$$\chi = \chi_{n_1, n_2, n_3, n_0}$$

$$\Big[\kappa^2 + 4\alpha^2 (n_1 + n_2 + n_3 - n_0 + 1)^2$$

$$+ 2\beta^2\kappa^2\left(n_1 + \tfrac{1}{2}\right)\Big]\chi_{n_1, n_2, n_3, n_0} \neq 0$$

Thus, there's no solution for $k_\mu k_\mu > 0$, because any solution ~~can~~ should be expanded into series of χ_{n_1,n_2,n_3,n_0} with non-zero coefficient, ~~which~~ (in fact that) is impossible

$k_\mu k_\mu < 0$ $\quad k_1 = k_2 = k_3 = 0, \quad k_0 = \varkappa$

$$\Big\{ -\varkappa^2 + \alpha^2\lambda^2\Big(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\Big)^2 + \beta^2\lambda^2\varkappa^2$$

$$\Big(-\frac{\partial^2}{\partial r_0^2} + \frac{1}{\lambda^4} r_0^2\Big)\Big]\chi(0,0,0,\varkappa; r_\mu) = 0$$

$$\Big\{ -\varkappa^2 + \frac{4\alpha^2}{\lambda^2}(n_1+n_2+n_3-n_0+1)^2 + 2\beta^2\varkappa^2\Big(n_0+\frac{1}{2}\Big)\Big\}\chi_{n_1 n_2 n_3 n_0} = 0$$

This is satisfied only ~~f~~ if
$$2\beta^2\Big(n_0+\frac{1}{2}\Big) < 1,$$

or $\quad n_0 < \Big(\dfrac{1}{2\beta^2} - \dfrac{1}{2}\Big)$

or $\quad n_0 < \dfrac{1}{2}\Big(\dfrac{1}{\beta^2} - 1\Big)$

~~$\chi_{n_1,n_2,n_3,0}$~~

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C, 2

Thus, among infinitely many states belonging to the same value of

$$n \equiv n_1 + n_2 + n_3 - n_0,$$

only those states with $n_0 < \frac{1}{2}\left(\frac{1}{\beta^2} - 1\right)$ can survive, if ~~we~~ introduce the coupling between external and internal ~~motions~~ degrees of freedom.

On the contrary, if ~~introduce~~ we adopt the perturbation theory, ~~only~~ all of the states with arbitrary n_1, n_2, n_3, n_0 ~~was~~ survive and the mass values ~~to~~ are changed a little bit, so that the degeneracy is removed partly ~~and~~

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C, 3

Field Equations with Interaction. ~~field~~

$$\left\{-\frac{\partial^2}{\partial r_\mu \partial X_\mu} + F'\right\}\varphi(X, r) = F\varphi(X, r)$$

$$= -g\sum_\alpha \bar{\psi}_\alpha(X+\tfrac{1}{2}r)\psi_\alpha(X-\tfrac{1}{2}r)$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x') = -g\int \psi(x'')(x''|\varphi|x')$$

$$\times dx''$$

$$\frac{\partial \bar{\psi}(x')}{\partial x'_\mu}\gamma_\mu - M\bar{\psi}(x') = g\int (x'|\varphi|x'')$$

$$\times \bar{\psi}(x'')\,dx''$$

8-dimensional Green's Function

$$F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi_\mu(X, r)$$

$$= \mu\,\varphi(X, r)$$

suppose that

This eigenvalue problem can be solved and, in particular, let us assume that

$$F = -\frac{\partial^2}{\partial X_\mu \partial X_\mu} + F'\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu\frac{\partial}{\partial r_\mu},\right.$$

$$\left. r_\mu \frac{\partial}{\partial X_\mu}, \frac{\partial^2}{\partial X_\mu \partial r_\mu}\right)$$

~~any~~
The eigenfunctions ~~φ_s~~ belonging to an eigenvalue μ_s be φ_s, one can expand an arbitrary fn

$$\varphi(X, r) = \sum c_s \varphi_s(X, r).$$

Eigenvalues μ_s are, in general, degenerate and ~~consist~~ constitute (both discrete and) continuous spectra. One had better write them as

$$\varphi_{k_\mu, n_\mu}(X, r) \equiv e^{ik_\mu X_\mu} \varphi_{n_\mu}(k_\mu, r_\mu)$$

where n_μ are discrete and k_μ are continuous and as eigenvalues

$$\mu(k_\mu, n_\mu)$$

Then one can define

$$\overline{G}(X_\mu, r_\mu; X'_\mu, r'_\mu) = \frac{1}{(2\pi)^4} \sum_{n_\mu} \int \frac{e^{-ik_\mu X_\mu} \tilde{\varphi}_{n_\mu}(k_\mu, r_\mu)\, e^{ik_\mu X'_\mu} \times \varphi_{n_\mu}(k_\mu, r'_\mu)}{\mu(k_\mu, n_\mu)} (dk)^4$$

Then

$$F\left(\frac{\partial}{\partial x_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right)\bar{G}(X_\mu, r_\mu; X'_\mu, r'_\mu)$$

$$= -\delta(X_\mu - X'_\mu)\,\delta(r_\mu - r'_\mu)$$

provided that F is ~~bilinear~~ even with respect to $\frac{\partial}{\partial x_\mu}$. (The particular examples as given above all satisfy this ~~argu~~ ~~condition~~ requirement)

Then

$$\int \bar{G}(X_\mu, r_\mu; X'_\mu, r'_\mu)\,\cancel{\varphi(X'_\mu, r'_\mu)}\, g\sum \bar{\psi}_\alpha(X'+\tfrac{1}{2}r')\,\psi_\alpha(X'-\tfrac{1}{2}r')\,(dX')^4(dr')^4$$

satisfies the field equation and the general solution for $\varphi(X, r)$ becomes

$$\varphi(X, r) = \varphi^{(f)}(X, r)$$
$$+ \int \bar{G}(X_\mu, r_\mu; X'_\mu, r'_\mu)$$
$$\times\, g\sum \bar{\psi}_\alpha(X'+\tfrac{1}{2}r')\,\psi_\alpha(X'-\tfrac{1}{2}r')\,dX'dr'$$

where $\varphi^{(f)}$ ~~is~~ is any solution of the free field equation for $\varphi(X, r)$.

$\bar{G}$ corresponds to $\bar{\Delta}$ in the ordinary theory.

~~Nature of Green's fn G.~~:

Our next question is whether it is possible to construct ~~a Green's~~ other fn's which correspond to Δ-fn and retarded and advanced Δ-functions in the local field theory.

If we take

$$\varphi_0(k_\mu, r_\mu) \equiv \varphi_{0,0,0,0}(k_\mu, r_\mu)$$

alone,

$$\equiv \frac{1}{\pi\lambda^2} \cdot e^{-\frac{1}{2\lambda^2}\left\{r_\mu r_\mu + \frac{2(k_\mu r_\mu)^2}{\kappa^2}\right\}}$$

$$\tilde{\varphi}_0(k_\mu, r_\mu)\varphi_0(k_\mu, r'_\mu)$$

$$= \frac{1}{\pi^2\lambda^4} \cdot e^{-\frac{1}{2\lambda^2}\left\{r_\mu r_\mu + r'_\mu r'_\mu + \frac{2(k_\mu r_\mu)^2}{\kappa^2} + \frac{2(k_\mu r'_\mu)^2}{\kappa^2}\right\}}$$

$$= \frac{1}{\pi^2\lambda^4} e^{-\frac{1}{2\lambda^2}\left[2\xi_\mu^2 + \frac{1}{2}\eta_\mu^2 + \frac{2}{\kappa^2}\left\{(k_\mu\xi_\mu)^2 + \frac{1}{2}(k_\mu\eta_\mu)^2\right\}\right]}$$

$$\frac{1}{2}(r_\mu + r'_\mu) = \xi_\mu \qquad (k_\mu r_\mu)^2 + (k_\mu r'_\mu)^2 = 2(k_\mu\xi_\mu)^2 + \frac{1}{2}(k_\mu\eta_\mu)^2$$

$$r_\mu - r'_\mu = \eta_\mu$$

$$2\xi_\mu^2 + \frac{1}{2}\eta_\mu^2 = r_\mu^2 + r'^2_\mu$$

Thus, if we may regard λ very small, $\hat{\varphi}_0 \varphi_0$ is small, except for very small values of r_μ, r'_μ or ξ_μ, η_μ, or, since

$$\left.\begin{aligned} x'_\mu &= X_\mu + \tfrac{1}{2} r_\mu \\ x''_\mu &= X_\mu - \tfrac{1}{2} r_\mu \end{aligned}\right\}$$

$$\left.\begin{aligned} x'''_\mu &= X'_\mu + \tfrac{1}{2} r'_\mu \\ x^{iv}_\mu &= X'_\mu - \tfrac{1}{2} r'_\mu \end{aligned}\right\}$$

except for $x'_\mu \simeq x''_\mu$, $x'''_\mu \simeq x^{iv}_\mu$

We have then approximately

$$G(X_\mu, r_\mu; X'_\mu, r'_\mu)$$

$$= -\underbrace{\cancel{\tfrac{\lambda}{\pi}}}_{\frac{1}{\pi^2\lambda^4}\,\frac{1}{(2\pi)^4}} \int \frac{e^{-ik_\mu X_\mu} e^{ik_\mu X'_\mu}}{k_\mu k_\mu + m_0^2} \underbrace{\delta(r_\mu)\delta(r'_\mu)}_{\left(\pi^2\lambda^2\right)^2}$$

$$= \frac{\lambda^4}{\pi^2} \cdot \bar{\Delta}_{m_0}(X_\mu - X'_\mu)\,\delta(r_\mu)\delta(r'_\mu)$$

Now one can construct

$$G(X_\mu, r_\mu; X'_\mu, r'_\mu)$$

$$= -\frac{1}{(2\pi)^4} \sum_{n_\mu} \cancel{\frac{\pi}{4}} \int e^{-ik_\mu X_\mu} \varphi_{n_\mu}(k_\mu, r_\mu) e^{ik_\mu X'_\mu} \times \varphi_{n_\mu}(k_\mu, r'_\mu) \cdot \varepsilon(k_\mu)\,\delta(\mu)(dk)^4$$

behaves similar to Δ-fn

which satisfies the free field equation and approximately satisfies the requirements for Δ-fn.

The commutation relation for the free field would then be

$$[\varphi^{(+)}(x_\mu, r_\mu), \varphi^{(+)}(x'_\mu, r'_\mu)]$$

$$= i\, G\,(x_\mu, r_\mu; x'_\mu, r'_\mu)$$

Now the remaining question is ~~whether one can~~ how to construct the Green fn corresp. to Δ^{ret}, Δ^{adv}.

(

(1)

素粒子の構造について I.

―― 質量スペクトルと収斂因子 ――

湯川秀樹

1. はしがき

~~非局所場の理論の意図する所は~~、素粒子に内部運動の自由度を想定し、それを非局所場という形で相対論的に記述することによって、一方では素粒子の質量スペクトルを内部運動に関する固有値問題から決定すると共に、他方では素粒子の相互作用にあらわれる形状因子を内部運動の固有函数から導き出すこと、~~これが私の~~ この意圖がどのようにして具体化できるか、どの程度まで成功の見込があるかについて詳しく検討して見たいと思う。

少

二三年前に非局所場の一般的性質を論じた時にも(1)、このような考えが漠然とした形で存在していたが、当時はまだパイ中間子より重い色々な質量の粒子について少ししか知られていなかったし、形状因子と場の理論の収斂性の関係についてもほとんど何もわかっていなかった。その後で、V, π, χ 等の諸粒子が続々と発見され、~~スペクトルの問題が漸次重要性を見とめられてくる~~ それに何よりも先ず非局所場という茫漠たる新しい対象に色々の制限を加えることによって、

今となっては素粒子に何か局所場では表現しきれない未知の自由度を与えなければ、質量スペクトル自身の解釈ができないばかりでなく、これ等の重い粒子がいずれも比較的に長い寿命を持っているという事実を説明するために必要な選択規則も導き出せないであろうということが、著しくはっきりしてきたように見える。それと同時に最近、局所場の間の色々な局所的相互作用をくりこみ理論の立場から分析することが名古屋グループによって試みられ(2)、その結果としてある種の局所的相互作用にくりこみ理論をどこまでも適用してゆけば結局非局所的相互作用になってしまうことが明らかとなった。一方において局所場の間の非局所的相互作用の問題をPeierls, Rayski, Bloch, 等によって詳しく検討され(3)、場の理論における発散と、非局所的相互作用にあらわれる形状因子の間の関係が大分わかってきた。特にMoller及びKristensen(4)は、核子と中間子の間の非局所的相互作用を適当に選べば、第一近似においては核子及び中間子の自己エネルギーを有限にできることを示した。しかし単に非局所的相互作用だけを問題にする段階では、形状因子の任意性が除去できないばかりでなく、質量スペクトルの問題とは何の関連も見

せないのである。 ~~自由粒子の~~ 質量スペクトルに関する

2、一般的考察

~~そこで~~ 所が非局所場という立場からこれ等の問題を新しく見直すと、~~次のような興味ある~~ 質量スペクトルの問題と ~~[illegible]~~ 収斂因子の問題の ~~次のような~~ 間に密接な関係があり得ることが明かとなるのである。~~[illegible]~~ 最初の着眼点は質量スペクトルの問題を非局所場によって記述される粒子の内部運動に対する固有値問題と解釈することにある。簡単のためにスカラー場 $(x'|\varphi|x'')$ を考え、これを重心座標(外部座標)

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu) \qquad \mu=1,2,3,4$$

及び相対座標(内部座標)

$$r_\mu = x'_\mu - x''_\mu$$

の函数

$$(x'|\varphi|x'') \equiv \varphi(X, r)$$

と見直すと、自由な場の方程式は一般に

$$F\left(\frac{\partial}{\partial X_\mu},\ r_\mu,\ \frac{\partial}{\partial r_\mu}\right)\varphi(X, r) = 0 \qquad (1)$$

という形に書けるものと推定される。F は $\frac{\partial}{\partial X_\mu}$, r_μ, $\frac{\partial}{\partial r_\mu}$ の或る相対論的に不変な函数で、特に四次元座標の原点の移動に対して不変であるためには、X_μ を含まないことが必要である。いいかえれば F は $\frac{\partial^2}{\partial X_\mu \partial X_\mu}$, $r_\mu r_\mu$, $\frac{\partial^2}{\partial r_\mu \partial r_\mu}$, $r_\mu \frac{\partial}{\partial X_\mu}$, $\frac{\partial^2}{\partial X_\mu \partial r_\mu}$, $r_\mu \frac{\partial}{\partial r_\mu}$

のある函数の積である。

今特に簡単な場合として、外部運動と内部運動とが分離できるものとして

$$F \equiv -\frac{\partial^2}{\partial X_\mu \partial X_\mu} + F^{(r)}\left(r_\mu r_\mu,\ \frac{\partial^2}{\partial r_\mu \partial r_\mu},\ r_\mu \frac{\partial}{\partial r_\mu}\right) \qquad (2)$$

という形に書けるとすると、場の方程式(1)は

$$\varphi(X, r) = u(X)\,\chi(r)$$

と置くことによって分離でき、u及びχはそれぞれ、分離常数をμとすると

$$\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu} - \mu\right) u(X) = 0 \quad 等 \qquad (3)$$

$$\left(F^{(r)} - \mu\right) \chi(r) = 0 \qquad (4)$$

の解であればよいことになる。従って $F^{(r)}$ なる演算子を 0 又は正の離散固有値しか持たないように選んでおけば、方程式(4)で定義された四次元的固有値問題の解として得られる μ の固有値 μ_n $(n = 1, 2, \ldots)$ から $\hbar = c = 1$ なる単位系での質量スペクトル

$$m_n = \sqrt{\mu_n} \qquad (5)$$

が決定されることになる。一般には固有値 μ_n は縮退しているので、そのために問題はもっと複雑化するが、ここでは先ず縮退はないものとして、μ_n に四次元的に属する規格化された固有函数を $\chi_n(r)$ とすると、自由場は一般に

$$\varphi(X, r) = \sum_n u_n(X)\,\chi_n(r) \qquad (6)$$

と書けることになる。ここで u_n は

$$\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu} - m_n^2\right) u_n(X) = 0 \qquad (7)$$

を満足しなければならぬから，これを m_n なる質量の粒子の外部運動を記述する局所場と見做すことができる．

すると非局所場 $\varphi(X,r)$ の量子化の問題は，局所場 $u_n(X)$ の量子化という問題に帰着し，従つて $u_n(X)$ を

$$\cancel{[u_n(X), u_{n'}(X')] = i\,\Delta_n}$$

$$[u_n(X), u_{n'}(X')] = i\,\delta_{nn'}\,\Delta_n(X-X') \qquad (8)$$

なる交換関係を満足する Hermite 型演算子と見做せば，非局所スカラー場 $\varphi(X,r)$ はある意味で種々の質量を持つた ~~ボーズ~~ 粒子の集りと同等になる．但し Δ_n は質量 m_n なる場合の不変デルタ函数である．

3．収斂因子の問題

次の問題は内部運動の固有函数 $\chi_n(r)$ がどういう役割を演ずるかという点にある．それを明らかにするには，相互作用のある場合の非局所場の方程式を考察する必要がある．順序として先ず自由場に対する方程式(1)が変分原理

$$\delta\int\cdots\int \tilde{\varphi}\cdot F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi\,(dX)^4(dr)^4 = 0 \qquad (9)$$

から導き出されることから，非局所自由場に対する Lagrange 密度を

$$L_{sc} = \frac{1}{2}\tilde{\varphi}(X,r)\cdot F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(X,r), \qquad (10)$$

の積分のいずれを取るか~~に~~るように $\chi_n(r)$ を選んだとすると、$u_n(X)$ は —— 量子化する前には —— 実函数であり、従って量子化す~~る Hermite 演算子となる様に~~した後には

$$[u_n(X), u_{n'}(X')] = i\delta_{nn'}\Delta_n(X-X') \qquad (10)$$

なる交換関係を満足する Hermite 演算子となる。但し Δ_n は質量 m_n なる場合の不変デルタ函数である。このような意味で非局所スカラー場 $\varphi(X,r)$ は種々の質量を持ったボース粒子の集りと同等であると考えられる。特に上の例のように ~~$\varphi(X,r)=$~~ φ として Hermite 型演算子を取れば、中性のボース粒子の集りと同等になる。

$$x_j' = x_j + v_j\left[\frac{x_j v_j}{v_j v_j}\left(\frac{1}{\sqrt{1-v^2}}-1\right) - \frac{x_0}{\sqrt{1-v^2}}\right]$$

$$x_0' = \frac{x_0 - v_j x_j}{\sqrt{1-v^2}} = \frac{1}{m}(k_0 x_0 - k_j x_j)$$

$$x_4' = i\frac{-k_\mu x_\mu}{\sqrt{-k_\mu k_\mu}}$$

$$k_j x_j - k_0 x_0$$

$$k_j = \frac{m v_j}{\sqrt{1-v^2}} \qquad k_0 = \frac{m}{\sqrt{1-v^2}}$$

$$\left\{\begin{aligned} x_j' &= x_j + \frac{k_j}{k_0}\left[\frac{k_0^2}{(k_l k_l)}\frac{(k_l}{k_0}x_l)\left(\frac{k_0}{m}-1\right) - \frac{k_0 x_0}{m}\right] \\ &= x_j + k_j\left[\frac{k_l x_l}{k_l k_l}\left(\frac{k_0}{m}-1\right) - \frac{x_0}{m}\right] \\ x_0' &= \frac{k_0}{m}\left(x_0 - \frac{k_l x_l}{k_0}\right) = \frac{-k_\mu x_\mu}{\sqrt{-k_\mu k_\mu}} \end{aligned}\right.$$

$(x'|\psi|x'') \equiv \psi(X, r)$ (6)

又はこれと同等な量で定義しておく。次に もう一つ別の非局所場を考える場合には、それに對する適當な Lagrange 密度を書き加えればよい。この二つの非局所場の間の相互作用は例えば ~~Lagrange 密度~~ 作用積分に

$$g\int (x'|\psi^*|x^{(2)})(x^{(2)}|\varphi|x^{(3)})(x^{(3)}|\psi|x^{(4)})\, dx'\, dx^{(2)}\, dx^{(3)}\, dx^{(4)} \qquad (11)$$

なる項をつけ加えることによって記述せられるであろう。但し g は結合常数で、今の例では ψ は 一般の Hermite 型でない非局所スカラー場と見なす。しかし以下の議論では Møller-Kristensen が詳しく考察した局所スカラー場と局所スピノル場の間の非局所的相互作用の場合との對應を明らかにしたいと思うので、ψ として局所スピノル場 $\psi_\alpha(x')$ を取ることとする。従って相互作用の Lagrange 密度も例えば

$$L_{int} = g\,(x'|\varphi|x'')\sum_\alpha \overline{\psi}_\alpha(x'')\psi_\alpha(x') \qquad (12)$$

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* 非局所場 $(x'|\varphi|x'') \equiv \varphi(X, r)$ が Hermite 型演算子であるという制限を加えておくと、$(x'|\varphi|x'')$ と $(x''|\varphi|x')$ とは ~~$(x'|\varphi|x'')$~~ 複素共軛でなければならぬことになり、従って

$$\tilde{\varphi}(X, r) = \varphi(X, -r) \qquad (\dagger)$$

なる關係が満足されるような函数に限らねばならぬことになる。所で固有函数 $\chi_n(r)$ は一般に偶函数か奇函数のいずれかに限定できるから、そのいずれであるかによって

$$\tilde{\chi}_n(r) = \chi_n(-r) = \pm\chi_n(r)$$

の諸條件のどちらかを取らねばならぬ。これ等の詳細は別の機會に論ずる。

(17)

と取ることができる.** すると変分原理から導き出される場の方程式は, F として (2) の表式をとれば,

$$\left\{-\frac{\partial^2}{\partial X_\mu \partial X_\mu} + F^{(r)}\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right)\right\}\varphi(X, r)$$
$$= -g \sum_\alpha \overline{\psi}_\alpha\left(X+\tfrac{1}{2}r\right)\psi_\alpha\left(X-\tfrac{1}{2}r\right) \qquad (13)$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x') = -g\int \psi(x'')(x''|\varphi|x')\,dx'', \qquad (14)$$

$$\gamma_\mu \frac{\partial \overline{\psi}(x')}{\partial x'_\mu} - M\overline{\psi}(x') = g\int (x'|\varphi|x'')\overline{\psi}(x'')\,dx'', \qquad (15)$$

となる.

所で一般の非局所場 $\varphi(X, r)$ は固有函数 $\chi_n(r)$ を使って (6) なる形に展開できるから, これを (13) の左辺に代入し, (4) を考慮して, 両辺に $\tilde{\chi}_n(r)$ を乗じて ~~[illegible]~~ 四次元の r 空間で積分すると

$$\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu} - m_n^2\right)u_n(X) = g\int \tilde{\chi}_n(r)\sum_\alpha \overline{\psi}_\alpha\left(X+\tfrac{1}{2}r\right)\psi_\alpha\left(X-\tfrac{1}{2}r\right)dr \qquad (16)$$

が得られる. 同様にして (14) の右辺を変形でき, 脚註の関係式 (7) を考慮すると, (14) は

** スピノル場を非局所化することは ~~スピンが半整数の粒子~~ のフェルミ粒子の質量スペクトルを導き出す上に必要であるが, 問題を単純化するためにスピノル場 ~~[illegible]~~ の方は通常の局所場を取ることにしておく. これに関しては片山泰久 ; 素粒子論研究 (Vol. 4. No. 9 (1952), 97頁) を参照されたい.

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x') = -g\sum_n \int \tilde{\chi}_n(x'-x'')\,u_n\left(\frac{x'+x''}{2}\right)\psi(x'')\,dx'' \qquad (17)$$

となる。これ等の方程式を Møller-Kristensen の得た方程式

$$\left(\frac{\partial^2}{\partial x''_\mu \partial x''_\mu} - m^2\right)u(x'') = \int \Phi(x', x'', x''')\sum_\alpha \bar{\psi}_\alpha(x')\psi_\alpha(x''')\,dx'\,dx''' \qquad (16)'$$

$$\gamma_\mu \frac{\partial \psi(x')}{\partial x'_\mu} + M\psi(x') = -\int \Phi(x', x'', x''')\,u(x'')\,\psi(x''')\,dx''' \qquad (17)'$$

と比較して見ると、両者の間の著しい類似性と同時に、本質的な相違が明瞭になる。即ち非局所スカラー場で統一的に記述される色々な質量のボーズ粒子の中で、特定の一つ、例えば質量 m_n なる粒子だけを取り出し、他はすべて無視してしまうと、(16)及び(17)はそれぞれ(16')及び(17')において

$$m = m_n, \qquad \Phi(x', x'', x''') = g\,\tilde{\chi}_n(x'-x'')\,\delta\left(\frac{x'+x'''}{2} - x''\right) \qquad (18)$$

とおいたのと同等になる。従って内部運動の固有函数 $\chi_n(r)$ が非局所的相互作用の理論における形状函数ないし収斂因子の役割をすることになる。

しかしある質量の場だけをとり出すことは、精々低いエネルギー領域における近似解を求めるのに役立つという以上の意味を持ち得ない。非局所場理論では質量スペクトルの問題が収斂因子の問題と切り離せない。これが局所場の非局所的相互作用の理論と本質的に異なる点である。

4. 振動子模型

以上の一般的考察が一々の具体的な場合においてどの程度まで正当化されるか、どのような複雑な問題が伏在しているかを明かにするために、最も簡単らしく見える振動子模型を実例としてとり上げることにしよう。即ち(2)の形に分離できる演算子Fの中で、特に

$$F \equiv -\frac{\partial^2}{\partial x_\mu \partial x_\mu} + \alpha\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right) \qquad (19)$$

なる形を仮定すると、内部運動の固有函数

$$\chi_{n_1, n_2, n_3, n_0}(r) = H_{n_1}(r_1/\lambda) H_{n_2}(r_2/\lambda) H_{n_3}(r_3/\lambda) H_{n_0}(r_0/\lambda) \exp\left\{-\frac{1}{2\lambda^2}\left(r_1^2 + r_2^2 + r_3^2 + r_0^2\right)\right\} \qquad (20)$$

に対応する質量の固有値 m_{n_1, n_2, n_3, n_0} は

$$(m_{n_1, n_2, n_3, n_0})^2 = \frac{2\alpha}{\lambda^2}(n_1 + n_2 + n_3 - n_0 + 1) \qquad (21)$$

となる。但し α は次元なしの常数、λ は長さの次元を持った常数、r_0 は ~~虚数 実数~~ 相対時間に比例する実変数で、虚数座標 r_4 との ~~$r_4 = i r_0$~~ 関係は

$$r_4 = i r_0$$

である．n_1, n_2, n_3, n_0 はそれぞれ 0 ~~又は~~及び正の整数値だけを取り得る量子数を表わし，$H_n(x)$ は変数 x に関する n 次の Hermite 多項式を表わす．

振動子模型の特徴の一つは，固有函数 χ_{n_1,n_2,n_3,n_0} が量子数 n_1, n_2, n_3, n_0 の如何にかかわらず，$|r_1|, |r_2|, |r_3|, |r_0|$ の大きな値に対して充分早く減少するばかりでなく，これ等の函数の Fourier 変換を作ると，全く同じ形を持つているから，運動量．エネルギー空間でも具合よく収斂してくれることである．* ここで直ぐ気がつくのは，(20) なる形の固有函数が相対論的に不変でないことではあるが，そのこと自身は別に理論の難点とはならない．ただこれに関連して次の二つの重要な困難を無視できない．

(i) 第一は (21) の右辺が負なる場合には，質量が虚数となり，物理的意味づけが困難になることである．

この点の改良は容易であつて，例えば演算子 F として (19) の代りに

$$F \equiv -\frac{\partial^2}{\partial x_\mu \partial x_\mu} + \frac{\alpha^2}{\lambda^2}\left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu\right)^2 \qquad (19)'$$

を取れば，内部運動の固有函数~~は~~として元の通り (20) を取つてよく，質量の固有値は

$$(m_{n_1,n_2,n_3,n_0})^2 = \frac{4\alpha^2}{\lambda^2}(n_1+n_2+n_3-n_0+1)^2 \qquad (22)$$

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* 振動子模型は Born の提唱した相反性の立場から見ると，自己相反性を満足する場合に当つており，従つて通常の四次元空間での表現と，運動量・エネルギー空間での表現とが当然同形になつてくる．しかし Born-Green の理論は(5) 素粒子の内部自由度を考慮したことになつていないから，場の理論の収斂性の問題とは関係がつかなかつた．

しかし非局所場理論ではある質量の場だけを他から切りはなして取上げることは，一種の近似としてしか許されない．高エネルギーの過程においては一般に近似は悪くなり，少なくとも幾つかの低い質量の場を同時に考える必要がある．局所場の非局所的相互作用の理

$$\beta^2\lambda^2\left\{-\left(\frac{\partial}{\partial x_\mu \partial r_\mu}\right)^2+\frac{1}{\lambda^4}\left(r_\mu\frac{\partial}{\partial x_\mu}\right)^2\right\}$$

$$k_\mu k_\mu$$

$$\left\{-\left(k_\mu\frac{\partial}{\partial r_\mu}\right)^2+\frac{1}{\lambda^4}\left(k_\mu r_\mu\right)^2\right\}$$

$$-(k_j r_j - k_0 r_0)^2 = -k_0'^2 r_0'^2$$

$$-\left(k_j\frac{\partial}{\partial r_j}+k_0\frac{\partial}{\partial r_0}\right)^2 = -k_0'^2\frac{\partial^2}{\partial r_0'^2}$$

$$(+k_\mu k_\mu)$$

$$\beta^2\lambda^2(-m^2)\frac{(2n_0'+1)}{\lambda^2}$$

$$= -m^2$$

$$m^2$$

$$-m^2+(\quad)^2+\beta^2 m^2(2n_0'+1)$$

$$m^2=\frac{(\quad)^2}{1-\beta^2(2n_0'+1)}\qquad n_0'=0,1,2,\cdots$$

である。H_n は n 次の Hermite 多項式を表わす。従って固有函数 χ_{n_1,n_2,n_3}

$$(m_{n_1,n_2,n_3,n_0})^2 = \frac{\frac{4\alpha^2}{\lambda^4}(n_1+n_2+n_3-n_0+1)^2}{1-2\beta^2(n_0+\frac{1}{2})} \qquad (27)$$

で与えられることになる。但し n_0 の値は、

$$1-2\beta^2(n_0+\tfrac{1}{2}) > 0$$

$$n_0+\tfrac{1}{2} < \frac{1}{2\beta^2}$$

$$n_0 < \tfrac{1}{2}\left(\frac{1}{\beta^2}-1\right)$$

を満足する 0 又は正の整数に限られる。例えば $\beta^2=\frac{1}{2}$ とおくと、$n_0=0$ だけが許されることになり、~~(27) は~~ 質量スペクトルは

$$m_{n_1,n_2,n_3,0} = \frac{\sqrt{8}\,\alpha}{\lambda^2}(n_1+n_2+n_3+1) \qquad (28)$$

で与えられ、従って

~~(ii) 実数~~

となるから，常数 α を ~~正の値~~ に取れば実数の質量だけしか現われないことになる．

(ii) 第二の困難はたとい質量を実数値（従つて 0 又は正の値と限つて差支えない）に制限しておいても，常に固有値が無限に縮退していることである．例えば (21)′ の場合について最小の質量 $m_0=0$ を取ると，この固有値に属する固有函数は

$$n_1+n_2+n_3-n_0=-1$$

を満す n_1, n_2, n_3, n_0 の組合せのどれを取つてもよいから，無限に縮退していることになる．

この第二の困難は第一の困難よりはるかに重大であつて，恐らく振動子模型だけの特性でなく，相対論的な固有値問題につきまとう困難らしく思われる.** この難点を除くには ~~次節に述べるように~~ 内部運動と外部運動の間の結合を考慮する必要がありそうである．その一例として，(19)′ なる演算子に更に

$$-\beta^2\lambda^2\left\{-\left(\frac{\partial}{\partial X_\mu \partial r_\mu}\right)^2+\frac{1}{\lambda^4}\left(r_\mu\frac{\partial}{\partial X_\mu}\right)^2\right\} \qquad (25)$$

なる項を附加したとすると，自由場の方程式 (1) は

$$\varphi(X,r)=e^{ik_\mu X_\mu}\chi(k_\mu, r_\mu) \qquad (26)$$

なる形の解を有することが直ちにわかる．そして χ の満足すべき方程式

但し β は実常数とする．

** Schrödinger が Born-Green の振動子模型に対する難点としてあげている諸点もこの問題と密接に関係している.(6)

$$\left[k_\mu k_\mu + \alpha^2 \underbrace{}_{(\lambda^2)} \left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu \right)^2 + \beta^2 \lambda^2 \left\{ -\left(k_\mu \frac{\partial}{\partial r_\mu} \right)^2 + \frac{1}{\lambda^4} (k_\mu r_\mu)^2 \right\} \right] \chi(k_\mu, r_\mu) = 0 \qquad (25)'$$

の解は(20)なる固有函数の完全系で展開することによつて得られる。~~一般の場合はやや複雑である~~，しかしこの手續きは次の事情を考慮することによつて，非常に簡単化される。即ち運動ベクトル k_μ が時間的である場合，いいかえれば $k_\mu k_\mu < 0$ なる場合にのみ(25)′が解を有することを考慮して，* 座標系の変換を行い，新しい座標系では $k_1 = k_2 = k_3 = 0$，$k_0 = \kappa$ となつたとすると，(25)′は

$$\left\{ -\kappa^2 + \alpha^2 \left(-\frac{\partial^2}{\partial r_\mu \partial r_\mu} + \frac{1}{\lambda^4} r_\mu r_\mu \right)^2 + \beta^2 \lambda^2 \kappa^2 \left(-\frac{\partial^2}{\partial r_0^2} + \frac{1}{\lambda^4} r_0^2 \right) \right\} \chi(0,0,0,\kappa; r_\mu) = 0, \qquad (26)'$$

となり，従つて κ が質量

$$-\kappa^2 + \frac{4\alpha^2}{\lambda^2} (n_1 + n_2 + n_3 - n_0 + 1)^2 + 2\beta^2 \kappa^2 \left(n_0 + \frac{1}{2} \right) = 0 \qquad (28)$$

なる関係を満す場合に限つて(20)が(25)′を満足することがわかる。従つて質量スペクトルは(27)の代りに

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* k_μ が空間的ベクトル，即ち $k_\mu k_\mu > 0$ の場合には，$|r_1|$, $|r_2|$, $|r_3|$, $|r_0|$ の大きな値に対して減少するような χ が存在しないことが容易に証明できる。

$$m_{n_1,n_2,n_3,n_0} = \frac{2\alpha}{\lambda}\frac{|n_1+n_2+n_3-n_0+1|}{\sqrt{1-2\beta^2(n_0+\frac{1}{2})}} \qquad (29)$$

となる。但し α は正の常数になり、n_0 の値は

$$n_0 < \frac{1}{2}\left(\frac{1}{\beta^2}-1\right) \qquad (30)$$

を満足する 0 又は正の整数に限られる。例えば $\beta=\frac{1}{\sqrt{2}}$ と取ると、$n_0=0$ だけが許されることになり、質量スペクトルは

$$m_{n_1,n_2,n_3,0} = \frac{2\sqrt{2}\,\alpha}{\lambda}(n_1+n_2+n_3+1) \qquad (29)'$$

となる。従つて固有値は一般に有限の縮退しか持たぬことになり、特に最小の固有値

$$m_{0,0,0,0} = \frac{2\sqrt{2}\,\alpha}{\lambda} \qquad (31)$$

は縮退しておらず、これに対應する固有函数は

$$\chi_{0,0,0,0}(r) = \exp\left\{-\frac{1}{2\lambda^2}(r_1^2+r_2^2+r_3^2+r_0^2)\right\} \qquad (30)$$

に限られることになる。

以上は外部運動ないし重心運動に関する静止系、いわゆる重心静止系から見た時の話であるが、元の座標系に戻つても、質量スペクトルは変化なく、固有函数の中の変数が座標変換を受けるだけである。例えば(30)なる固有函数は、元の座標系では

$$\chi_{0,0,0,0}(k_\mu, r_\mu) = \exp\left\{-\frac{1}{2\lambda^2}\left(r_\mu r_\mu + 2\frac{(k_\mu r_\mu)^2}{k_\mu k_\mu}\right)\right\} \qquad (30)'$$

となる。

尚(29)がパイ中間子の質量 m_π と同じ位の質量を与えるためには、α として1の程度の数、長さの次元を持つ常数 λ としてパイ中間子のCompton波長(通常の単位で $\hbar/m_\pi c$)の程度の値をとればよい。*
すると(27)'からその整数倍の質量の粒子がすべて同様に存在しなければならぬことになるが、それ等はパイ中間子よりもっと複雑な内部構造を持ち、従って色々違った性質を持ったボーズ粒子と考えられる。**
このような結論が正しいかどうか今の所判定できない。
もしも λ として核子のCompton波長に近い小さい値をとるならば、α は1よりずっと小さくなければならぬことになる。

5. むすび

以上振動子模型という特定の場合に関連して得られた結果が、どの程度の一般性を持つかの推定は容易で

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* 特に例えば $\alpha = 1/\sqrt{2}$ ととれば、λ が古典電子半径(通常の単位で $e^2/m_e c^2$, m_e は電子質量)である場合に

$$m_{0,0,0,0} = 274\, m_e$$

となる。

** 今までに知られている不安定粒子の質量は近似的に整数法則に従うように見えるが(7)自己エネルギーによる補正がどの位か推定できなければ、(27)'の正否の判定は不可能である。

ないが、~~[illegible]~~望ましくない縮退を取除くために外部運動と内部運動を組合せすことが一般に必要だと假定すると、第三節で内部運動を平均化して非局所場を非局所的相互作用に歸着せしめた手續きは、そのままでは使えないことになる。從つて非局所場理論の收斂性の問題も簡單に見透しがつきにくくなるが、この問題の詳細な検討は次の機會に譲ることにして、一先ず擱筆する[8]。

や、内部~~場~~自由度と関係した新しい選擇規則の導出等~~[illegible]~~

文　献

(1) H. Yukawa, Phys. Rev. 77 (1950), 219; 80 (1950), ~~[illegible]~~ 1047; D. R. Yennie, Phys. Rev. 80 (1950), 1053.

となる。

5. 結論

以上 振動子模型~~について~~ ~~検討から明らかになった~~ ~~ことは~~ 論じたことは、~~恐らく~~ の場合に関連して得られた ~~一般~~ 結論は、恐らくたいした本質的な変化なしに他の色々な模型にも適用できる

ただどの位の一般性をもつかの推定は容易でないが、

若し(27)'で与えられる質量スペクトルが ものとすると、αとして1の程度の常数、λとしてパイ中間子のCompton波長（通常の単位で $\hbar/m_\pi c$）ないしは古典的

(29)が ~~通常~~ パイ中間子の質量と一致するためには、例えば $\alpha = 1/\sqrt{2}$ とした場合に、長さの次元をもつ常数として λ が古典電子半径（通常の単位で $e^2/m_e c^2$, m_e：電子質量）を取っておけばよいことになる。もっと一般に(29)がパイ中間子の質量と一致 ~~させるためには~~ するとし、(27)'で与えられる質量スペクトルから パイ中間子の ~~整数~~ 倍の質量の粒子が同時に存在 を認めることになる しなければならぬことになる。このような結論が果して正しいか

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$$[(x'|\varphi|x''), (x'''|\varphi|x'''')] \qquad r' = x''' - x^{IV}$$

$$= \left[\sum_n u_n(X)\chi_n(r), \sum_{n'} u_{n'}(X')\tilde{\chi}_{n'}(r')\right]$$

$$= i\sum_n \chi_n(r)\tilde{\chi}_n(r')\Delta_n(X - X'),$$

$$= \int \frac{e^{ik_\mu X_\mu}\chi_n(r)\, e^{-ik_\mu X'_\mu}\tilde{\chi}_n(r')}{\varepsilon(k_\mu)\,\delta(k_\mu k_\mu + m_n^2)}\, N\,(dk)^4$$

$$=$$

$$F \cdot e^{ik_\mu X_\mu}\chi_n(r) = \mu\, e^{ik_\mu X_\mu}\chi_n(r)$$

$$\mu = k_\mu k_\mu + m_n^2$$

$$\int \varphi_{k,n}(X,r)\,\tilde{\varphi}_{k,n}(X',r')\,\frac{\delta(\mu)}{\varepsilon(k)}\,(dk)^4$$

Structure ~~of~~ and Mass Spectrum of Elementary Particles

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New York, New York

if it into ~~existed~~ exists at all,

A unified field theory of elementary ~~particles~~, is expected to account for the mass spectrum of a great variety of stable and unstable particles on the one hand and to be free from the pathological divergences on the other. The non-local field was ~~introduced in order~~ supposed to describe, in a relativistically invariant way ~~an entity which could be~~ elementary ~~system~~ with its own ~~internal structure~~ ~~a system~~ an object which ~~is ad~~ ~~was not~~ ~~a con~~ ~~did not~~ was elementary in the sense that it could not be decomposed into more elementary ~~syst~~ objects, but was complex enough to ~~have~~ ~~take~~ be in various kinds of states with different masses, spins and other intrinsic properties. In the previous papers(1), the author tried to decompose a general non-local field into irreducible parts, by imposing as ~~many restrictions as possible~~ a number of compatible relations upon ~~this~~ it. However, the conclusions reached were very unsatisfactory for a number of reasons.(2) Among other things, the ~~possible masses~~ mass levels of the particles associated with each of the irreducible fields remained completely arbitrary and ~~simple~~ plausible ~~simple~~ assumptions on the interaction between these fields did not result in the expected convergence of self-energies. ~~Since the~~ This may well be due to the imposition of too many ~~restrictions~~ field equations upon ~~the~~ a single non-local field. Instead, one could have started from one field equation of the following type, ~~without~~ laying aside the question of reduction of ~~of paying too much attention to the reduction of fields~~ a field into irreducible parts.

Let us take, just for simplicity, a scalar (or pseudoscalar) non-local field

$$(x'_\mu|\varphi|x''_\mu) \equiv \varphi(X_\mu, r_\mu),$$

where x'_μ, x''_μ $(\mu = 1, 2, 3, 4)$ stand for two ~~sto~~ sets of space-time parameters ~~with imaginary time~~ and

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu)$$

~~free~~, $r_\mu = x'_\mu - x''_\mu$.*

The field equation is supposed to have a general form

$$F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(X_\mu, r_\mu) = 0, \quad (1)$$

where the operator F is a certain invariant function of $\frac{\partial}{\partial X_\mu}$, r_μ and $\frac{\partial}{\partial r_\mu}$. It ~~cannot~~ must be ~~dependent~~ in~~dependent~~ ~~on X_μ~~ of X_μ in order to be invariant under the translation of the ~~coord~~ origin of the coordinate system. In particular, if ~~one~~ we assume that F is linear in $\frac{\partial^2}{\partial X_\mu \partial X_\mu}$ and separable, i.e.

$$F \equiv -\frac{\partial^2}{\partial X_\mu \partial X_\mu} + f\left(r_\mu, \frac{\partial}{\partial r_\mu}\right), \quad (2)$$

* In contradistinction to the previous papers, $\mu = 4$ corresponds to an ~~we use~~ imaginary time parameter ~~instead~~, ~~For~~ instance, $r_4 = i r_0$, where r_0 is the real relative time.

we have ~~eig an ei~~ eigen-solutions of the forms $\varphi(x_\mu, r_\mu) = u(x_\mu)\chi(r_\mu)$
where u and χ satisfy

$$\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \mu\right) u = 0 \quad (3)$$

$$\left(f\left(r_\mu, \frac{\partial}{\partial r_\mu}\right) - \mu\right)\chi = 0, \quad (4)$$

μ being the separation constant. Thus, the masses of the particles associated with the non-local field φ are given as the eigen-values of μ of the equation (4) for ~~χ.~~ the internal eigenfn χ. If it is possible to choose the operator f such that the eigenvalues are all discrete, ~~μ_n say,~~ then ~~the~~ an ~~free field φ can be expand~~ arbitrary non-local field φ can be expanded

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(1)

A Field Theory of ~~Elementary~~ Non-Local Systems ~~Particles~~ ~~(with Internal Structure)~~ I.

By Hideki Yukawa*

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Abstract

Abstract

~~Introduction~~

* On leave of absence from Columbia University in the city of New York.

~~The introduction~~ implication of the concept of non-local field, ~~implies~~ ~~that some new insight~~ which is ~~one~~ one simple way of describing ~~a system~~ relation of elementary ~~with the internal structure~~ was twofold.

1. Introduction

~~The original intention of the author, when he tried to extend introduced the notion non-local of field, in such a way that the elementary particles with internal structure was to extend the field theory twofold.~~

On the one hand by ~~introdu~~ taking into account the possible internal structure of ~~the~~ elementary particles and formulating it ~~relativ.~~ in term of the non-local field, the mass-spectrum of elementary particles was expected to come out ~~as the result~~ by solving a certain eigen-value problem. On the other hand, it was hoped that the eigenfunction which was associated with each of the ~~for the~~ eigenvalues internal motion of the mass might ~~serve~~ play the role of the convergence factor in the interaction between elementary particles.

In the previous papers[1], however, these points were not discussed. On the contrary, ~~the~~ main effort was concentrated on restricting the possible forms of the

non-local field, which was too vague a notion and ~~too~~ broad to be dealt with effectively. For that purpose, the author ~~took the advantage of~~ adopted the concept of reciprocity in a certain particular way and ~~unfortunately~~, decomposed the general non-local field into irreducible parts.
It ~~was found~~ turned out, however, that the restrictions thus imposed on the non-local field in order to obtain irreducible fields were so severe that an irreducible ~~for~~ non-local field ~~reduced to the local field~~ became equivalent to a certain type of local field as first pointed out by Fierz[(2)]. For that reason, it ~~was felt necessary to give up the red complete reduction of~~ is inappropriate to stay to find a correspondence between each of ~~an~~ irreducible non-local fields and ~~the each~~ each ~~one~~ type of elementary particles. This means

that we have to drop, at least, one of the field equations ( ) for the non-local field, ~~or replace it~~ so that we can ~~admit~~ admit the ~~mixture~~ reducible fields as ~~the model~~ possible models for the el. particles. ~~We can start from the following general equation or one can~~ For instance, one can assume only one field eq. for the non-local field instead of a set of three simultaneous equations. From this, one can formulate the general eigenvalue problem for the mass as shown in the next section.

In the mean-time non-local interaction between local fields has been investigated by a number of authors(2). Among them, a recent paper by Møller and ~~Rosenfeld~~ Kristensen(4) is of particular interest in that the self-energies of both of the meson and nucleon could be made ~~cover~~ finite, at least ~~in~~ in the ~~second~~ lowest order in the coupling constant by ~~introducing~~ choosing ~~the~~ a suitable form function, ~~into the interact for the inter.~~ in certain type of meson theory.

In this connection, it ~~In this it is also~~ will be shown in the third section that ~~the eigenfunctions for the internal motion~~, if one ignores the possible degeneracy of eigenvalues for the mass, the ~~field~~ equations for ~~the~~ a non-local field interacting with some other field can be brought to the form, in which the eigenfunction for the internal motion plays the ~~form~~ role similarly to the form function in the theory of non-local interact. between local fields. However, there is an important difference between these two theories. Namely, ~~such~~ non-local field is ~~shown~~ equivalent to a certain mixture of infinite number of local-fields with different masses and the form functions are the uniquely defined in connection with the eigenvalue problem for the mass spectrum.

This has its own ~~ad~~ disadvantage as well as advantage compared with

the theory of non-local interaction.
It is more satisfactory in that ~~the~~ we ~~for~~ have the means of determining the form function, ~~whereas~~ On the other hand, however, the ~~question~~ problem of convergence of self-energies ~~and~~ becomes the quantities like much more ~~complicated~~ involved due to the appearance of infinitely many terms already in the lowest approximation.
~~The complication~~ There is a further complication which we ~~have~~ ignored ~~in the~~ in order to ~~show~~ see show ~~the general~~ feature of the more clearly the essential points of our arguments. some of Namely, the four-dimensional in the case eigenvalue problem with ~~of~~ which we are going to deal with, ~~the~~ degeneracy is ~~so~~ very The difficulty of the infinite serious. In order to make clear the situation, we take up the

simple case of the four-dimensional harmonic oscillation in §4. This is the one which were investigated by Born and Green[5] in connection with the problem of mass-spectrum, although their interpretation was ~~essent~~ quite different from that of the present paper.

In §5, ~~the~~ a tentative method of removing the infinite degeneracy by introducing the coupling between external and internal motion will be discussed. ~~The introduction of the coupl~~ This, however, leads us to the new situation that the field equations for the interacting non-local fields can no longer be brought to the form which is similar to the theory of ~~int~~ non-local ~~fields~~ interaction. ~~In~~ Thus in order to carry out the formal integration of field equations, we must adopt a new method which will be discussed in the ~~sh~~ next paper.

2. General considerations on the four dimensional eigenvalue problem

An arbitrary scalar non-local field, which is ~~characterized~~ a scalar $(x'_\mu|\varphi|x''_\mu)$, function of two sets of space-time parameters x'_μ, x''_μ $(\mu = 1, 2, 3, 4)$, can be written as in the previous paper as a function

$$(x'_\mu|\varphi|x''_\mu) \equiv \varphi(X_\mu, r_\mu)$$

of two sets of ~~vari real~~ variables

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu)$$

$$r_\mu = x'_\mu - x''_\mu .$$

For convenience, let us call them external and internal variables respectively.

We adopt the usual ~~notations~~ conventions: ~~and~~ the fourth component is always pure imaginary and related to the real time variables which is characterized by the index 0. For instance,

$$\cancel{x'_4 = i x'_0,} \quad X_0 = i \quad r_4 = i r_0$$

where r_0 denotes the ~~relative~~ internal time in natural units in which $c = \hbar = 1$.

The field equation for the free non-local field $\varphi(X, r)$ is supposed to have the general form

$$F\left(\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi(X, r) = 0, \quad (1)$$

where F is a certain invariant function of $\frac{\partial}{\partial X_\mu}, r_\mu, \frac{\partial}{\partial r_\mu}$. Obviously F ~~cannot~~ should be independent of X_μ in order that it is invariant with respect to the translation of the origin of the Lorentz frame-work. The ~~explicit~~ dependence of F both on r_μ and $\frac{\partial}{\partial r_\mu}$ is ~~not only admissible,~~ ~~but also~~ essential for the derivation of the discret mass-spectrum. ~~This~~ The dependence of F on r_μ is admissible, because r_μ denotes the internal or relative coordinates, ~~while~~ Born ~~and Green's theory~~[1] ~~is to be reinterpreted~~ It seems to the author that this is the only reasonable reinterpretation of ~~the~~ ~~Bor~~ the eigen-value problem problem proposed by Born and Green[2].

Thus, in general, F ~~is~~ can be a function of $\frac{\partial^2}{\partial X_\mu \partial X_\mu}, r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu \frac{\partial}{\partial r_\mu}; r_\mu \frac{\partial}{\partial X_\mu},$

$\frac{\partial^2}{\partial x_\mu \partial r_\mu}$. Since the last two operators refer to the coupling between external and internal motions, let us ignore them for the moment and ~~assume~~ discuss the simple case ~~F ≡~~ in which

$$F \equiv -\frac{\partial^2}{\partial X_\mu \partial X_\mu} + F^{(r)}\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r_\mu}, r_\mu \frac{\partial}{\partial r_\mu}\right), \quad (2)$$

so that the field equation (1) can be readily separated by ~~of~~ putting

$$\varphi(X, r) = u(X)\chi(r).$$

The external and internal wave functions $u(X)$ and $\chi(r)$ satisfy the equations

$$\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu} - \mu\right) u(X) = 0 \quad (3)$$

$$\left(F^{(r)} - \mu\right)\chi(r) = 0 \quad (4)$$

respectively, where μ is the separation constant, Dec, 1952

Lagrangian Formalism in N. L. F. T.

$\varphi(x_\mu)$: spin mass: arbitrary, continuous

$\varphi(x'_\mu, x''_\mu)$ spin mass: discrete

$$L\left(x'_\mu, x''_\mu, \frac{\partial}{\partial x'_\mu}, \frac{\partial}{\partial x''_\mu}\right)\varphi(x'_\mu, x''_\mu) = 0$$

$$L\left(r_\mu, X_\mu, \frac{\partial}{\partial X_\mu}, \frac{\partial}{\partial r_\mu}\right)\varphi(X_\mu, r_\mu) = 0$$

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu), \quad r_\mu = x'_\mu - x''_\mu$$

$$L = L^{(ext)}\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu}\right) + L^{(int)}\left(r_\mu, \frac{\partial}{\partial r_\mu}\right)$$

$$\varphi(X_\mu, r_\mu) = \sum_n \varphi_n^{(ex)}(X_\mu)\varphi_n^{(in)}(r_\mu)$$

$$L^{(int)}\left(r_\mu, \frac{\partial}{\partial r_\mu}\right)\varphi_n^{(in)}(r_\mu) = -\frac{1}{2}m_n^2\varphi_n^{(in)}(r_\mu)$$

$$L^{(ext)}\left(\frac{\partial^2}{\partial X_\mu \partial X_\mu}\right) = \frac{1}{2}\frac{\partial^2}{\partial X_\mu \partial X_\mu}$$

$m_n^2 \geq 0$: $L^{(int)}$ has positive eigenvalues.

Interaction

$$L_1 = g\int\bar{\psi}(x'x'')\varphi(x''x''')\psi(x'''x^{IV})\,dx'dx''dx'''dx^{IV} = \iiint g\,\bar\psi_{n'}^{(ex)}(X')\varphi_{n''}^{(ex)}(X'')\psi_{n'''}^{(ex)}(X''')$$

$$\Phi_{n'n''n'''}(X', X'', X''')\,dX'dX''dX'''$$

$$\Phi_{n'n''n'''} = \iint \bar\psi_{n'}^{(in)}(r')\varphi_{n''}^{(in)}(r'')\psi_{n'''}^{(in)}(r''') \times dr'$$

$$= \int \bar\psi_{n'}(r')\varphi_{n''}(2(X'-X'')-r') \times \psi_{n'''}(r' + 2(X''-X''') - 2(X'-X''))\,dr'$$

$$= \Phi_{n'n''n'''}(X'-X'', X''-X''')$$

$$r' + r'' = 2(X'-X'')$$

$$r'' + r''' = 2(X''-X''')$$

$$r''' - r' = 2(X''' + X' - 2X'')$$

$$r' + r''' = 2(X''' - X')$$

Lagrangian formalism :

$$\bar{L}_0 = \int \varphi(x'x'')\{L^{(ex)} + L^{(in)}\}\varphi(x''', x^{iv})\, dx'dx''dx'''dx^{iv}$$

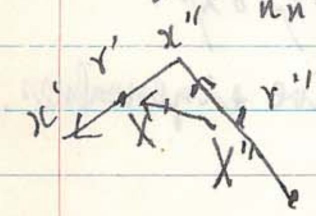
$$[\varphi p_\mu][p_\mu \varphi]\} + L^{(in)}(\{p_\mu \varphi\}, [x_\mu, \varphi]\})$$

$\varphi(x', x''$

$$\varphi^{(ex)}_{n'}(X')\varphi^{(in)}_{n'}(r') \qquad \varphi^{(ex)}_{n'''}(X''')\varphi^{(in)}_{n'''}(r''')$$

$$\bar{L}_0 = Tr\left[\frac{1}{2}[p_\mu\varphi][p_\mu\varphi] + \frac{1}{2}\{p_\mu\varphi\}_+\{p_\mu\varphi\} + \frac{1}{2\lambda^4}[x_\mu\varphi][x_\mu\varphi]\right]$$

$$= \iint\left(\frac{1}{2}\sum_{n'n''}\frac{\partial\varphi^{(ex)}_{n'}(X')}{\partial X'}\frac{\partial\varphi^{(ex)}_{n''}(X'')}{\partial X''}\int\varphi^{(in)}_{n'}(r')\varphi^{(in)}_{n''}(r'')\,dr' + \frac{1}{2}m^2\varphi^{(ex)}(X')\varphi^{(ex)}(X'')\right)dX'dX''$$



$$r' + r'' = 2(X' - X'')$$

$$r'' = 2(X' - X'') - r'$$

$$\varphi_n^{(in)}(-r) = \pm\varphi_n^{(in)}(r)$$

$$\bar{L}_0 = \frac{1}{2}\int\left(\frac{\partial\varphi^{ex}_{n'}(X')}{\partial X'}\frac{\partial\varphi^{ex}_{n''}(X'')}{\partial X'} + \kappa^2_{n'n''}\varphi^{ex}_{n'}(X')\varphi^{ex}_{n''}(X'')\right)$$

$$\kappa_{n'n''} = \varphi^{(in)}_{n'}(r')\varphi$$

$$\times F_{n'n''}(X' - X'')\,dX'dX''$$

$$F_{n'n''}(X' - X'') = \int\varphi^{(in)}_{n'}(r')\varphi^{(in)}_{n''}(2(X' - X'') - r')\,dr'$$

Mass Factor :

Normalization factor : $F_{n'n''}(X' - X'')$

Form factor : $\Phi_{n'n''n'''}(X' - X'', X'' - X''')$

$$\int\left\{\frac{\partial\varphi(r'_\mu)}{\partial x'_\mu}\cdot\frac{\partial\varphi(r''_\mu)}{\partial x''_\mu}+r'_\mu\varphi(r')r''_\mu\varphi(r'')\right\}dr'$$

$$r''=2(X'-X'')-r'$$

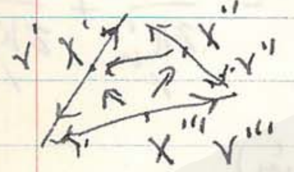
$$\bar{L}_0=\iint(x'|L_0|x'')\,\delta(x'-x'')\,dx'dx''$$

$$=\frac{1}{2}\int\left(\left(\frac{\partial\varphi_n(X')}{\partial X'}\right)^2+m^2\varphi_n^2(X')\right)\overset{\pm\delta_{n'n''}}{\bar{F}_{n'n''}}\,dX'$$

$$\bar{F}_{n'n''}=\int\varphi^{(in)}_{n'}(r')\,\varphi^{(in)}_{n''}(-r')\,dr'=\pm\delta_{n'n''}$$

$$\varphi^{(in)}_{n''}(-r')=\pm\varphi^{(in)}_{n''}(r')$$

$$\bar{L}_1=g\int\bar{\psi}(x'x'')\varphi(x''x''')\psi(x''',x^{iv})\,\delta(x',x^{iv})\,dx'\cdots dx^{iv}$$



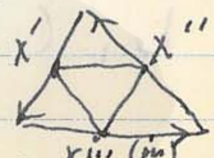
$\cancel{X'-X''}$ −

$$r'+r''+r'''=0 \qquad = 2($$

$$2r'=2(X'''-X'+X'''+X'-2X'')$$

$$\Phi_{n'n''n'''}=\int\bar{\psi}_{n'}(r')\varphi_{n''}(2(X'-X'')-r')\psi(2(X''-X')-r')$$

$$r'=2(X'''-X'')$$



$$\Phi_{n'n''n'''}=\int\bar{\psi}^{(in)}_{n'}(2(X'''-X''))\varphi^{(in)}_{n''}(2(X'-X''))\psi^{(in)}_{n}(2(X''-X'))$$

$$\text{if}\quad \bar{\psi}^{(in)}_{n}\propto\delta(2(X'''-X'')),\quad \psi^{(in)}_{n}\propto\delta(2(X''-X'))$$

$$\Phi_{n'n''n'''}\propto\varphi^{(in)}_{n''}(2(X'-X''))\,\delta(2(X'''-X''))\,\delta(\ )$$

$$\tilde{L}_1 = g \underbrace{\iiint}_{\Phi_{n'n''n'''}} \psi^{(ex)}_{n'}(X)\varphi_{n''}(X)\varphi_{n'''}(X)\,dX$$

One cannot go over to the limit
$\varphi_{n''} \to \delta$ or $\varphi \cdot \psi \to \delta$
because then ~~the~~ L will be infinite
(one has too many δ-fns.)

$L_0\left([p_\mu, \varphi],\ \frac{1}{2}[p_\mu, \varphi]_+,\ [x_\mu, \varphi] \right)$

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu)$$

$$r_\mu = x'_\mu - x''_\mu \qquad \frac{\partial}{\partial r_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x'_\mu} - \frac{\partial}{\partial x''_\mu}\right)$$

$$\left.\begin{aligned} x'_\mu &= X_\mu + \tfrac{1}{2} r_\mu \\ x''_\mu &= X_\mu - \tfrac{1}{2} r_\mu \end{aligned}\right\} \qquad \frac{\partial}{\partial X_\mu} = \frac{\partial}{\partial x'_\mu} + \frac{\partial}{\partial x''_\mu}$$

$$L_0 = L^{(ex)}([p_\mu, \varphi]) + L^{(in)}\left(\tfrac{1}{2}[p_\mu, \varphi]_+,\ [x_\mu, \varphi]\right)$$

$L_1(\varphi, \psi, \tilde{\psi})$

Møller - Kristensen
Bloch
Katayama

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}\left(\frac{\partial^2}{\partial r_\mu \partial r_\mu} - r_\mu r_\mu\right) - \left(k_\mu \frac{\partial}{\partial r_\mu}\right)\left(k_\mu \frac{\partial}{\partial x_\mu}\right) + (k_\mu r_\mu)(k_\mu r_\mu)$$

$$= \frac{\partial^2}{\partial x_\mu \partial x_\mu}\left(\frac{\partial^2}{\partial r_\nu \partial r_\nu} - r_\nu r_\nu\right) - \frac{\partial^2}{\partial x_\mu \partial r_\mu}\frac{\partial^2}{\partial x_\nu \partial r_\nu} + \left(r_\mu \frac{\partial}{\partial x_\mu}\right)\left(r_\mu \frac{\partial}{\partial x_\mu}\right)$$

$$= \left(\frac{\partial^2}{\partial x_\mu \partial r_\nu}\right)^2 - \left(\frac{\partial^2}{\partial x_\mu \partial r_\mu}\right)^2 + r_\mu r_\mu$$

$$= \frac{\partial^2}{\partial x_\mu \partial r_\nu}\frac{\partial^2}{\partial x_\mu \partial r_\nu} - \frac{\partial^2}{\partial x_\mu \partial r_\mu}\frac{\partial^2}{\partial x_\nu \partial r_\nu}$$

$$\frac{\partial^2}{\partial x_1 \partial x_1}\left(\frac{\partial^2}{\partial r_\mu \partial r_\mu} - \frac{\partial^2}{\partial r_1 \partial r_1}\right)$$

$$*\, 2\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial^2}{\partial r_1 \partial r_2}\right)$$

$$\underline{(k_\mu r_\mu)\varphi = 0}$$

or $\underline{r_\mu \frac{\partial}{\partial x_\mu}\varphi = 0}$

$$\underline{(k_\mu r_\mu)^2 \varphi = 0} \qquad \varphi \propto \delta((k_\mu r_\mu)^2) \propto \frac{\delta(k_\mu r_\mu)}{k_\mu r_\mu}$$

the form for

$$\Phi_{n'n''n'''} = \varphi^{(in)}_{n'}(2(X''-X'''))\,\varphi^{(in)}_{n''}(2(X'-X'''))$$
$$\times\,\varphi^{(in)}_{n'''}(2(X''-X'))$$

is not invariant by itself.

$\Phi_{n'n''n'''}$

In order to have only the particles with real masses, we have to take, for instance:

$$h^{(in)} = \left\{\left(\frac{1}{2}\{p_\mu, \varphi^{(in)}\}_+\right)^2 + \frac{1}{\lambda^4}[x_\mu, \varphi^{(in)}]^2 \right\}^2$$

~~together with supplem. cond.~~, $\varphi^{(ex)}\varphi^{(ex)}$

eigenvalues:

$$\mu^2 n^2 = \mu^2(n_1+n_2+n_3-n_0)^2$$

degenerated

$n=0$: $\quad n_1=n_2=n_3=n_0=0$.

~~Rotational~~

Supplementary condition

~~or Rotational energy~~

$$f\left(r_\mu \frac{\partial}{\partial x_\mu}\right)\varphi = 0$$

$$\varphi \propto e^{ik_\mu x_\mu}\,\varphi^{(in)}_{k_\mu}(r_\mu)$$

$$k_\mu r_\mu \cdot \varphi_{k_\mu} = 0$$

$$\varphi^{(in)}_{k_\mu}(\cdots)\,\delta(k_\mu r_\mu)$$
$$= \varphi^{(in)'}_{k_\mu}(r_1', r_2', r_3')\,\delta(r_4')$$

r_4' ~~$\cdots$~~ r_μ' $\rightarrow$

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$$\left(\frac{\partial^2}{\partial r_4'^2} + \frac{1}{\lambda^4} r_4'^2\right) \varphi^{(in)}_{k_\mu}(r'_\mu) = \bigcirc\, \varphi^{(in)}_{k_\mu}(r'_\mu)$$

$n_0 = 0$:

$$\mu^2 n^{2} = \mu^2 (n_1 + n_2 + n_3)^{2}$$

$n = 0$: $\quad n_1 = n_2 = n_3 = 0$

$$r_4' = r_\mu' \frac{\partial}{\partial x_\mu}$$

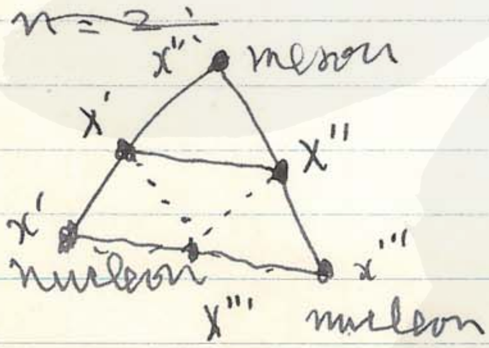
$$\cancel{\frac{\partial}{\partial r_4'} = \frac{1}{\mu^3}\frac{\partial}{\partial r_\nu'} \cdots \frac{\partial}{\partial x_\nu}}$$

$$\frac{1}{\mu^2}\left\{\left(\frac{\partial^2}{\partial r_\mu \partial x_\mu}\right)^2 + \frac{1}{\lambda^4}\underbrace{\left(r_\mu \cdot \frac{\partial}{\partial x_\mu}\right)^2}_{-\,\mathrm{const}}\right\}\varphi(X, r) = 0$$

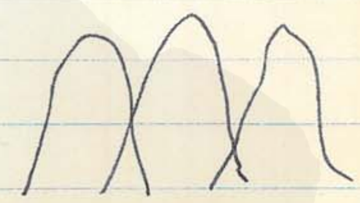
$\boxed{\text{degenerate}}$

$n = 1$:
$$\begin{cases} n_1 = 1, & n_2 = n_3 = 0 \\ n_1 = 0, & n_2 = 1, \quad n_3 = 0 \\ n_1 = 0, & n_2 = 0, \quad n_3 = 1 \end{cases}$$

~~$n = 2$:~~



$$\varphi^{(in)}_{1,0,0}(x' - x''') \propto (x' - x''')$$



Selection rule $\qquad n' + n'' + n'''$: even

$$\cancel{\frac{1}{\chi_4},\ \varphi^{(m)}}.$$

$$\cancel{\left(\frac{\partial^2}{\partial r_4'^2} + \frac{1}{\chi^4}\frac{\partial}{\partial r_4'^2}\right)\varphi^{(m)}_{k\mu}}(r_\mu') = 0.$$

0 Space-Time and the structure of the Elementary Particle
Modified ~~M~~-Minkowski world

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$$\xi_\mu = x_\mu\left(1 - \frac{\lambda^2}{x_1^2+x_2^2+x_3^2+x_4^2}\right)$$

$\mu = 1, 2, 3, 0 \qquad x_0 = ct$

$$\rho^2 = \xi_1^2+\xi_2^2+\xi_3^2+\xi_0^2$$

$$= R^2\left(1-\frac{\lambda^2}{R^2}\right)^2$$

$$\frac{\rho^2}{R^2} = \left(1-\frac{\lambda^2}{R^2}\right)^2$$

$$1-\frac{\lambda^2}{R^2} = \pm\frac{\rho}{R}$$

$$\frac{\lambda^2}{R^2} \pm \frac{\rho}{\lambda}\cdot\frac{\lambda}{R} - 1 = 0$$

(+)

$$\frac{\lambda}{R} = \frac{1}{2}\left(\mp\frac{\rho}{\lambda} \pm \sqrt{\left(\frac{\rho}{\lambda}\right)^2+4}\right)$$

(+) $$\frac{\lambda}{R} = \frac{1}{2}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)$$

$$x_\mu = \xi_\mu \Big/ \left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2\right)$$

~~$\rho \to 0 \qquad x_\mu$~~

$$\frac{R}{\lambda} = \frac{2}{\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}}$$

$\rho \to 0 \qquad R \to \lambda.$

To the point $x, \xi_\mu = 0$, the sphere

$$x_1^2 + x_2^2 + x_3^2 + x_0^2 = \lambda^2$$

corresponds.

Minkowski world

$$ds^2 = \varepsilon_{\mu\nu} d\xi_\mu d\xi_\nu$$

$$d\xi_\mu = dx_\mu \left(1 - \frac{\lambda^2}{x_\mu x_\mu}\right) + \frac{\lambda^2}{(x_\mu x_\mu)^2} 2x_\mu x_\nu dx_\nu$$

$$= \left\{\left(1 - \frac{\lambda^2}{x_\nu x_\nu}\right)\delta_{\mu\lambda} + \lambda^2 \frac{2x_\mu x_\lambda}{(x_\nu x_\nu)^2}\right\} dx_\lambda$$

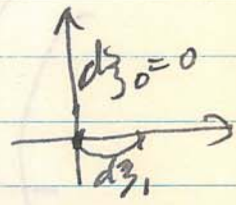
$$\begin{pmatrix} 1 - \frac{\lambda^2}{(x_\nu x_\nu)^2}(x_\nu x_\nu - 2x_\mu^2) & \frac{2\lambda^2 x_1 x_2}{(x_\nu x_\nu)^2} & , & \\ \frac{2\lambda^2 x_1 x_2}{(x_\nu x_\nu)^2} & 1 - \frac{\lambda^2}{(x_\nu x_\nu)^2}(x_\nu x_\nu - 2x_2^2) & . & \\ \frac{2\lambda^2 x_1 x_3}{(x_\nu x_\nu)^2} & \frac{2\lambda^2 x_2 x_3}{(x_\nu x_\nu)^2} & , & , \\ \frac{2\lambda^2 x_1 x_0}{(x_\nu x_\nu)^2} & \frac{2\lambda^2 x_2 x_0}{(x_\nu x_\nu)^2} & , & \end{pmatrix}$$

$$d\xi_\mu = \frac{\partial \xi_\mu}{\partial x_\lambda} dx_\lambda$$

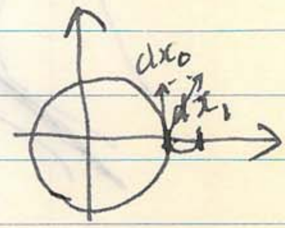
$$\varepsilon_{\mu\nu} d\xi_\mu d\xi_\nu = \varepsilon_{\mu\nu} \frac{\partial \xi_\mu}{\partial x_\lambda} \frac{\partial \xi_\nu}{\partial x_\kappa} dx_\lambda dx_\kappa$$

$$g_{\mu\nu}\frac{\partial\xi_\mu}{\partial x_\lambda}\frac{\partial\xi_\nu}{\partial x_\kappa} = g_{\lambda\kappa}$$

$\|\delta_{\mu\nu} - \eta_{\mu\nu}\|$ ~~[illegible]~~ $|\eta_{\mu\nu}| \leq 1$



$R \gg x_i$, $\|\delta_{\mu\nu} - \eta_{\mu\nu}\| \cong 1$



$x_2 = x_3 = x_0 = 0$ $\qquad |x_1| \geq \lambda$

$$1 + \frac{\lambda^2}{x_1^4}x_1^2 = 1 + \frac{\lambda^2}{x_1^2}$$

$$\begin{vmatrix} 1+\frac{\lambda^2}{x_1^2} & 0 & 0 & 0 \\ 0 & 1-\frac{\lambda^2}{x_1^2} & 0 & 0 \\ 0 & 0 & 1-\frac{\lambda^2}{x_1^2} & 0 \\ 0 & 0 & 0 & 1-\frac{\lambda^2}{x_1^2} \end{vmatrix} > 0$$

$d\xi_1 = \left(1+\frac{\lambda^2}{x_1^2}\right)dx_1$

$d\xi_2 = \left(1-\frac{\lambda^2}{x_1^2}\right)dx_2$

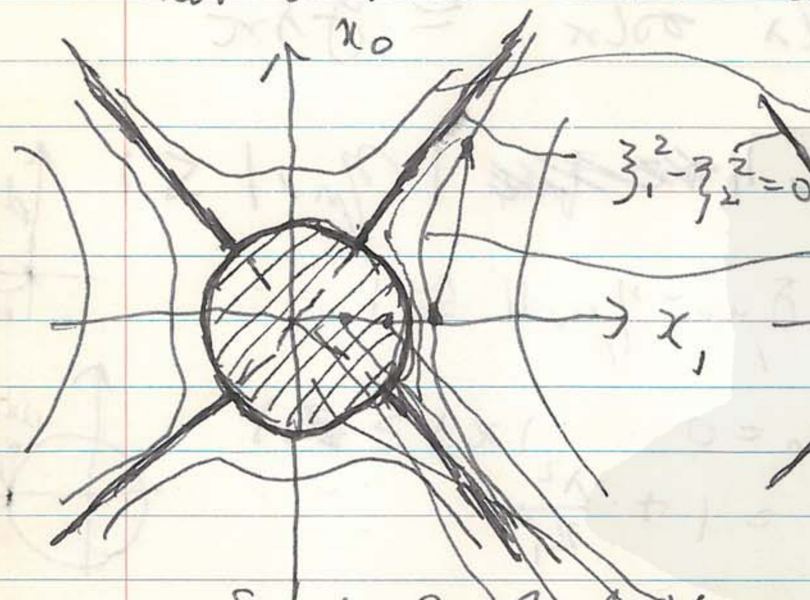
$\vdots$

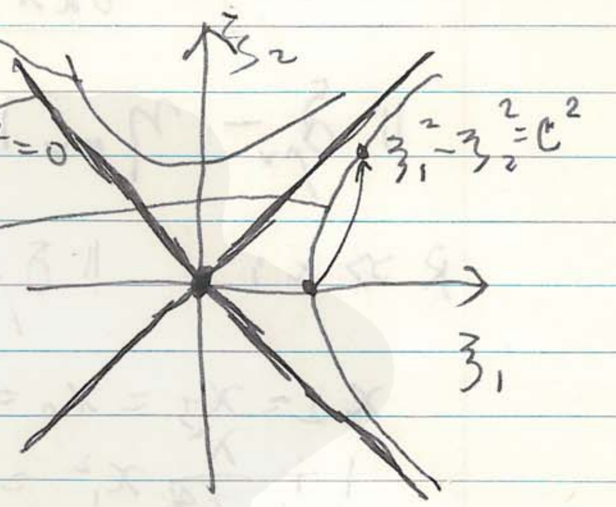
$d\xi_0 = \left(1-\frac{\lambda^2}{x_1^2}\right)dx_0$

for $x_1^2 > \lambda^2$

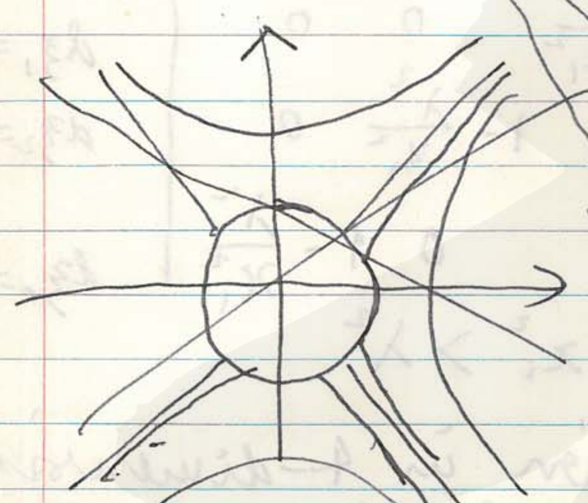
Since the rotation in 4-dimensional (Euclidian) world has no zero Jacobian and transforms any point with $R \geqq \lambda$ into another point with $R \geqq \lambda$, the Jacobian for the transformation $x_\mu \to \xi_\mu$ or its inverse is different from zero, provided that $R > \lambda$.

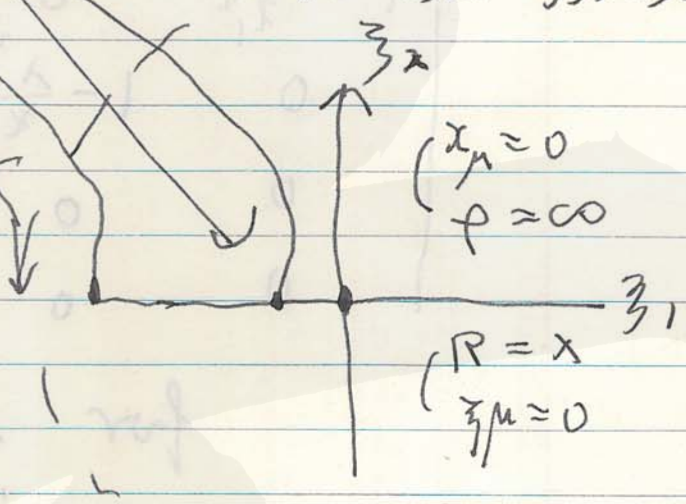
Let us take two dimensional case:





$$\frac{\delta(x_1^2+x_2^2+x_3^2+x_0^2-\lambda^2)}{\lambda^2} \qquad \delta(\xi_1)\delta(\xi_2)\delta(\xi_3)\delta(\xi_0)$$





$$\Box\,\varphi(x_\mu) = -\delta(x_\mu x_\mu - \lambda^2) \qquad \Box\,\Phi = -\delta$$

$$f(x_1, x_2, x_3, x_0)$$

$$d\xi_1 d\xi_2 d\xi_3 d\xi_0 = \sqrt{\left(1+\frac{\lambda^2}{R^2}\right)\left(1-\frac{\lambda^2}{R^2}\right)^3}\cdot dx_1 dx_2 dx_3 dx_0$$

$$\delta(\xi_1)\delta(\xi_2)\delta(\xi_3)\delta(\xi_0) \approx \frac{\delta(x_1^2+x_2^2+x_3^2+x_0^2-\lambda^2)}{\sqrt{\left(1+\frac{\lambda^2}{R^2}\right)\left(1-\frac{\lambda^2}{R^2}\right)^3}\cdot\lambda^2}$$

Modified Lorentz transformation

$$\xi'_\mu = a_{\mu\nu}\xi_\nu$$

$$x'_\mu(\xi'_\mu) = x'_\mu(a_{\mu\nu}\xi_\nu)$$
$$= x'_\mu(a_{\mu\nu}\xi_\nu(x_\lambda))$$

$x_\mu x_\mu >> \lambda^2$: $\xi_\nu \cong x_\nu$

$$x'_\mu = x'_\mu(a_{\mu\nu}x_\nu)$$
$$x'_\mu x'_\mu > \lambda^2$$

$$\varphi(x', x'') = \varphi(X, r)$$

X : Minkowski space
r : modified space

$x' = X + \frac{r}{2}$ $\quad x'' = X - \frac{r}{2}$

$e^{iK_\mu X_\mu} f(r_\mu)$

limit to the point particle
$f(r_\mu) = 0$ for $r_\mu r_\mu > \lambda^2$

Momentum space

$$f(x_\mu) = \int\!\!\int e^{ik^\mu x_\mu} g(k^\mu)\cdot dk$$

Distance of two points in x-space is approximately

$$\cancel{x_\mu x^\mu}\,(x_\mu - y_\mu)(x^\mu - y^\mu),$$

if $(x_\mu - y_\mu)(x_\mu - y_\mu)$ is large compared with λ^2.

On the contrary, if two points x_μ, y_μ are close to each other, ~~the~~ distance is given by

$$(\xi_\mu - \eta_\mu)(\xi^\mu - \eta^\mu),$$

where

$$\cancel{\xi_\mu = x_\mu\left(1 - \cdot\,\lambda^2\right.}$$

$$\xi_\mu - \eta_\mu = (x_\mu - y_\mu)\left(1 - \frac{\lambda^2}{(x_\mu - y_\mu)(x_\mu - y_\mu)}\right)$$

If $(x_\mu - y_\mu)(x_\mu - y_\mu) \ll \lambda^2$, the distance could be ~~very~~ large again compared with λ^2.

The metric in x-space is very complicated, but for any two points x_μ, y_μ: $\left(\begin{array}{l} x_\mu = X_\mu + \frac{1}{2} r_\mu \\ y_\mu = X_\mu - \frac{1}{2} r_\mu \end{array}\right.$

X_μ - space : Minkowski
r_μ - space : modified ~~Minkowski~~ Minkowski space
~~$F(X_\mu) \cdot f(r_\mu)$~~

$$\Phi(x_\mu - y_\mu)\,\Psi(x_\mu)\,\varphi(y_\mu)$$

$$x_\mu = X_\mu + \tfrac{1}{2} r_\mu$$
$$y_\mu = X_\mu - \tfrac{1}{2} r_\mu$$

$$X'_\mu = a_\mu + a_{\mu\nu} X_\nu$$

$$r'_\mu = r'_\mu(r_\mu)$$
$$r_\mu r_\mu\ ,\ r'_\mu r'_\mu \geq \lambda^2$$

$$x'_\mu = a_\mu + a_{\mu\nu} X_\nu + \tfrac{1}{2} r'_\mu(r_\mu)$$
$$y'_\mu = a_\mu + a_{\mu\nu} X_\nu - \tfrac{1}{2} r'_\mu(r_\mu)$$

$$x'_\mu = a_\mu + a_{\mu\nu} \frac{x_\mu + y_\mu}{2} + \tfrac{1}{2} r'_\mu(x_\mu - y_\mu)$$
$$y'_\mu = a_\mu + a_{\mu\nu} \frac{x_\mu + y_\mu}{2} - \tfrac{1}{2} r'_\mu(x_\mu - y_\mu)$$

~~$d\xi_\mu \to 0$~~

Ordinary space-time world

$$dx_\mu = d\xi_\mu \Big/ \left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2 + 4} - \frac{\rho}{\lambda}\right)^2\right)$$

$$+ \xi_\mu \cancel{\frac{\frac{1}{4\lambda^2}\frac{\xi_\nu d\xi_\nu}{\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4}}}{\left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2\right)^2}} \, 2($$

$$+ \xi_\mu \frac{\frac{1}{2}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)}{\left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2\right)^2} \left\{ \frac{\frac{1}{\lambda^2}\xi_\nu d\xi_\nu}{\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4}} - \frac{\xi_\nu d\xi_\nu}{\lambda \cdot \rho} \right\}$$

$$= \frac{d\xi_\mu}{\left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2\right)}$$

$$+ \frac{\frac{1}{2}\frac{1}{\lambda^2}\,\xi_\mu \xi_\nu d\xi_\nu}{\left(1 - \frac{1}{4}\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2\right)^2} \; \frac{-\left(\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} - \frac{\rho}{\lambda}\right)^2}{\sqrt{\left(\frac{\rho}{\lambda}\right)^2+4} \cdot \frac{\rho}{\lambda}}$$

$$g_{\mu\nu}\, dx_\mu dx_\nu = \Gamma_{\mu\nu}\, d\xi_\mu d\eta_\nu$$

$\Gamma_{\mu\nu}$: singular at $\rho = 0$ or $\xi_\mu = 0$

if $g_{\mu\nu}$ is finite (non-zero)

on $x_\mu x_\mu = \lambda^2$.

Connection between Two Spaces

(i) There is one-one correspondence between the whole space of ξ_μ and the space of x_μ outside of the sphere $x_\mu x_\mu = \lambda^2$.

(ii) As for the metric properties of ξ_μ (or x_μ), one may either assume that $g_{\mu\nu}$ vanishes on $x_\mu x_\mu = \lambda^2$ in such a way that $\Gamma_{\mu\nu}$ is regular at $\xi_\mu = 0$ or $g_{\mu\nu}$ is regular on $x_\mu x_\mu = \lambda^2$ but does not behave so as to give regular $\Gamma_{\mu\nu}$ at $\xi_\mu = 0$.

If x_μ is ~~our actual space~~, the quantity which corresponds to the relative coordinates of two particles, one can say that it is ~~mea~~ impossible that two particles come closer than λ four dimensionally (in any coordinate system and irrespective of the metric). This is clear, if ξ_μ is a Minkowski space. If, on the contrary, x_μ is a space which is not too singular on $x_\mu x_\mu = \lambda^2$, then ξ_μ-space may be singular at $\xi_\mu = 0$.

Then, if we construct a field theory in x-space and then go over to the limit $\lambda = 0$, we may ~~not~~ have a field theory which is quite different from the ordinary field theory.

$$\xi_\mu = x_\mu \sqrt{1 - \frac{\lambda^2}{x_1^2 + x_2^2 + x_3^2 + x_0^2}}$$

$$d\xi_\mu = dx_\mu \sqrt{1 - \cdots} + \frac{x_\mu \cdot \frac{\lambda^2 x_\nu dx_\nu}{(x_1^2 + \cdots)^2}}{\sqrt{1 - \frac{\lambda^2}{(\quad)}}}$$

$$= \frac{\left(1 - \frac{\lambda^2}{x_\mu x_\mu}\right) dx_\mu + \frac{\lambda^2 x_\mu x_\nu dx_\nu}{(x_\mu x_\mu)^2}}{\sqrt{1 - \frac{\lambda^2}{(x_\mu x_\mu)}}}$$

$$\begin{pmatrix} \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_\nu x_\nu}}} \left\{ 1 - \frac{\lambda^2}{(x_\nu x_\nu)^2} (x_\nu x_\nu - x_1^2) \right\} & \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_\nu x_\nu}}} \frac{x_1 x_2}{(x_\nu x_\nu)^2} & \cdot \\ \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_\nu x_\nu}}} \left\{ \frac{x_1 x_2}{(x_\nu x_\nu)^2} \right. & \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_\nu x_\nu}}} \left\{ 1 - \frac{\lambda^2}{(x_\nu x_\nu)^2} (x_\nu x_\nu - x_2^2) \right\} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$x_2 = x_3 = x_0 = 0$.

$$\begin{pmatrix} \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_1^2}}} & 0 & \\ 0 & \frac{1}{\sqrt{1 - \frac{\lambda^2}{x_1^2}}} \left(1 - \frac{\lambda^2}{x_1^2}\right) & \\ & & \cdot \\ & & & \cdot \end{pmatrix}$$

$$= \frac{\left(1-\frac{\lambda^2}{x_1^2}\right)^3}{\left(\sqrt{1-\frac{\lambda^2}{x_1^2}}\right)^4} = 1-\frac{\lambda^2}{x_1^2} > 0$$

for $x_1^2 > \lambda^2$.

$$\rho^2 = \xi_\mu \xi_\mu$$
$$= x_\mu x_\mu \cdot \left(1-\frac{\lambda^2}{x_\mu x_\mu}\right)$$

$$\frac{\rho^2}{x_\mu x_\mu} = 1-\frac{\lambda^2}{x_\mu x_\mu} \rightarrow = \frac{\rho^2}{\rho^2+\lambda^2} = \frac{1}{1+\frac{\lambda^2}{\rho^2}}$$

$$\frac{\rho^2+\lambda^2}{x_\mu x_\mu} = 1 \qquad x_\mu x_\mu = \rho^2+\lambda^2,$$

$$\xi_\mu = x_\mu \sqrt{1-\frac{\lambda^2}{\rho^2+\lambda^2}}$$

$$x_\mu = \frac{\xi_\mu}{\sqrt{1-\frac{\lambda^2}{\rho^2+\lambda^2}}} = \xi_\mu \frac{\sqrt{\rho^2+\lambda^2}}{\rho} \quad \left(\frac{1}{\sqrt{\frac{\rho^2}{\rho^2+\lambda^2}}} \right.$$

$$= \xi_\mu \sqrt{1+\frac{\lambda^2}{\rho^2}}.$$

$\rho \to 0$: $\quad x_\mu x_\mu \to \lambda^2$

$$\varepsilon_{\mu\nu} \xi_\mu \xi_\nu = \varepsilon_{\mu\nu} x_\mu x_\nu \left(1-\frac{\lambda^2}{x_\mu x_\mu}\right)$$

$$d\xi_1 d\xi_2 d\xi_3 d\xi_0 = \sqrt{1-\frac{\lambda^2}{x_\mu x_\mu}}\, dx_1 dx_2 dx_3 dx_0$$

$$\delta(\xi_1)\delta(\xi_2)\delta(\xi_3)d(\xi_0)$$
$$= \text{const.} \frac{\delta(x_\mu x_\mu - \lambda^2)}{\sqrt{1-\frac{\lambda^2}{x_\mu x_\mu}}}$$

$$\varphi(\xi_1, \xi_2, \xi_3, \xi_0) = f(x_1, x_2, x_3, x_0)$$

$f(x_\mu)$ may depend

$$\varphi(\varepsilon_{\mu\nu}\xi_\mu \xi_\nu) = \varphi\left(\varepsilon_{\mu\nu} x_\mu x_\nu \left(1-\frac{\lambda^2}{x_\mu x_\mu}\right)\right)$$

$\xi_\mu \to 0$. $\quad x_\mu x_\mu \to \lambda^2$

Any function of the ~~form~~

$$f(x_1, x_2, x_3, x_4)\delta(x_\mu x_\mu - \lambda^2)$$

can be an invariant. However,

$$\int \cdots \int f(x_1, x_2, x_3, x_4)\,\delta(x_\mu x_\mu - \lambda^2)\, dx_1 dx_2 dx_3 dx_4$$

depends on the distribution of f on the sphere $x_\mu x_\mu = \lambda^2$.

Jan. 1953

Carl Levinson, S-Matrix in Local and Non-local Field Theories

$$A_H(x) = U(-\infty, t)\, A_I(x)\, U^{-1}(-\infty, t)$$

$$A_M(x) \equiv S\, A_H(x) = U(\infty, t)\, A_I(x)\, U(t, -\infty)$$

$$A_M(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} T(H_1 \cdots H_n, A_I(x))\, dx_1 \cdots dx_n$$

$$\left(\because \int_{\sigma(t)}^{\sigma(+\infty)} \overbrace{H_1(x_1)\,\theta(x_1, x_2)\, H_2(x_2) \cdots \theta(x_{j-1}, x_j)}^{H(x_j)\; dx_1 \cdots dx_j} A(x)\, \cancel{\theta(x_j, x_{j+1})} \times \int_{\sigma(-\infty)}^{\sigma(t)} H_{j+1}(x_{j+1}) \cdots H_n(x_n)\, dx_{j+1} \cdots dx_n\right.$$

where j can be $1, 2, \dots n$, ~~depending according as~~ if we take together various term in the product of two φ's which are proportional to g^n.) The net result is

$$\sum_j \int_{-\infty}^{+\infty} H_1^{(int)}(x_1)\,\theta(x_1, x_2)\, H_2^{(int)}(x_2) \cdots \theta(x_j, x)\, A(x)\, \theta(x, x_{j+1})\, H_{j+1}(x_{j+1}) \cdots dx_1 \cdots dx_n \Big)$$

$$= \frac{1}{n!} P \sum_j \int_{-\infty}^{+\infty} H_1^{(int)}(x_1)\,\theta(x_1, x_2)\, H_2^{(int)}(x_2) \cdots$$

① Units $\hbar = c = 1$

S Matrix in Local Theory – Carl Levinson

First Proof – Valid only for Scaler Coupling

For any field operator (time derivatives of field operators excluded) we have:

$$\boxed{A_H(x) = U(-\infty, t)\, A_I(x)\, U^{-1}(-\infty, t)}$$

$A_H(x)$ = Heisenberg Representation

$A_I(x)$ = Interaction Representation

$U(a,b)$ = Dyson's "U" Matrix

Also:

(1) $A_M(x) \equiv S\, A_H(x) = U(\infty, t)\, A_I(x)\, U(t, -\infty)$ { Dyson's "Mixed" Representation

(2) $A_M(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} T(H_1, \cdots H_n, A_I(x))\, dx_1 \cdots dx_n$ { Proven in Dyson's First Paper

$H_1 = H_I^{int}(x_1)$ the interaction representation Hamiltonian.

Define $H_M(x) \equiv \sum_{n=1}^{\infty} H_M^{(n)}(x)$ where $H_M^{(n)}(x)$ is the term that goes as g^n in an expansion of $H_M(x)$ in a power series in g.

$$\therefore\ H_M^{(n)}(x) = \frac{(-i)^{n-1}}{(n-1)!} \int T(H_1 \cdots H_{n-1}, H(x))\, dx_1 \cdots dx_{n-1} \qquad n = 1, 2, \cdots$$

?

(2)

Now Dysons Expansion for the S Matrix is:

$$(3)\quad S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int T(H_1, \cdots H_n)\, dx_1 \cdots dx_n =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} \left(\frac{-i}{n}\right) \int T(H_1 \cdots H_{n-1}, H_n)\, dx_1 \cdots dx_{n-1}\, dx_n$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{-i}{n}\right) \int H_M^{(n)}(x)\, dx$$

but $H_M(x) = S\, H_H(x)$ $\quad\therefore\quad H_M^{(n)}(x) = \left(S\, H_H(x)\right)^{(n)}$

whe H_H = interaction part of the Hamiltonian in the Heisenberg representation.

$$\therefore\quad S = 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{n}\right) \int \left(S\, H_H(x)\right)^{(n)} d^4x$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{n}\right)\left(S \int H_H(x)\, dx\right)^{(n)}$$

set $\int H_H(x)\, dx = -\int \mathcal{L}_H^{int}(x)\, dx = -I$

$\mathcal{L}_H^{int}(x)$ = interaction part of Lagrangian in the Heisenberg representation:

$$\therefore\quad S = 1 + \sum_{n=1}^{\infty}\left(\frac{+i}{n}\right)\left(S I\right)^{(n)}$$

or $\quad S^{(n)} = \frac{i}{n}\left(S I\right)^{(n)} \qquad n = 1, 2, \cdots$

(4) or $\quad n\, S^{(n)} = i\left(S I\right)^{(n)} \qquad n = 0, 1, 2 \cdots$

Now $\left(g\frac{\partial S}{\partial g}\right)^{(n)} = N S^{(n)}$

$$\therefore \left(g\frac{\partial S}{\partial g}\right)^{(n)} = i\,(S I)^{(n)}$$

since the expressions are equal term by term they are equal in general.

$$\therefore \boxed{g\frac{\partial S}{\partial g} = i S I}$$

This follows also from Schwinger's

$$\boxed{\delta S = i S\, \delta W}$$

P.R. 82, 916 Equation 2.7

where $U^{-1}_{\infty,-\infty} = S$

where $\delta W = \frac{\delta g}{g}\int \mathcal{L}^{int}_{H}(x)\,dx$ is the change in the Lagrangian due to replacing g by $g+\delta g$.

Lemma $[I, S]_{-} = 0$

We now ~~read~~ repeat the above proof except that we transform everything by $S^{-1} \cdots S$.

$(1)'$ $A'_H(x) = A_H(x)\,S$ where $S^{-1} A_{in} S = A_{out}$

$(2)'$ $A'_H(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} T(H_1^{out}\cdots H_n^{out}, A^{out}(x))\,dx_1\cdots dx_n$

$(3)'$ $S^{-1} S S = S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int T(H_1^{out}\cdots H_n^{out})\,dx_1\cdots dx_n$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{n}\right)\int H_H^{(n)}(x)\,dx$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{n}\right)\left(\int H_H(x)\,dx\; S\right)^{(n)}$$

(4)' $\therefore \quad n\, S^{(n)} = i(I\,S)^{(n)}$

comparing (4) and (4)' we see that $(SI)^{(n)} = (I\,S)^{(n)}$ or $[I, S]_- = 0$ Q.E.D.

Second Type of Proof Applicable to Derivative Coupling theorys.

We shall prove the above theorem for the case of Pseudo-scaler mesons with Pseudo-vector coupling. First we must verify Schwinger's variation formula.

$$\delta \Psi_H(x, g) = -i\frac{\delta g}{g}\left[\int_{-\infty}^{t} \mathcal{L}_H^{int}(x)\, d^4x,\ \Psi_H(x, g)\right]_-$$

ie $\delta\Psi \equiv$ the change in $\Psi_H(x)$ due to a change in the equations of motion when g is increased to $g+\delta g$. (As a boundary condition we have taken $\delta\Psi = 0$ for $t = -\infty$.)

$$\mathcal{L}_H^{int}(x) = i g\, \bar{\Psi}_H(x)\, \gamma_5 \gamma_\mu\, \Psi_H(x) \frac{\partial}{\partial x_\mu} \phi_H(x)$$

$$\left.\begin{aligned}
(\gamma_\mu \partial_\mu + m)\, \Psi_H(x) &= i g\, \gamma_5 \gamma_\mu\, \Psi_H(x) \frac{\partial}{\partial x_\mu} \phi_H(x) \\
(\Box - \mu^2)\, \phi_H(x) &= i g \frac{\partial}{\partial x_\mu}\left(\bar{\Psi}_H(x)\, \gamma_5 \gamma_\mu\, \Psi_H(x)\right) \\
(\gamma_\mu^T \partial_\mu - m)\, \bar{\Psi}_H(x) &= -i g\, \bar{\Psi}_H(x)\, \gamma_5 \gamma_\mu \frac{\partial}{\partial x_\mu} \phi(x)
\end{aligned}\right\}\ \text{Equations of Motion}$$

(5)

Canonical Commutation Relations-

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi + g\bar\psi\gamma_5\gamma_4\psi$$

$$[\phi(\underset{\sim}{x},t), \pi_\phi(\underset{\sim}{x}',t)]_- = i\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$[\phi(\underset{\sim}{x},t), \phi(\underset{\sim}{x}',t)]_- = 0$$

$$[\psi^*_\alpha(\underset{\sim}{x}\,t), \psi_\beta(\underset{\sim}{x}',t)]_- = \delta_{\alpha\beta}\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$[\psi_\alpha(\underset{\sim}{x},t), \pi_\phi(\underset{\sim}{x}',t)]_- = 0$$

$$[\psi_\alpha(\underset{\sim}{x},t), \phi(\underset{\sim}{x}',t)]_- = 0$$

Hamiltonian: $H^{int}(x) = -\mathcal{L}^{int}(x) + \frac{1}{2}g^2(\bar\psi\gamma_5\gamma_4\psi)^2$

We must verify that if:

$\psi' = \psi + \delta\psi \qquad \bar\psi' = \bar\psi + \delta\bar\psi \qquad \phi' = \phi + \delta\phi$

then $\psi', \bar\psi', \phi'$ satisfy the Heisenberg equations of motion with g replaced by $g+\delta g$ (up to first order in δg of course) This is a straight-forward calculation and will not be given here.

Therefore, given that $\delta\psi_H(x,g) = -\frac{i\delta g}{g}\left[\int_{-\infty}^{t}\mathcal{L}^{int}(x)dx, \psi_H(x,g)\right]_-$

~~we find~~ set $\int_a^b \mathcal{L}^{int}(x)\,dx = I_b^a$

then:
$$\delta\psi_H(\underset{\sim}{x},\infty;g) = -\frac{i\delta g}{g}\left[I_{-\infty}^{\infty}, \psi_H(\underset{\sim}{x},\infty;g)\right]_-$$

or
$$\delta\psi^{out}(\underset{\sim}{x},\infty;g) = -\frac{i\delta g}{g}\left[I_{-\infty}^{\infty}, \psi^{out}(\underset{\sim}{x},\infty;g)\right]_-$$

(6)

The last equation implies:

$$\delta\varphi^{out}(x,g) = -i\frac{\delta g}{g}\left[I_{-\infty}^{\infty},\ \varphi^{out}(x,g)\right]_-$$

Proof. Both sides are equal for $t=\infty$ & both sides satisfy $(\gamma_\nu\partial_\nu+m)F(x)=0$ Q.E.D.

Now $S^{-1}\varphi_{in}(x)S = \varphi^{out}(x,g)$

$$\therefore\ \frac{\partial}{\partial g}\left(S\varphi^{out}(x,g)S^{-1}\right) = 0$$

or $$\frac{\partial}{\partial g}\varphi^{out}(x,g) = \left[\varphi^{out}(x,g),\ S^{-1}\frac{\partial S}{\partial g}\right]_- \qquad (A)$$

but $$\frac{\partial}{\partial g}\varphi^{out}(x,g) = \left[\varphi^{out}(x,g),\ \frac{i}{g}I\right]_- \qquad \text{proven above. } (B)$$

We now prove that (A) and (B) imply $S^{-1}\frac{\partial S}{\partial g} = \frac{i}{g}I$

Proof: define $Q^{-1}\frac{\partial Q}{\partial g} \equiv \frac{i}{g}I \qquad Q=1$ for $g=0$

then $Q^{-1}\varphi_{in}(x)\,Q \equiv \varphi_Q(x,g)$

or $$\frac{\partial\varphi_{in}}{\partial g} = 0 = \frac{\partial}{\partial g}\left(Q\,\varphi_Q(x,g)\,Q^{-1}\right)$$

or $$\frac{\partial\varphi_Q(x,g)}{\partial g} = \left[\varphi_Q(x,g),\ Q^{-1}\frac{\partial Q}{\partial g}\right]_- = \left[\varphi_Q(x,g),\ \frac{i}{g}I\right]$$

hence $\varphi_Q(x,g) = \varphi_{out}(x,g)$ for $g=0$

& both satisfy the same linear differential equation

$\therefore\ \varphi_Q = \varphi_{out}$ & hence $Q = S$ since $Q^{-1}\varphi_{in}Q = \varphi_{out}$ defines S.

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we must now prove that $\{S, I\}_- = 0$

Proof:

We have verified (elsewhere) that:

$\delta \Psi_H(x,g) = [\Psi_H(x,g), \frac{i\delta g}{g} I_a^\pm]_-$ where a is any given time. We now choose $a = +\infty$ and set $t = -\infty$ then we get:

$$\delta \Psi_H(x, -\infty, g) = -[\underset{t=-\infty}{\Psi_H(x,g)}, \frac{i\delta g}{g} I]_- \qquad (A')$$

with the boundary condition $\delta\Psi_H = 0$ for $t = \infty$

under these ~~t~~ kinds of boundary conditions we are holding $\Psi_{out}(x)$ fixed & varying $\Psi_{in}(x)$ therefore we write $S^{-1}\Psi_{in}(x,g)S = \Psi_{out}(x)$

or $\frac{\partial}{\partial g}\left(S^{-1}\Psi_{in}(x,g)S\right) = 0$

or $\frac{\partial}{\partial g}\Psi_{in}(x,g) = [\frac{\partial S}{\partial g}S^{-1}, \Psi_{in}(x,g)]_-$

also $\frac{\partial}{\partial g}\Psi_{in}(x,g) = [\frac{i}{g} I, \Psi_{in}(x,g)]_-$ from (A')

~~Again we identify~~

Again we can identify

$$\frac{i}{g} I = \frac{\partial S}{\partial g} S^{-1}$$

or $\frac{i}{g} I S = \frac{\partial S}{\partial g}$.

Q.E.D.

Relativistic Two-Body Problem

M. Gell-Mann and F. Low, P. R. 84 (1951), 350.

$$\frac{\partial F}{\partial x_\mu} = i[F, P_\mu]$$

$$P_\mu \Psi_n = p_\mu^n \Psi_n$$

$$P_0 = H = H_0(t) + H_I(t)$$

$t=0$: interaction repr. $O(t)$ and Heisenberg representation $O(t)$ coincide, where

$$O(t) = \exp(iH_0(0)t)\,O(0)\,\exp(-iH_0(0)t)$$
$$O(t) = \exp(iHt)\,O(0)\,\exp(-iHt)$$

or

$$O(t) = U(t,0)\,O(0)\,U^{-1}(t,0)$$
$$U(t,0) = \exp(iH_0(0)t)\exp(-iHt)$$
$$i\frac{dU(t,0)}{dt} = H_I(t)U(t,0)$$

with $U(0,0)=1$

True vacuum state: Ψ_0, (eigenstate of H with the lowest energy $p_0^0 = E_0$)
vacuum of free particles: Φ_0, (eigenstate of $H_0 = H_0(0)$ with the lowest value ε_0)

First Method;

1. Assume that at $t=-\infty$, coupling vanishes and
$$H(-\infty) = H_0(-\infty)$$
so that $\Psi_0 = \Phi_0$

2. Second method; Interact. repr. is defined at $t=0$ and
$$c\Psi_0 = [1 + (H_0 - \varepsilon_0)^{-1}(1-V)(H_I(0) - E_0 + \varepsilon_0)]^{-1}\Phi_0$$

where V is the projection operator on Φ_0.

(∵ $(H_0+H_I)\Psi_0 = E_0\Psi_0$

$\Psi_0 = a\Phi_0 + \Phi'$ $\qquad V\Psi_0 = a\Phi_0$

$(1-V)\Psi_0 = \Phi'$

$H_0\Phi_0 = \varepsilon_0\Phi_0$

$a(\varepsilon_0 + H_I - E_0)\Phi_0 + (H_0+H_I-E_0)\underbrace{\cancel{(\Psi_0 - a\Phi_0)}}_{(1-V)\Psi_0}$
$= 0$

or

$\cancel{\Phi_0 = a\Psi_0 +}$

$$c\Psi_0 = U^{-1}(\pm\infty,0)\Phi_0/(\Phi_0, U^{-1}(\pm\infty,0)\Phi_0)$$

Define Feynman's two-body kernel:

$$K(x_1 x_2; x_3 x_4) = \varepsilon(\Psi_0, P[\psi(x_1)\phi(x_2)\bar\psi(x_3)\bar\phi(x_4)]\Psi_0)$$

$\varepsilon = -1$ $\quad$ perm. is ~~odd~~ even

$+1$ $\quad$ odd

$t_1 \geq t_2 \geq t_3 \geq t_4$:

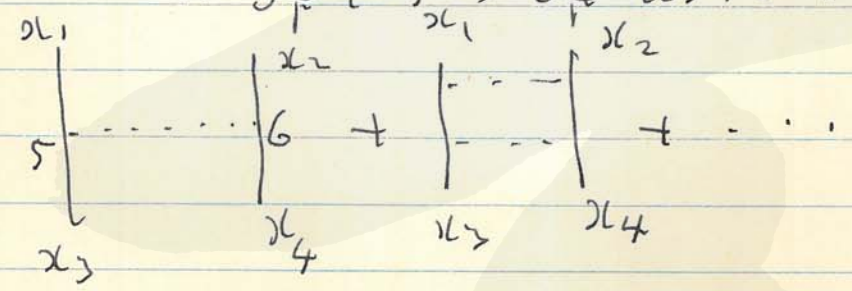
$K(x_1 x_2; x_3 x_4) = -(\Phi_0, U(+\infty,0)U^{-1}(t_1,0)$
$\psi(x_1)U(t_1,0)U^{-1}(t_2,0)\phi(x_2)\cdots$
$\times\bar\phi(x_4)U(t_4,0)U^{-1}(-\infty,0)\Phi_0)/$
$(\Phi_0, U(\infty,0)U^{-1}(-\infty,0)\Phi_0)$

$$U(t,0)\,U^{-1}(t',0) = U(t,t')$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} d\tau_1 \cdots d\tau_n \, P[H_I(\tau_1) \cdots H_I(\tau_n)]$$

$$K = \varepsilon \frac{(\Phi_0, P[U(\infty,-\infty)\psi(x_1)\phi(x_2)\bar{\psi}(x_3)\bar{\phi}(x_4)]\Phi_0)}{(\Phi_0, U(\infty,-\infty)\Phi_0)}$$

where
$$P[U(\infty,-\infty)\psi\phi\bar{\psi}\bar{\phi}]$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d\tau_1 \cdots d\tau_n \, P[H_I(\tau_1) \cdots H_I(\tau_n) \times \psi\phi\bar{\psi}\bar{\phi}]$$

$$K(1,2,3,4) = S_F^P(1,3)\,S_F^N(2,4)$$
$$- g^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} d\omega_5\, d\omega_6\, S_F^P(1,5)\,S_F^N(2,6)\,\Delta_F(5,6)$$
$$\times S_F^P(5,3)\,S_F^N(6,4) + \cdots$$



$$S_F^P(x-y) = \varepsilon(x-y)(\Phi_0, P[\psi(x)\bar{\psi}(y)]\Phi_0)$$
$$S_F^N(x-y) = \varepsilon(x-y)(\Phi_0, P[\phi(x)\bar{\phi}(y)]\Phi_0)$$
$$\Delta_F(x-y) = (\Phi_0, P[A(x)A(y)]\Phi_0)$$

$$K(1,2;3,4) \cong S_F^P(1,3)S_F^N(2,4)$$

$$-g^2\int_{-\infty}^{+\infty} dw_5\, dw_6\, S_F^P(1,5)\, S_F^N(2,6)\Delta_F(5,6)\, \times K(5,6;3,4)$$

(Bethe-Salpeter)

$$K(1,2;34) = \sum_n (\Psi_0, \varepsilon(1,2)P[\psi(x_1)\phi(x_2)]\Psi_n) \times (\Psi_n P[\bar{\psi}(x_3)\bar{\phi}(x_4)]\varepsilon(4,3)\Psi_0)$$

$$= \sum_n \chi_n(1,2)\tilde{\chi}_n(3,4)$$

$$\chi_n = V^{-\frac{1}{2}}\exp[i(p_\mu^n - p_\mu^0)X_\mu]f_n(x)]$$

$$\chi_n(1,2) = \cancel{-g^2\int dw_5 dw_6 S}$$

$$-\int dw_5\, dw_6\, dw_7\, dw_8\, S_F'^P(1,5)\, S_F'^N(2,6)\, G(5,6;7,8)\chi_n(7,8)$$

C. Gregory, Phys. Rev. 89 (1953)
1199 (March 15)

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Fusion theory of Bosons and Non-local Field
by Satosi Watanabe
Letter to the editor

GRUNBERG → ~~Fred~~ M. Johnson

$$(D(x') + mE_5)\Psi(x', x'')(D(x'') - mE_5)$$
$$= g^2 W(x'-x'') O'^{\alpha} \Psi(x', x'') O''^{\alpha}$$
$$D(x) = -i \frac{\partial}{\partial x_\mu} \cdot E_\mu$$
$$O'' = J \bar{O}' J^{-1}$$

$$\left.\begin{array}{ll} \bar{\Psi} = -\psi^* \gamma_5 & E_a = i\gamma_5 \gamma_a \\ E_5 = \gamma_5 & E_0 = -J = \gamma_4 \gamma_5 \end{array}\right\}$$

$$\Psi_{rs} \to \psi_r(x') \bar{\psi}_s(x'')$$

$$(D(x') + D(x'') + \kappa E_5)\Psi(x', x'') = 0$$

($W - \Delta_F$ in ordinary theory P.T.P. 5 (1950), 614)

$$\Psi = (iU + U^\mu E_\mu E_5) + (VE_5 + V^\mu E_\mu + \tfrac{1}{2} i V^{\mu\nu} E_\mu E_\nu E_5)$$
$$O' = -O'' = E_5$$
$$W = -\gamma_\mu \gamma^\mu \qquad g^2 = 4/\lambda^4$$

$$\left\{\begin{array}{l} [p_\mu [p^\mu U]] + \kappa^2 U = 0 \\ \{p_\mu \{p^\mu U\}\}^2 + (\kappa - 2m)^2 U \end{array}\right.$$

$$+\frac{16}{\lambda^4}[x_\mu\{x^\mu U\}]=0$$

$$\{p_\mu\{p^\nu U\}\}-\{p^\mu[p_\nu U]\}=0$$

$$U^\mu=\kappa\frac{\partial U}{\partial x_\mu}$$

$\frac{\partial U}{\partial x_\mu}$: time-like

$$U\propto\exp(r_\mu r^\mu/\chi^2)$$

$$r_\mu\propto k_\mu$$

$$\kappa=2m\pm 4\sqrt{2}/\lambda.$$