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Spring Session

Feb. 14, 53

Part I.

Application of Group Theory to Quantum Mechanics

Preliminary ~~was~~ considerations:

1. Why is group theory useful for quantum mechanics?

Let us start from ~~that~~ simple practical problems. We know that most of the Schrödinger equations for physically interesting system are too complicated to be solved rigorously. What we could do was to try to find approximate solutions by making use of perturbation theory or of variational methods. However, we also know that some of the characteristics ~~and~~ properties of quantum mechanics could be learned without our knowledge on the ~~det~~ exact form of Schrödinger equation.

For instance, let us take a system of ~~parts~~ n-electrons in ~~non~~-relativistic ~~particles~~ limit in the central field. ~~charged~~ Let us first ignore the spins of the particles and the interaction between them other than the Coulomb repulsion. Then the Hamiltonian for the system reduces to

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$$H = \frac{1}{2m}\sum_{j=1}^{n} p_j^2 + \sum_j V_j(r_j) + \sum_{j<k} V_{jk}(r_{jk})$$

Energy levels of this system or the eigenvalues of the Hamiltonian consist of continuous and discrete eigenvalues ~~and~~ in general. Let us ~~restrict~~ confine our attention to discrete energy levels and take any one of the discrete eigenvalues E_s, which is, in general, degenerate with n_s - eigenfunctions

~~$\Psi_1^{(s)}, \Psi_2^{(s)}$~~

$\Psi_1^{(s)}(r_1, r_2, \ldots r_n)$ . . .

$\Psi_{\nu_s}^{(s)}(r_1, r_2, \ldots r_n)$.

Any one of them satisfies S. eq.

$$H\Psi_\nu^{(s)} = E_s \Psi_\nu^{(s)} \qquad \nu = 1, 2, \ldots \nu_s$$

Except for a very ~~simple~~ particular cases for one particle system ($n=1$), ~~but it is~~ we cannot find the ~~rigorous~~ solutions for the eigenfunctions. However, we know that a number of important properties of stationary states of such a general n-electron system could be determined without knowing the explicit form of the eigenfunctions.

One of the most important characteristics of the stationary states is the

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eigenvalue of the angular momentum operator. Namely, we know that the total angular momentum operator

$$\mathbf{M} = \sum_{j=1}^{n} \mathbf{m}_j$$

with $\mathbf{m}_j = [\mathbf{r}_j \times \mathbf{p}_j]$

$$m_{jx} = y_j p_{zj} - z_j p_{yj}$$

is commutative with the Hamiltonian, because $\mathbf{m}_j$ is commutative with $\mathbf{p}_j^2$ and $\mathbf{r}_j^2$ or $\mathbf{r}_j$ and

$\mathbf{m}_j + \mathbf{m}_l$

is commutative with $\mathbf{r}_{jl}$.

(∵ $[x_j, m_{jx}] = 0$

$[y_j, m_{jx}] = -i\hbar z_j$

$[z_j, m_{jx}] = i\hbar y_j$

Hence

$$[(x_j - x_l)^2 + (y_j - y_l)^2 + (z_j - z_l)^2, m_{jx}]$$

$$= (y_j - y_l)[(y_j - y_l), m_{jx}] + [(y_j - y_l), m_{jx}](y_j - y_l)$$
$$+ (z_j - z_l)[(z_j - z_l), m_{jx}] + [(z_j - z_l), m_{jx}](z_j - z_l)$$
$$= 2(y_j - y_l)(-i\hbar z_j) + 2(z_j - z_l)(i\hbar y_j)$$

similarly,

$$[(\ )^2 + (\ )^2 + (\ )^2, m_{lx}] =$$
$$= 2(y_j - y_l)(+i\hbar z_l) + 2(z_j - z_l)(-i\hbar y_l))$$

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Thus, ~~the~~ stationary states of a system with the Hamiltonian which depends only on such quantities as p_j^2, r_j, r_{jk} could be at the same time the eigenstates of total a.m.o. ~~The~~ In other words, one can classify the ~~stationary~~ energy levels into S, P, D etc. according to the eigenvalues of the total ~~eigen~~ a.m. operator

$$m^2 = \hbar^2 l(l+1)$$
$$l = 0, 1, 2, \ldots$$

~~This theor is true if and only if the Hamiltonian is~~ This is equivalent to say that the stationary states can be classified by the q.n. l if and only if the Hamiltonian is invariant with respect to the rotation of c.s., or it does not change by any ~~trans~~ contravariant transformation

$$\left.\begin{aligned} x_j' &= a_{xx} x_j + a_{xy} y_j + a_{xz} z_j \\ y_j' &= a_{yx} x_j + \\ z_j' &= a_{zx} x_j + \end{aligned}\right\} (1)$$

with the coefficients satisfying

$$\left.\begin{aligned} a_{xx}^2 + a_{yx}^2 + a_{zx}^2 &= 1 \quad \text{etc} \\ a_{xx}a_{xy} + a_{yx}a_{yy} + a_{zx}a_{zy} &= 0 \\ &\text{etc.} \end{aligned}\right\} (2)$$

$(x_j')_r = \sum a_{rs} (x_j)_s$ $\quad a_{rx}a_{sx} = \delta_{rs}$

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From these arguments, it is now clear that the angular momentum operator must have some intimate connection with the rotation of the coordinate system. Conversely, one can imagine that if we investigate the nature of a quantum mechanical system with respect to rotation, one may be able to draw some useful general conclusions on physical characteristics of the system.

Now, let us consider a very general problem. Let us take a q.m.s. with the Hamiltonian which is invariant with respect to rotation*. If, in a certain coordinate system,

$$\Psi_1^{(s)}(x_1, \dots x_n) \dots \Psi_{\nu_s}^{(s)}(x_1, \dots x_n) \quad (3)$$

are the eigenfunctions ~~for~~ a energy value belonging to E_s, obviously

$$\Psi_1^{(s)}(x_1', \dots x_n') \dots \Psi_{\nu_s}^{(s)}(x_1', \dots x_n') \quad (4)$$

which are the same functions of new coordinates $x_1' \dots x_n'$ are also the eigenfunctions belonging to the same energy E_s. On the other hand we know that any eigenfunction must be a

* One can now include the spins of particles easily, as will be shown in the following.

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linear combination of (B), so that

$$\Psi_\mu^{(S)}(x_1' \cdots x_n') = \sum_{\nu=1}^{\nu_S} C_{\mu\nu} \Psi_\nu^{(S)}(x_1 \cdots x_n) \qquad (5)$$

$$\mu = 1, 2, \ldots \nu_S$$

In particular, if E_S is not degenerate, and we have only one function

$$\Psi^{(S)}(x_1 \cdots x_n)$$

and

$$\Psi^{(S)}(x_1' \cdots x_n') = C\, \Psi^{(S)}(x_1 \cdots x_n)$$

Hence $\Psi^{(S)}$ must be invariant, except for a multiplicative factor C. Thus, it must represent an S-state with the spherical symmetry.

In other cases, if one can determine in some way or other the coefficients $C_{\mu\nu}$ for an arbitrary rotation, one can hope that the properties of the states belonging to any one of the energy levels. Here comes the usefulness of group theory for the following reasons:

First of all, the totality of rotation of coordinate system constitutes what is called in mathematics the group of rotation. It is a kind of transformation group.

(See Note 1. on page. 8)

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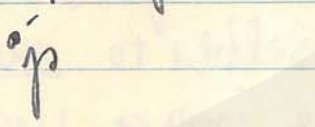
~~Definition of a group.~~

2. Transformation Group

(Weyl, Chapter III.)

Let us imagine a domain of elements which we call points. A transformation can operate on such a point-field, which is the totality of a finite number of elements or an infinite set, in particular a continuum such as space (and/or) time. A mapping or correspondence S of the point-field on itself is determined by a law which associates with each point p of the field a point p' as image:

$$p \to p' = Sp.$$



Two correspondences ~~are~~ Sp, Tp are identical if for all points p, the two image points Sp and Tp coincide. If the point-field contains only a finite number of elements, the correspondence S can be defined by giving explicitly the image point for each point p. For infinite sets, we need a law which determines the correspondence between individual points.

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In general, there is a particular one among these correspondences which associates with each point p the point p itself: $p \to p$. It is called <u>identity</u>, I. Two correspondences can be applied successively. First $p \to p' = Sp$, then $p' \to p'' = Tp'$. The correspondence resulting from the composition of the two is defined by the association $p \to p'' = T(Sp)$ and is denoted by TS. The resultant depends, in general, on the order of the two factors S and T. (In order that such a composition is possible, the correspondences are to map the point-field on itself, and not on another point field.)

We shall restrict ourselves to <u>one-to-one</u> correspondences S: the image points $p' = Sp$ shall each appear as the image of one and only one of the points p. Thus, the correspondence S determines a second, the <u>inverse</u> S^{-1}: $p' \to p$:

$$S^{-1}(Sp) = p \quad) \qquad S^{-1}S \cdot p = p \quad)$$
$$S(S^{-1}p') = p' \quad) \qquad S S^{-1} \cdot p' = p' \quad)$$
$$S^{-1}S = I \quad)$$
$$S S^{-1} = I \quad)$$

The inverse of S^{-1} is S and I is its own inverse.

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The resultant TS of two one-to-one corresps. S, T is itself one-to-one, and its inverse is $(TS)^{-1} = S^{-1}T^{-1}$, because on inverting the correspondence $p \overset{S}{\to} p' \overset{T}{\to} p''$, we have $p'' \overset{T^{-1}}{\to} p' \overset{S^{-1}}{\to} p$. Henceforth we shall consider only one-to-one correspondence and call it <u>transformation</u> or <u>substitution</u>.

Example 1. n-elements which are numbered $1, 2, \ldots n$. The numbering can be regarded as a one-to-one reciprocal relation between the elements of the point-field and the integers $1, 2, \ldots n$. (or possible position q in the series) A permutation consists in the transition from one such arrangement to another. One may either think that the positions as fixed compartments into which the movable elements can be laid (inserted), or may conversely think that the elements are fixed and shift the movable numbers about. In any case, each permutation is associated with a one-to-one correspondence $p - p'$. etc.

Example 2. Motion of a rigid body. point-field is the space itself. The one-to-one correspondence $p \to p'$ connects the initial

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and final states. This is equivalent to the congruent transformation of space into itself, in which the distance between any two points is not altered.



In particular, if we fix any one point in the rigid body, we obtain the group of rotation.

The definition of the group of transformations O_f:

1. It contains the identity
2. If S belongs to O_f, its inverse S^{-1} does also.
3. The resultant TS of any transf. S, T of O_f is also a transformation of O_f.

One important property of the transformation is that the associative law

$$(AB)C = A(BC)$$

holds for any three three transformations.

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3. Abstract Group.

An abstract group is obtained by extracting from the concept of the group of transformations those features which are related to the mutual relationship between transf. without regard to the point-field in the background. Thus, it is a system of elements within which a law of composition (multiplication) is defined * and which satisfies the following requirements:

1. The associative law
$$(ba)c = b(ac)$$
2. There exists an element $\mathbb{1}$ which satisfies (unit element)
$$\mathbb{1}a = a\mathbb{1} = a$$
for arbitrary element a,
3. To each a, a^{-1} exists (an inverse)
$$aa^{-1} = a^{-1}a = \mathbb{1}.$$

---

* For two elements a, b, ba is uniquely defined and belongs to the system.

Realization of an abstract group by transformation groups: To each element a of the abstract group a transformation $T(a)$ of the point-field is associated in such a way that to the composition (product) ba of a, b is associated with $T(ba)$ $T(ba) = T(b)T(a)$, which is obtained as the product

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It follows from this that

$$\begin{array}{ccc} | & \to & I \\ a^{-1} & \to & T(a^{-1}) = T^{-1}(a) \end{array}$$

The realization is said to be faithful if to distinct elements of the group correspond distinct transformations

$$T(a) \neq T(b) \quad \text{when } a \neq b.$$

The necessary and sufficient condition for faithfulness is that $T(a)$ shall be the identity only if a is the unit element.

Reason: For any a, b, the relation

$$T(a) = T(b)$$

implies that

$$T(a)T^{-1}(b) = T(a)T(b^{-1})$$
$$= T(ab^{-1}) = I$$

Thus ~~$a b^{-1}$ from~~ the above condition it follows that

$$a = b.$$

In the study of transformation group, we always deal with two manifolds.

1. Structureless point-field
2. The manifold of the group elements, the structure of which is expressed by the law of composition.

Thus, our problem is twofold

1. examination of various possible

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group structures.

2. Examination of possibilities of realizations of a given (abstract) group by transformation a given point-field.

~~Another~~ In connection with the second problem, ~~However, there~~ is the first important problem, which is crucial in the application of group theory to q.m., ~~That~~ is the problem of,

Representation of a group by linear transformations.

In the particular case of rotation groups, we know that ~~the~~ any rotation in the ordinary space is given by the linear transformation (1) ~~satisfy~~ with coef. ~~a~~ satisfying the relations (2).

By a representation in general of a group, we mean that to each element S of a group, G_f is associated a linear transformation $U(S)$, which is a space of ~~one~~ a certain dimension n. To the product ST is associated the transformation as given by

$$U(ST) = U(S)\,U(T)$$

If we use a definite coordinate system $U(S)$ is represented by a square matrix, so that the ~~most~~ product of two transformations is (~~given by~~ represented the matrix product of corresp.

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matrices. great
Now there can be a number of
different representations of the same
group. In fact, in the case of
the rotation group, to each rotation
a linear transformation (5') is associated
in such a way that to the product of
two rotations a linear transf.
given characterized by the product
matrix is associated. Thus, we
have a ν_s-dimensional representation
in which the ν_s eigenfunctions belonging
to the energy value E_s are the
basic vectors. Now the properties of
the ν_s-states with respect to the rotation
is characterised by the matrix $C_{\mu\nu}$,
so that ~~we~~ if ~~we know from~~ the
group ~~theory~~ theory gives us ~~the~~ all
possible representations, ~~we~~ that will
helps us a great deal. This clearly
shows how useful will be theory
of representation of transformation
group for q.m. ~~It should~~ It can be
easily verified that the transformation
like (5) is actually a representation:

$$\Psi^{(s)}_{\pi}(r_1'' \dots r_n'') = \sum_{\mu=1}^{\nu_s} b_{\mu\nu} x_{\mu} \Psi^{(s)}_{\mu}(r_1' r_2' \dots r_n')$$

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$$\Psi_\kappa^{(s)}(x_1'' \cdots x_n'') = \sum_\nu a_{\kappa\nu}\, \Psi_\nu^{(s)}(x_1 \cdots x_n)$$

$$a_{\kappa\nu} = \sum_\mu b_{\beta\kappa}\, c_{\mu\nu}$$

Note 1,

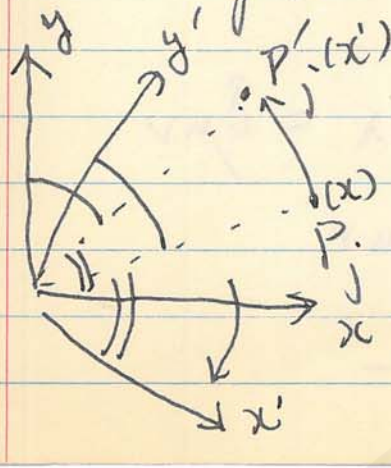
It should be noticed that what was done by a transformation (1) can be interpreted either as

(i) coordinate transformation in which the same point P_j in ~~vct~~ space changed its name from (x_j, y_j, z_j) to (x_j', y_j', z_j') (rotation of the coordinate axes)

or

(ii) actual rotation, by which a point P_j as given by the coordinates (x_j, y_j, z_j) ~~going~~ goes over to another point P_j' with the coordinates (x_j', y_j', z_j') in the same coordinate system.

Obviously, there is a very simple relation between these two interpretations. Namely, if in the latter interpretation (ii) $P_j \to P_j'$, the ~~co~~ interpretation (i) with the same $a_{\kappa\nu}$ etc. gives the rotation of the coordinate axes in the opposite direction (inverse).



Thus the eigen-function $\Psi_\nu^{(s)}(x)$ ~~relation between~~

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at a point $P(x)$ ~~goes over~~ to $\Psi(x')$ on another point $P'(x')$ in $3n$-dimensional space is given by the linear relation of type (5), where $(C_{\mu\nu})$ now corresponds to the ~~inverse~~ rotation which is ~~inverse of~~ the coord. transf. (1). characterized by

Inverse rotation:

$$x_{\mu r} = b_{\mu\nu\, rs}\, x'_{\nu s}$$

$$x'_{\mu r} = a_{\mu\nu\, rs}\, x_{\nu s}$$

Hence we have

$$b_{\mu\kappa\, rt}\, a_{\kappa\nu\, ts} = \delta_{\mu\nu\, rs}$$

and

$$a_{\mu\kappa\, rt}\, b_{\kappa\nu\, ts} = \delta_{\mu\nu\, rs}$$

$$a_{\kappa\mu\, tr}\, b_{\kappa\nu\, ts} = \delta_{\mu\nu\, rs}$$

Since the inverse is unique,

$$b_{\mu\kappa\, rt} = a_{\kappa\mu\, tr}.$$

As for $c_{\mu\nu}$, if we take Ψ's orthog. and normalized,

$$\tilde{c}_{\mu\kappa}\, c_{\mu\lambda}\, \delta_{\kappa\lambda} = \delta_{\mu\nu}$$

or

$$\tilde{c}_{\mu\kappa}\, c_{\mu\kappa} = \delta_{\mu\nu}$$

Unitary representation

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$$\Psi_\mu^{(s)}(r_1^{\#} r_2^{\#} \cdots r_n^{\#}) = \sum_\nu d_{\mu\nu}\, \Psi_\nu^{(s)}(r_1' \cdots r_n')$$

$$= \sum d_{\mu\nu} c_{\nu\kappa}\, \Psi_\kappa^{(s)}(r_1 \cdots r_n)$$

$$d_{\mu\nu} c_{\nu\kappa} = \delta_{\mu\kappa}$$

or $$\Psi_\mu^{(s)}(r_1' \cdots r_n') = \sum_\kappa c_{\mu\kappa} d_{\kappa\nu}\, \Psi_\nu^{(s)}(r_1' \cdots r_n')$$

$$c_{\mu\kappa} d_{\kappa\nu} = \delta_{\mu\nu}$$

or $$c_{\nu\kappa} d_{\kappa\mu} = \delta_{\mu\nu}$$

Thus,

$$d_{\kappa\mu} = \tilde{c}_{\mu\kappa}.$$

Product:

(i) $b_{\mu\kappa} a_{\kappa\nu} \rightarrow d_{\mu\kappa} a_{\kappa\nu}$

(ii) ($b_{\kappa\mu} a_{\nu\kappa} \rightarrow \tilde{d}_{\kappa\mu} \tilde{c}_{\nu\kappa}$

or $a_{\nu\kappa} b_{\kappa\mu} \rightarrow \tilde{c}_{\nu\kappa} \tilde{d}_{\kappa\mu}$.

$b'_{\mu\kappa} a'_{\kappa\nu} \rightarrow d'_{\mu\kappa} c'_{\kappa\nu}$)

where $b'_{\mu\kappa} = b_{\kappa\mu}$, $a'_{\kappa\nu} = a_{\nu\kappa}$

$d'_{\mu\kappa} = d_{\kappa\mu}$, $c'_{\kappa\nu} = \tilde{c}_{\nu\kappa}$

Thus, if we interchange the rows and columns and take the complex conjugate

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in the 2nd interpretation, we obtain the corresponding representation in the first interpretation. (order of multiplication must be reversed at the same time.

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4. General Properties of the groups.

1. Isomorphism: For the abstract groups, what is essential is its structure, the natures of its elements being immaterial. Thus, if we have two groups G, G' and there is a correspondence (one-to-one) between each element a of G and an element a' of G' such that

$$(ba)' = b'a',$$

$(a \rightleftarrows a' \quad b \rightleftarrows b' \quad ba \rightleftarrows (ba)')$

then the two groups are said to be (simply) isomorphic. Simply isomorphic abstract groups offer no means of distinguishing one from the other.

The concept of isomorphism can be applied to transformation groups, too. Two isomorphic transformation groups can be considered as faithful representations of one and the same abstract group, because they have the same structure.

A group may be isomorphic with itself; it is then said to be automorphic. This is the particular case of isomorphism where G and G' coincide.
(Homomorphism. See Example 2, below)

2. Every abstract group possesses (at least one) faithful realization. Namely, one can take as the point field the group manifold

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itself. Then to each element a of the group one can associate the transformation

$$s \to s' = as,$$

which is the transformation of the group manifold into itself. This "left-translation" t_a is a faithful realization of the original group, because

1. $a: \; s \to s' = as$

$b: \; s' \to s'' = bs'$

$ba: \; s \to s'' = b(as) = (ba)s$

due to the associative law of for the original groups.

2. If a and b are distinct,

$$s \to s' = as$$

and

$$s \to s' = bs$$

are different as can be seen for $s = 1$

On the other hand, the "right-translation" behaves otherwise:

$$s \to s' = sa \qquad t^*(a)$$

$$t^*(ba) = t^*(a)t^*(b)$$

($\because$ $s \to s' = sa$: a

$s' \to s'' = s'b$: b

$s \to s'' = s'b = sab$: ba)

---

a : $s \to s' = sa^{-1}$ $\quad (s = s'a)$

b : $s' \to s'' = s'b^{-1}$ $\quad (s' = s''b)$

ba : $s'' \to s' = s'b^{-1} = sa^{-1}b^{-1} = s(ba)^{-1}$ $\quad (s = s''ba)$

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~~4. Gentral Properties of the Group~~

3. Sub. group Gf' of Gf is a set of elements in Gf which itself fulfils the requirements for a group. The requirement can be reduced to one condition:

If a, b are in Gf', then ba^{-1} also in Gf'.

(∵ $a=b$, $ba^{-1}=aa^{-1}=1$ is in Gf',
$b=1$, $ba^{-1}=a^{-1}$ is in Gf',
If b, a are in Gf', a^{-1} is in Gf', so that
$b(a^{-1})^{-1}=ba$ is in Gf'.)

Examples:

In the group of motion of a rigid body in euclidian space, the group of rotations and the group of translations are subgroups.

Unitary transformations constitute a subgroup of the complete group of all homogeneous linear transformations.

$$\Psi(x_1'\cdots x_n') \equiv \Psi'_\mu(x_1\cdots x_n) = c_{\mu\nu}\Psi_\nu(x_1\cdots x_n)$$

$$\tilde{\Psi}'_\mu = \tilde{c}_{\mu\nu}\tilde{\Psi}_\nu$$

$$\int \tilde{\Psi}'_\mu \Psi'_\nu \, d\tau = \tilde{c}_{\mu\kappa} c_{\nu\lambda} \int \tilde{\Psi}(x')\Psi(x')\,dx_1'\cdots dx_n' = \tilde{c}_{\mu\kappa} c_{\nu\lambda} \int \tilde{\Psi}\Psi\, dx_1 dx_2\cdots dx_n$$

$$\tilde{c}_{\mu\kappa} c_{\nu\kappa} = \delta_{\mu\nu}$$

In general, such transformations in Gf that leaves a point p or a figure as a whole or each point on the figure unchanged constitute a subgroup.

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4. Invariant of a group.

One employs invariant functions instead of invariant figures.
If $\psi(p)$ is any function of position on the point-field, we can associate with the transformation

$$S: p \to p'$$

the function ψ' defined by

$$\psi'(p') = \psi(p),$$

and say that it is obtained from ψ by the transformation S.
If $p' = Sp$, $p'' = Tp'$, the equations

$$\psi(p) = \psi'(p') = \psi''(p'')$$

show that the composition of the transitions

$$\psi \to \psi'$$

and

$$\psi' \to \psi''$$

result in the transition

$$\psi \to \psi''$$

associated with TS.
Now, if, we consider all transformations S of G which carry $\psi(p)$ over into itself, i.e. for which

$$\psi(Sp) = \psi(p)$$

is an identity in p, they constitute a sub-group G' of G and $\psi(p)$ is an invariant of G'.
For instance, we can separate the rotations from the homogenous

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linear transformations by requiring the invariance of the unit quadratic form:

$p: \quad x^2+y^2+z^2$

$p'=Sp: \quad x'^2+y'^2+z'^2$

Rotation $S: \quad x'^2+y'^2+z'^2=x^2+y^2+z^2$.

5. Cyclic-subgroup; (a)

a: arbitrary element of G_f.

$\ldots a^{-2}, a^{-1}, a^0=1, a, a^2, \ldots$

(definition: $a^0=1, \quad a^{u+1}=a^u\cdot a$)

(a) is the smallest subgroup that contains a, i.e. its elements are common to all subgroups which contain a. The elements of (a) may be ~~all~~ either all distinct, or (it must be for a finite group G_f) $a^h=1$, for a certain h, so that only the elements

$1, a, a^2, \cdots a^{h-1}$

are distinct. h is all the order of the element a,

the order of a finite group is the number of its elements. Of course, h is the order of the subgroup (a).

If $ab=ba$ is satisfied for any two elements of a group, the group is call commutative or Abelian.

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Cyclic groups are Abelian.

6. Equivalents of a runs through the subgroup $\mathfrak{H}'$ of $\mathfrak{G}$, the associated transformation

$$t_a : \quad s \to as$$

where s runs through the group $\mathfrak{g}$, ~~then~~ is isomorphic with $\mathfrak{H}'$, ~~then~~ elements s, s' of $\mathfrak{G}$ are said to be equivalent to each other with respect to $\mathfrak{H}'$, if the relation

$$s' = as$$

holds, where a is an element in $\mathfrak{H}'$,

$$s' \equiv s \quad \text{with respect to } \mathfrak{H}'.$$

In this way the elements of $\mathfrak{G}$ are divided into sets of elements which are equivalent with respect to $\mathfrak{H}'$.

For instance, the elements of $\mathfrak{H}'$ itself is equivalent to 1, because

$$a = a \cdot 1$$

for any a in $\mathfrak{H}'$.

If the number of such sets is finite, it is called index of $\mathfrak{H}'$ in $\mathfrak{G}$.

If, ~~further~~, $\mathfrak{G}$ is finite, the number of elements in each of these sets is equal to the order of $\mathfrak{H}'$, because different translations t_a send s into different elements: $as \neq bs$ if $a \neq b$.

(Number can not be larger than the order of $\mathfrak{H}'$, because, if s' is equival

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(which is equ to s) to s, it must be equivalent to a fixed element s_0 in Gf. Namely

$$s' = as$$

$$s_0 = bs \qquad s = b^{-1}s_0$$

$$s' = (ab^{-1})s_0.$$

Thus all elements equivalent to each other is obtained from a fixed s_0 by multiplying all elements of Gf.)

Accordingly, the order of Gf' is a divisor of the order of Gf, and the quotient is the index of Gf' in Gf.

7. Conjugation

There is another way (not necessary faithful) of realization of the abstract group Gf:

$$a: \quad s \to s' = asa^{-1}$$

s: manifold of group Gf.

This correspondence is called conjugation K_a, which is reciprocal one-to one correspondence and has the inverse

$$s = a^{-1}s'a.$$

The law of composition is obeyed:

$$a: \quad s \to s' = asa^{-1}$$

$$b: \quad s' \to s'' = bs'b^{-1}$$

$$ba: \quad s'' \to s = basa^{-1}b^{-1} = bas(ba)^{-1}.$$

Two elements s, s' of Gf are said to be conjugate, if they are equivalent with respect to the group of all conjugations.

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(By conjugations, in general, a certain part of the elements of G_f changes into each other, they do not go outside this part by conjugation.)
Accordingly the whole group is divided into classes. The elements of one class is equivalent to each other with respect to conjugations.
For instance, the unit element 1 is a class by itself:

$$1 = a\,1\,a^{-1}.$$

The realization by conjugations is not faithful, in general, but is a contracted. (homomorphic, not isomorphic)
In particular, if a commutes with all elements s of G_f, conjugations by a coincides with the identity;

$$a:\ s \to asa^{-1} = s$$

a; element of subconstruct group $\to$ 1 in conjugations

Totality of such elements a is called central of the group. A central is a subgroup, because, if a, b commute with all elements of G_f, ab must do so. Also they commute with each other. Thus it is an Abelian subgroup of G_f.

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van der Waerden, p. 31

~~homomorphic~~ (not simply)

Conjugation is an isomorphic correspondence within the group which leaves the unit element invariant.
It associate with each subgroup $\mathfrak{g}'$ of $\mathfrak{g}$ another subgroup

$$a\mathfrak{g}'a^{-1},$$

which is called conjugate subgroup.

$$\mathfrak{g}' \to a\mathfrak{g}'a^{-1}$$

($a\mathfrak{g}'a^{-1}$ is a subgroup, because, if b, c are in $\mathfrak{g}'$, then
$aba^{-1} \cdot aca^{-1} = abca^{-1}$
~~are also~~ in $a\mathfrak{g}'a^{-1}$ as well as aba^{-1}, aca^{-1}. ~~How~~ Unit element is there, because $a1a^{-1} = 1$ and 1 was in $\mathfrak{g}'$.
~~If b is~~ With aba^{-1}, $(aba^{-1})^{-1}$ $= ab^{-1}a^{-1}$ belongs to $a\mathfrak{g}'a^{-1}$, because, ~~b, b^{-1}~~ if b was in $\mathfrak{g}'$; b^{-1} was to be in $\mathfrak{g}'$ too.)
$\mathfrak{g}'$ is called a self-conjugate subgroup or invariant subgroup, if it coincides with all its conjugate subgroups.

Theorem:
If $\mathfrak{g}'$ is an invariant subgroups and
if $s' \equiv s$ (with respect to $\mathfrak{g}'$)
$t' \equiv t$ (" ")
then $s't' \equiv st$ (")

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Proof:

$$s' = a\,s \qquad t' = b\,t \qquad (a, b \text{ in } g')$$
$$s't' = a\,s\,b\,t = a(sbs^{-1})st$$
$$= ac \cdot st$$

where $c = sbs^{-1}$ in g'.

The elements of G which are equivalent to each other with respect to G' can be regarded as not differing from each other. By regarding so, the group properties of G is still retained, provided that G' is self-conjugate. The group thus obtained by from G by identifying all elements which are equivalent with respect to G' is called the factor group. $g = G/G'$. Its order is the index of the invariant sub-group.

Now let the ~~transformation~~ $T(a)$ ~~of~~ a given field ~~on itself~~ correspond to the element a ~~by composition~~ of the abstract group G;

$$a \to T(a) \qquad a' \to T(a')$$

then $T(a) = T(a')$ if and only if ~~the relation~~ $e = a'a^{-1}$

($a' = ea$) ~~e is an element such that~~ ~~is satisfied for where~~ $T(e)$ ~~is identity~~, of G

because

$$T(a') = T(e)\,T(a) = T(a)$$

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if satisfyed, if and only if such e with $T(e) = I$ exists.

Such elements e which satisfy

$$T(e) = I$$

constitute a subgroup $\mathcal{G}'$ of $\mathcal{G}$, because if

$$T(e) = I, \quad T(e') = I$$

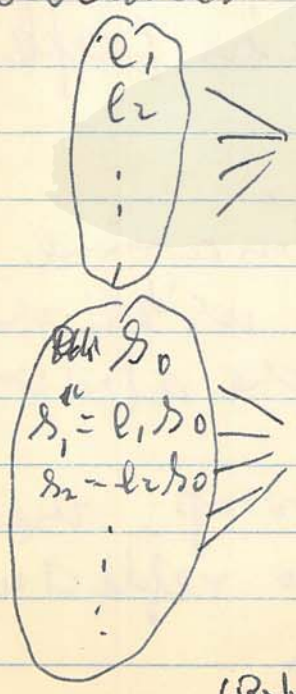
$$T(ee') = T(e)T(e') = I$$

It is an invariant subgroup because

$$T(aea^{-1}) = T(a)T(e)T^{-1}(a) = T(a)T^{-1}(a) = I,$$

(homomorphic)

Thus, in any realization of an abstract group $\mathcal{G}$ by a group of transformations, the elements of $\mathcal{G}$ which satisfy $T(e) = I$, constitute an invariant subgroup of $\mathcal{G}$. ~~Conversely~~, Two different elements will be associated with the same transformation,



if and only if they are equivalent with respect to $\mathcal{G}'$, The group of transformations is consequently a faithful realization of the factor group $\mathcal{G}/\mathcal{G}'$.

~~Example. A group which is the abstraction of rotation group.~~ plus reflection
~~Realization by reflection.~~ rotation.

~~(Reflection is a self-conjugate and invariant subgroup)~~

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5. General considerations ~~on Representation~~

Examples: ~~of~~ †

1. Whole group of linear transformations which leave the distance

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$$

between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) invariant, includes combinations

(i) translation

(ii) rotation (proper)

(iii) reflection

If we restrict ourselves to homog linear transformations, translations are excluded. Proper rotation is characterized by the ~~transformation~~ transf. with

$$\|a_{rs}\| = 1,$$

while improper rotation (reflection) is by

$$\|a_{rs}\| = -1.$$

Now one can ~~can~~ imagine an abstract group with the same structure as the above transf. group.

In this abstract group, the elements, corresponding to reflection with (E)

$$a_{rs} = -\delta_{rs}$$

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commutes with all elements of the whole group G, so that the subgroup G' consisting of 1 and e is invariant.

Thus, a realization of the group G in which

$$T(1) = I \quad T(e) = I \quad \Big\} \quad T(s) = T(es) \text{ etc.}$$

($T(aea^{-1}) = I$ for any a in G)

is homomorphic to G and is a faithful realization of G/G'. The latter being isomorphic to rotation group.

Example 2. $G_{n!}$, (the symmetric group)

The group of all permutation of n objects of order $n!$, has an invariant subgroups a homomorphic realization: To each even permutation there is a group G' which is holomorphic to G: To each even permutation in G corresponds $+1$ in G', to each odd permutation in G corresponds -1 in G'.

† Problem 1. What is the factor-group G/G' in this case. order 2; homog.

Show that, in the whole G of linear transformations of three variables (x_1, x_2, x_3) which leave $x_1^2 + x_2^2 + x_3^2$, the proper rotations with $\|a_{rs}\| = 1$ constitute a subgroups. Also show that identity and reflection $(-1, -1, -1)$ constitute an invariant subgroup of

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with non-vanishing determinant

5. General Considerations on Representation

Representation of a group by a homogeneous linear* transformations in n-dimensional vector space $\mathfrak{R} = \mathfrak{R}_n$.

n-Dimensional representation or a (*linear → affine) representation of degree n.

$a \quad \rightarrow \quad U(a) = A$

$b \quad \rightarrow \quad U(b) = B$

$ab \quad \rightarrow \quad U(ab) = U(b)U(a) = BA$

$U(s)$: square matrix of n-rows and n-columns.

Weyl: Chapt 1. (See Remark (*) below)

A linear transformation or mapping or correspondence of the space $\mathfrak{R}_n$ on itself can defined in the following way: Take a definite coordinate system and introduce basic unit vectors with the components

$e_1 = (1, 0, \ldots 0)$

$e_2 = (0, 1, \ldots 0)$

$\vdots$

$e_n = (0, \ldots 1)$

Then, arbitrary point in $\mathfrak{R}_n$ is given by the vector

$x = x_1 e_1 + \ldots + x_n e_n$

A linear correspondence A is expressed by

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A: $x \rightleftarrows x'$ for each x

with

$$x' = x_1' e_1 + \cdots + x_n' e_n$$

$$x_j' = \sum_{k=1}^{n} A_{jk} x_k \qquad \|A_{jk}\| \neq 0$$

B: $x' \rightleftarrows x''$

$$x'' = x_1'' e_1 + \cdots + x_n'' e_n$$

$$x_j'' = \sum_{k=1}^{n} B_{jk} x_k' \qquad \|B_{jk}\| \neq 0$$

BA: $x \rightleftarrows x''$

$$x_j'' = \sum_l \left(\sum_k B_{jk} A_{kl} \right) x_l$$

$$\|B_{jk} A_{kl}\| = \|B_{jk}\| \cdot \|A_{kl}\| \neq 0.$$

Coordinate Transformation (see (*))

S: $(e_1, e_2, \cdots e_n) \to (e_1', e_2' \cdots e_n')$

$$e_j' = \sum_k S_{jk} e_k \qquad \|S_{jk}\| \neq 0$$

$$r = \sum_j x_j e_j = \sum_j y_j e_j'$$

$$r' = \sum_j x_j' e_j' = \sum_j y_j' e_j'$$

$r \rightleftarrows r'$: $x_j \rightleftarrows x_j' = \sum_k A_{jk} x_k$

$$r = y_j S_{jk} e_k \qquad r' = y_j' S_{jk} e_k$$

$\times (S^{-1})_{kl}$ $\quad \sum_k y_j' S_{jk} = \sum_{k,m} A_{jk} S_{kl}^{-1} (y_m S_{mk})$

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S^{-1}: $\sum_k (S_{jk})(S^{-1})_{kl} = \delta_{jl}$

$$y_i' = y_m S_{mj} A_{kj} (S^{-1})_{kl}$$

If one defines instead the matrix S_{jk}:

$$e_j' = \sum_k e_k S_{kj}$$

~~Then $e_j' = S_{jk} S_{kl} e_l$~~
~~$S_{jk} S_{kl} = \delta_{jl}$.~~

Then $e_j' = S_{jk} S_{kk} e_l'$
$\sum_k S_{jk} S_{kk} = \delta_{jl}$

$(S^{-1})_{kl} = S_{kk} S_{lk}$ }
$(S^{-1})_{kl} = S^{-1}_{lk}$.

~~$y_l' = y_m (S^{-1})_{mj} A_{kj} S_{kl}$~~

$y_l' = y_m (S^{-1})_{jm} A_{kj} S_{lk}$
$= (SAS^{-1})_{lm} y_m$

Thus in the new coordinate system:
$$a \to SAS^{-1}$$
while in the old coordinate system:
$$a \to A.$$
These two representations are essentially

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the same, differing only in the choice of the coordinate system in terms of which they are described. Thus, we may say they are equivalent to each other.

Example 1. One dimensional representation. This is merely to assign to each element a of the group a non-vanishing number $\chi(a)$ in such a way that

$$\chi(ab) = \chi(a)\chi(b).$$

In particular, for $a = 1$

$$\chi(1) = 1$$

because $\chi(b) = \chi(1)\chi(b)$.

A most trivial 1-dimensional representation is obtained by assigning the number 1 to each element. ($\chi(a) = 1$) This is called the identical representation.

Example 2. In particular, if we consider the symmetric group, the group π_f:

$$a \to \varepsilon_a = \pm 1$$

according as a is even or odd permutation defines a 1-dimensional representation (alternating representation) of π, because

$$\varepsilon_{ab} = \varepsilon_a \cdot \varepsilon_b.$$

Example 3. C_f is a finite cyclic group of order h, consisting of

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elements

$$1, a, a^2, \cdots a^{h-1}$$

$(a^h = 1)$

If we choose

$$\chi(a) = \varepsilon,$$

then we have the one-dimensional representation

$$1, \varepsilon, \varepsilon^2, \cdots \varepsilon^{h-1}$$

and

$$\varepsilon^h = 1$$

Thus, ε must be one of the h^{th} root of unity.

$$a^r \to \varepsilon^r \quad (r = 0, 1, \cdots)$$

There will be no other one-dimensional representation of a finite cyclic group.

Now let us consider a linear vector space R. A linear sub-space R' can be obtained from all those points x in R, which satisfy the following conditions:

If x, x' are in R', then

$$cx + c'x'$$

are also in R' for arbitrary numbers c, c'. (The space can be, in general, complex. One can restrict to real space, if one likes.) If R' has a dimension $n'(<n)$, one can choose in R the basic vectors $e_1, e_2, \cdots e_n$, so that $x = \sum x_j e_j$ $(j = 1, \cdots n')$ for x in R'.

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Consider a correspondence U of R on to itself. A linear subspace R′ is said to be invariant under U, if the vectors of R′ are transformed into vectors of R′ by U. In that case, ~~Thus~~ U gives rise to a correspondence U′ of R′ on to itself. We say that U induces U′ in R′.

(Weyl. p. 4:

~~If~~ Two vectors x and y in R are said to be congruent modulo R′.

$$x \equiv y \pmod{R'}$$

if their difference lies in R′.

Every vector is congruent to itself. If $x \equiv y \pmod{R'}$, then $y \equiv x \pmod{R'}$; If $x \equiv y$ and $y \equiv z$ then $x \equiv z$. Thus, in a sense two congruent vectors are equivalent to each other and regard them identical.

By this abstraction ~~A~~ p, which is called projection with respect to R′, a $n-n'$-dimensional space R is obtained. Namely, if we choose the basic vectors

$$e_1, e_2, \cdots e_n$$

such that

$$e_1, e_2, \cdots e_{n'} \quad (n' < n)$$

belongs to R′, the ~~pro~~ $n-n'$-space R′

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is obtained by taking out of

$$x = \sum_{j=1}^{n} x_j e_j ,$$

$$x^{(R)} = \sum_{j=n'+1}^{n} x_j e_j ,$$

because the part $\sum_{j=1}^{n'} x_j e_j$ ~~does not~~ has no effect in characterizing the point in R. R is also a vector space, because, if

$$x_1 \equiv x_2 \qquad y_1 \equiv y_2 \pmod{R'}$$

$$a x_1 \equiv a x_2$$

$$x_1 + y_1 \equiv x_2 + y_2 \pmod{R'}$$

Thus, if R' is invariant under U, the space R is invariant also, because one can disregard the part $\sum_{j=1}^{n'} x_j e_j$ of the whole vector, which does not change under U, and regard the correspondence between ~~point~~ vectors in R simply as the correspondence between two vectors in R'.)

If we consider a system Σ of correspondences U, R' is said to be invariant under Σ, ~~if~~ when it is invariant under every U.

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describing $\mathfrak{R}$ in terms of a coordinate system in which $\mathfrak{R}'$ is pulled back by first n'- basic vectors, all matrices U of the system Σ reduce to the form

$$x = \sum_{j=1}^{n} x_j e_j$$

$$U_j: \quad x \to x'$$

$$x_j' = \sum_k U_{jk} x_k$$

$x_j' = 0$ for $j > n'$, if $x_k = 0$ for $k > n'$

Namely,

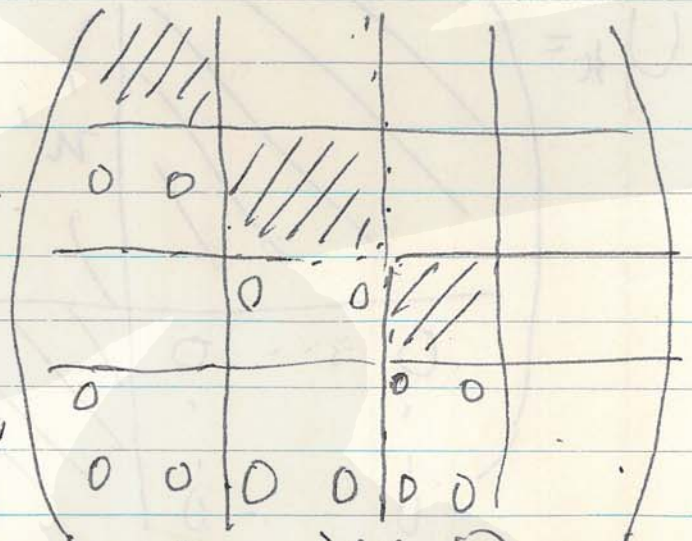
$U_{jk} = 0$ for $j > n'$
$k \leq n'$ }

$$U_{jk} = \left(\begin{array}{c|c} \multicolumn{2}{c}{\overbrace{\quad\quad}^{n'}} & \\ \hline & & \\ & n' & \\ & & \\ \hline 0 \cdots 0 & & \\ \vdots \quad \vdots & & \\ 0 \cdots 0 & & \end{array}\right) \left(\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{n'} \\ \hline x_{n'+1} \\ \vdots \\ x_n \end{array}\right)$$

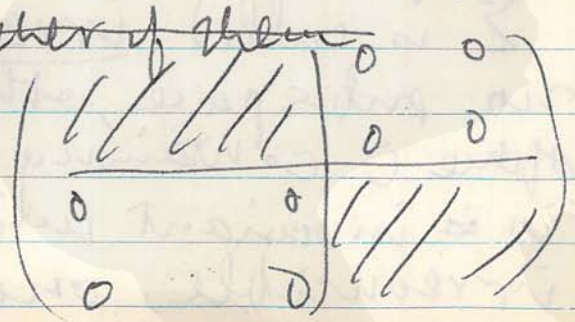
Σ is called <u>irreducible</u> if $\mathfrak{R}$ contains no subspace, other than $\mathfrak{R}$ itself, and the space 0 containing only the vector 0, which is is invariant under Σ. (If $\mathfrak{R}'$ is not irreducible, one can choose construct

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a sequence (composition series)
$0, R_1, R_2, \cdots R_r = R$
in such a way that R_i (mod R_{i-1}) is irreducible. Namely, if R' is an invariant subspace in R under Σ, transformation of vectors of R' as induced by Σ may be reducible or irreducible. If it is reducible still, one can find smaller subspace R'' and so on. Thus, one arrives at the smallest subspace R_1. By adding a number of basic vectors, one will obtain a next smallest space R_2 etc. up to R itself.)

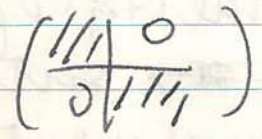


Σ is said to be completely reducible if R can be decomposed into two subspaces R' + R'', each of which are invariant under Σ.



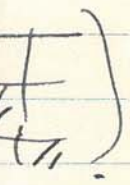
~~such that neither of them~~
This means that if one uses ~~the~~ a suitable coordinate

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system, all of the matrices U of Σ assume the form  simultaneously:

$$U_{jk} = 0 \quad \text{for} \quad \begin{cases} j > n', & k \le n' \\ j \le n', & k > n' \end{cases}$$

One proceeds further and may finally arrive at the complete decomposition of the R:

$$R = R_1 + R_2 + \cdots R_k$$ 

with dimensions $n_1, n_2, \cdots n_k$ respectively where $n_1 + n_2 + \cdots + n_k = n$

~~where~~ and R's are invariant subspaces none of which is completely reducible.

In many cases, reducibility implies completely reducibility. This is actually the case for Σ which is a system of unitary transformations as will be shown later.

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Remark (*) on vector space

Consider the totality $\mathfrak{R}$ of vectors x, x', ... satisfying the following requirements: †

(i) cx belongs to $\mathfrak{R}$, if x does so, for arbitrary number c. (numerical multiplication was defined)

(ii) $x_1 + x_2$ belongs to $\mathfrak{R}$, if x_1, x_2 do so. (Addition was defined commutatively)

(Combining them, it follows that

$c_1 x_1 + c_2 x_2$ belongs to $\mathfrak{R}$, because $c_1 x_1, c_2 x_2$ do so by (i) and $c_1 x_1 + c_2 x_2$ does so by (ii).

If, among $x_1, x_2, \cdots x_m$, there is a relation

$$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m = 0, \quad (1)$$

(0-vector was necessary and defined $x_1 + 0 = x_1$ as $x_1 - x_1 \equiv 0$)

then we say they are linearly dependent.

If there are vectors $x_1, x_2, \cdots x_n$ in $\mathfrak{R}$, which are linearly independent (there is no relation of the form (1) for $m = n$), but any $n+1$-vectors in $\mathfrak{R}$ are linearly dependent, we call $\mathfrak{R}$ n-dimensional vector space. One can take n vectors in $\mathfrak{R}$ which are linearly independent and call them

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$e_1, e_2, \cdots e_n$, then arbitrary vector x can be written as

$$x = x_1 e_1 + \cdots x_n e_n$$

Thus, one may call $(x_1 \cdots x_n)$ the coord. of the vector x in the coordinate system in which $e_1, \cdots e_n$ are basic vectors.

Any other set $e_1', e_2' \cdots e_n'$ of n-vector which are linearly independent to each other can be obtained by a linear transf.

$$e_k' = \sum_{j=1}^{n} e_j S_{jk} \qquad \|S_{jk}\| \neq 0.$$

(because, ~~in order that~~ if e_k' are linearly independent, e_j can be expressed in terms of e_k'. In other words S must have its inverse S^{-1}).

According we have the coordinate transformation from $(x_j) \to (y_k)$

$$x = \sum_j x_j e_j = \sum_k y_k e_k'$$

$$= \sum_k y_k e_j S_{jk}$$

$$x_j = \sum_k S_{jk} y_k$$

A linear correspondence A:

$$A: \quad x \to x'$$

$$x' = \sum_j x_j' e_j'$$

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$$x_j' = \sum_k A_{jk} x_k$$

The corresponding correspondence in the new coord. system:

$$\mathbf{x} = \sum_k y_k' \mathbf{e}_k' \quad (= \sum x_j \mathbf{e}_j)$$

$$\Rightarrow \mathbf{x}' = \sum_k y_k' \mathbf{e}_k'$$

$$= \sum y_k' \mathbf{e}_j S_{jk}$$

$$= \sum x_j \mathbf{e}_j$$

$$= \sum_{j,k} A_{jk} x_k \mathbf{e}_j$$

~~$y_k' S_{jk} = A_{jk} x_k$~~

$$S_{jk} y_k' = A_{jk} x_k$$

$$x_j \mathbf{e}_j = y_k \mathbf{e}_j S_{jk}, \quad x_l = \delta_{lk} y_k$$

$$(S^{-1})_{ij} S_{jk} y_k' = A_{jk} S_{kl} y_l$$

$$y_i' = (S^{-1})_{ij} A_{jk} S_{kl} y_l$$

$$= \sum_l (S^{-1} A S)_{il} y_l$$

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(23)

[One can go further back and start from p.1 of Weyl's book:
vector x is defined by a set of n-numbers $(x_1 \cdots x_n)$
$x = (x_1 \cdots x_n)$
$y = (y_1 \cdots y_n)$
Definition of operations
$ax = (ax_1, ax_2, \cdots ax_n)$
$x + y = (x_1 + y_1, \cdots x_n + y_n)$
Fundamental Rules:
(α) Addition
1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. Subtraction
~~$x - y = z$~~ $z = x - y$
(From $y + z = x$, one can define uniquely z, from x and y)
(β) Multiplication
1. $(a + b)x = ax + bx$
2. $a(bx) = (ab)x$
3. $1x = x$
4. $a(x + y) = ax + ay$
Abstraction → vector space of any dimension → linear independence → Definition of dimension
→ Introduction of coordinates

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Dual vector space →
~~Unitary space~~ Scalar (or inner) product:

$$(x,y) = \tilde{x}_1 y_1 + \tilde{x}_2 y_2 + \cdots \tilde{x}_n y_n$$
$$\equiv \bar{x}\, y$$

(Unitary space)

$$(x,x) = \bar{x}\, x = \tilde{x}_1 x_1 + \cdots \tilde{x}_n x_n .$$

By a correspondence (transformation)

$$x \to x'$$
$$x = \sum_j x_j e_j \to \sum_j x'_j e_j$$
$$x'_j = \sum_k A_{jk} x_k ,$$

$\bar{x}$ undergoes simultaneously a transformation

$$\bar{x} \to \bar{x}'$$
$$\bar{x} = \sum_j \tilde{x}_j \tilde{e}_j \to \bar{x}' = \sum_j \tilde{x}'_j \tilde{e}_j$$

(regarding $\bar{x}$ as a vector in another (dual) vector space)

where
$$\tilde{x}'_j = \sum_k A_{jk}\, \tilde{x}_k$$

In order that the scalar product (x,y) is invariant, the transformation must be unitary.

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In order that the ~~transformation~~ new representation

$$A' = S^{-1} A S$$

is again unitary, the coordinate transformation

$$\Phi'_k = \sum_j \Phi_j S_{jk}$$

$$(\Phi'_k = \sum \Phi_j S_{jk})$$

itself must be unitary.

$$S S^* = S^* S = 1.$$

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Problem.

Symmetric group π_3 of order $3! = 6$. can be represented by linear transformations in three dimensional space: The three basic vectors e_1, e_2, e_3 undergo interchange between themselves:

1 : $e_1 \to e_1$, $e_2 \to e_2$, $e_3 \to e_3$

(12) : $e_1 \to e_2$, $e_2 \to e_1$, $e_3 \to e_3$

(123) : $e_1 \to e_2$, $e_2 \to e_3$, $e_3 \to e_1$

etc.

In general,

$$e_j' = \sum_{k=1}^{3} S_{jk} e_k$$

1 : $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

(12) : $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(123) : $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

etc.

$x \to x'$

$x = \sum_j x_j e_j$

$\to x' = \sum_j x_j' e_j'$

Reduce completely the representation into two dimension and one dimensional representations;

† Alternatively, one can establish the association

$x \to x'$ $\qquad x = \sum x_j e_j \to x' = \sum x_j' e_j$ etc.

$(1,2)$: $e_1 \to e_2$, $e_2 \to e_1$, $e_3 \to e_3$

$(x_1, x_2, x_3) \to (x_1' = x_2, x_2' = x_1, x_3' = x_3)$

$x_j' = a_{jk} x_k$: $a_{jk} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc.

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$$e_1' = \frac{1}{3}(e_1+e_2+e_3)$$
$$e_2' = e_1 - e_2$$
$$e_3' = e_1 - e_3$$
$$e_k' = \sum e_j S_{jk}$$

$$S_{jk} = \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix}$$

$$(S^{-1})_{jk} = \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix}$$

$e_1 = \varphi_1(1)\varphi_2(2)\varphi_3(3)$
$e_2 = \varphi_2(1)\varphi_1(2)\varphi_3(3)$
$e_3 = \varphi_1(1)\varphi_1(2)\varphi_2(3)$
~~$+\varphi_2(1)\varphi_1(2)\varphi_3(3)$~~
$e_2' = (\varphi_1(1)\varphi_2(2) - \varphi_2(1)\varphi_1(2))\varphi_2(3)$

$$(1,2) \to \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix}$$
$$= \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$

$$(1,2,3) \to \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\left(\begin{array}{c|cc} 1/3 & 1 & 1 \\ \hline 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{array}\right)$$

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$$\simeq \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right)$$

$$(1,3) \to \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc} 1/3 & 1 & 1 \\ \hline 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -1 & -1 \end{array}\right)$$

$$(1,3,2) \to \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$(2,3) \to \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 1 \\ 1/3 & -1 & 0 \\ 1/3 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

$$(1,2) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (1,3) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad (2,3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(123) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (132) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Relation of Group Representation to Quantum Mechanics

Let us consider a quantum mechanical system consisting of n-particles. Let us take a ~~set~~ complete set of orthogonal and normalized eigenfunctions

$$\psi_1(r_1 \cdots r_n),\ \psi_2(r_1 \cdots r_n) \cdots$$

as defined by the eigenvalue problem

$$H(r_1 \cdots r_n, \text{grad}_1, \cdots \text{grad}_n) \equiv H(r_1 \cdots r_n, \frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial z_n})$$

$$H \psi_{jr} = E_r \psi_r,$$

the eigenvalues $E_1, E_2, \cdots$ being assumed to be all discrete.

Now let G be the group of rotations (including reflection) in 3-dimensional space.

$$A: \quad x'_j = a_{jk} x_k \qquad (r \to r')$$

This will induce a ~~rota~~ simultaneous rotation of n-points

$$x_j^{(\nu)\prime} = a_{jk} x_k^{(\nu)} \qquad (\nu = 1, 2, \cdots n)$$

With this, any one of the eigenfunctions $\psi_r(r_1 \cdots r_n)$ at the point $r_1, r_2, \cdots r_n$ will be changed into

$$\psi_r(r_1', r_2' \cdots r_n') \equiv \psi'_r(r_1, r_2 \cdots r_n),$$

which can be regarded as a certain function of $r_1 \cdots r_n$.

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Now one can expand

$$\psi_r' = \sum_s c_{rs} \psi_s$$

It is clear that this is a kind of representation, ~~However,~~ in fact it is a repres. in the space of infinite dimension (Hilbert space) It is a unitary representation, if ψ's are orthogonal and normalized,

However, we have no need to worry about the complications due to the fact that the space has infinite dimensions, if ~~Hamil~~ the operator H, which can be identified with the Hamiltonian in q.m.,, ~~to~~ is invariant with respect to rotation, ~~be Hav~~ cause one can decompose the space into invariant subspaces, ~~each~~ of which has a finite dimensions.

Namely, if one takes the eigenfunctions belonging to an eigenvalue E_r, then these eigenfuns transforms among themselves under rotations. Thus the matrix c_{rs} of an arbitrary rotation A is reduced to the form,

///	0	0	0
0	///	0	0
0	0	//	0
0	0	0	///

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Thus, one can deal with ~~the~~ each of these finite dimensional representations separately. Since each of them must be a certain representation of the rotation group, one can deduce all possible ~~the~~ properties of eigenfunctions with respect to rotations, if one knows all the representations of the rotation group.

Such a general consideration already throws some light on the question of spin in connection with the rotation. Namely, it can be shown that there is a peculiar type of representation which does not appear in the above arguments, in which the spins of the particles in question was ignored.

Let us take a two dimensional representation instead of the three dimensional

$$A^{(3)}: x_j \to x'_j \Rightarrow \sum_k a_{jk} x_k$$

$$A^{(2)}: \quad \left.\begin{aligned} \psi_1' &= c_{11}\psi_1 + c_{12}\psi_2 \\ \psi_2' &= c_{21}\psi_1 + c_{22}\psi_2 \end{aligned}\right\}$$

we shall show later that actually $A^{(2)}$ becomes a representation (homomorphic) of rotation group, if we ~~chose the~~ define $c_{\mu\nu}$ appropriately.

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6. Group C_n and its Representations (28)

~~Linear form, Quadratic form etc.~~

Group $C = C_n$ of all linear and homogeneous transformations in n-variables $x_1, x_2, \cdots x_n$ with non-vanishing determinants. One may regard each set of $(x_1 \cdots x_n)$ as a vector in an n-dimensional vector space $R = R_n$.

Consider a linear form

$$\sum_j \alpha_j x_j$$

which transforms into

$$\sum_j \alpha'_j x'_j$$

by a linear transformation

$$x'_j = \sum_k s_{jk} x_k \qquad \|s_{jk}\| \neq 0. \quad (1)$$

then

$$\sum_j \alpha'_j s_{jk} x_k = \sum_k \alpha_k x_k$$

or

$$\alpha_k = \sum_j \alpha'_j s_{jk}$$

Since $\|s_{jk}\| \neq 0$, there is a transf.

$$\alpha'_j = \sum_k S^{(1)}_{jk} \alpha_k .$$

In this case, the transform. $S^{(1)}$ is contragredient to s. ~~Both~~ $S^{(1)}$ is another n-dimensional representation of C_n

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Next, one can take up a quadratic form

$$\sum_{j,k} a_{jk} x_j x_k$$

which transforms into

$$= \sum_{j,k} a_{jk}' x_j' x_k'$$

(like symmetric tensor of the second rank)

As $x_j x_k$ transforms linearly, which we call ~~$[C]^2$~~. a_{jk} transforms contragradiently to ~~$[C]^2$~~, ~~which we call $S^{(2)}$~~. (There are $n(n+1)/2$ coefficients a_{jk}) ~~More generally~~ Both are $n(n+1)/2$ dimensional representation of C_n.

More generally the totality of monomial of f

$$x_1^{f_1} x_2^{f_2} \cdots x_n^{f_n},$$

where

$$f_1 + f_2 + \cdots + f_n = f,$$

constitute the substratum a representation ~~S^f~~ $[C]^f$.

Instead of a quadratic form, one can start from the product $x_j y_k$. If both x, y transform simultaneously according to (1), then the transformation of $x_j y_k$

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gives the n^2-dimensional representation $(C)^2$ of C. (This is the transformation of a tensor of the second rank.)

Now the important thing is that $[C]^2$ is contained in $(C)^2$ in the sense that $(C)^2$ induces a transformation of a symmetric tensor into another sym. tensor, so that totality of symmetric tensors constitute an invariant subspace of in the ~~space~~ n^2-dimensional space.

On the other hand, the totality of anti-symmetric tensors transformed in itself by $(C)^2$. In particular from two vectors x_j, y_k one can construct a surface element

$$x\{jk\} = x_j y_k - x_k y_j$$

with $n(n-1)/2$ independent components. Let's call this transformation $\{S\}_2$ $S \to \{S\}_2$ is a representation $\{C\}^2$ with the dimension $n(n-1)/2$.

Thus the representation $(C)^2$ can be reduced into the representations $[C]^2$ and $\{C\}$, because any tensor F_{jk} can be written uniquely.

$$F_{jk} = \frac{1}{2}(F_{jk} + F_{kj}) + \frac{1}{2}(F_{jk} - F_{kj})$$

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with the dimensional relation

$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2}$$

One can further construct ~~a~~ antisymmetric (as well as symmetric) tensors of any rank from three or more vectors. For instance, the 3-dimensional volume element with

$$\{n\}_3 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

components from x, y, z

$$x\{jkl\} = \begin{vmatrix} x_j & x_k & x_l \\ y_j & y_k & y_l \\ z_j & z_k & z_l \end{vmatrix}$$

is an antisymmetric tensor of the third rank. One can go ~~over~~ up to n-th rank.

We shall see later that the ~~symmetric~~ representations of C by means of symm. and anti-symmetric tensor of order f are irreducible and shall be able to solve the problem of complete reduction of $(C)^f$.

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7. Addition and Multiplication of Representations

Direct Product: X

The tensor concept shows us that it is convenient to introduce the convention of direct product of X of two matrices, two spaces, two groups etc.

If m-variables x_j undergo a transformation

$$A: \quad x_j' = \sum_{k=1}^{m} A_{jk} x_k$$

and n-variables y_μ

$$B: \quad y_\mu' = \sum_{\nu=1}^{n} B_{\mu\nu} y_\nu,$$

then mn-variables $x_j y_\mu$ undergo a transformation

$$A \times B: \quad x_j' y_\mu' = \sum_{k=1}^{m} \sum_{\nu=1}^{n} A_{jk} B_{\mu\nu} x_k y_\nu$$

where the matrix

$$(A \times B)_{j\mu;\, k\nu} = A_{jk} B_{\mu\nu}$$

with mn-rows and columns is call the direct product of two matrices A and B which have m-rows and columns and n-rows and columns respectively.

Correspondingly, one may call the space as give by

$$x \times y$$

with mn-components $x_j y_\mu$ the product space $R_m \times R_n$.

Two representations of G:

$$by: \quad a \to U(a), \qquad by': \quad a' \to U'(a')$$

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in R_m and R_n respectively, give rise to a new $m+n$-dimensional representation

$$D_f \times D_{f'} : \quad a \to U(a) \times U'(a)$$

This presents a general method of obtaining a new representation from two given representations.

By using ~~this~~ notation $\times$ for the direct product, one may write for instance

$$(\mathbb{C}) \times (\mathbb{C}) = (\mathbb{C})^2$$
$$(\mathbb{C}) \times (\mathbb{C}) \times (\mathbb{C}) = (\mathbb{C})^3$$
etc.

Addition of two representations +

If (x_j, y_μ) ~~with components~~ are regarded as the components of a single vector z in a $m+n$-dimensional space. Suppose that these variables undergo transformations

$$A: \quad x_j \to x_k$$
$$A': \quad y_\mu \to y_{\nu'}$$

then ~~(x_j, y_μ) are said to~~ the representation we obtain

$$D_f + D_{f'} : \quad a \to [U(a), U'(a)] \quad (m+n\text{-dimensional})$$

This is obviously the inverse of the process of reduction of the representation $D_f + D_{f'}$ into components D_f and $D_{f'}$.

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Another important method of obtaining representations of any abstract group is the following: (including $u \to U$, u irr. Γ)

Γ: any representation Γ_N in N-dimension of the linear group C_n in n-dimension

Γ_G: $S \to u$: n-dimensional representation of the group G_f with elements s.

~~For~~ By Γ, the association

$$u \to U$$

is established, where

u : linear transf. in n-dimension

U : " in N-dimension *

Thus $S \to U$ through

$S \to u \to U$

is an N-dim. repr. of G_f: $\Gamma(G_f)$.

Example: Γ be a n^2-dimensional representation of C_n in which the substratum is the space of tensors of the 2nd rank. Then we obtain $G_f \times G_f$ etc.

(~~Take~~* Take out a portion of Γ, which for which u is in G_f.

Summing up the above considerations:

(1) Addition + (inverse: reduction)

(2) Multiplication ×

(3) Γ process

(4) G_f: subgroup of G. Any repr. of G contains a repr. of G'. Thus one

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obtains a representation of G_f' from any representation of G_f.

Direct Product of Two Groups

Given two groups G_f, G_f'.

s in G_f, s' in G_f'

The pairs (s, s') can be considered as an element of

$G_f \times G_f'$: direct product of $G_f \times G_f'$

Multiplication law:

$$(s, s')(t, t') = (st, s't')$$

order of $G_f \times G_f'$ is the product of the orders of G_f and G_f', if they are finite.

If $\quad L_f : \; s \to U(s)$

n-dimensional repres. of G_f

$L_f' : \; s' \to U'(s')$

n'-dimensional repres of G_f',

Then

$$(s, s') \to U(s) \times U'(s')$$

is obviously a representation in nn'-dimensions of $G_f \times G_f'$

We write this representation by

$L_f \times L_f'$

This procedure can be decomposed into steps (i) representation

two $\{ (s, s') \to U(s) \quad$ of $G_f \times G_f' \}$

(ii) $\{ (s, s') \to U'(s') \quad$ of $G_f \times G_f' \}$

(ii) $(s, s') \to U(s) \times U'(s')$

So we may write $L_f \times L_f'$ instead of $L_f \times L_f'$.

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8. Unimodular Group in Two Dimensions

$G = C_2$:

S: $x' = ax + by \qquad y' = cx + dy$

Let us impose a restriction

$$ad - bc = 1$$

$f+1$-monomials:

$$x^f,\ x^{f-1}y,\ \cdots\ xy^{f-1},\ y^f \qquad (1)$$

transform linearly ($[S]_f$) and we have $[C]_f$ or C_f

$$S \to [S]_f$$

Let us take the subgroup C_1 in C consisting of principal transformations

$$x' = ax, \qquad y' = \frac{1}{a}y \qquad (a \neq 0) \quad (2)$$

C_1 is Abelian. By this, the monomials (1) are multiplied by

$$a^f,\ a^{f-2},\ \cdots\ a^{-(f-2)},\ a^{-f} \qquad (3)$$

By associating the number a^r with (2), one obtains 1-dimensional representation of C_1: $C^{(r)}$ r; any integer. This is a special case of obtaining of representations $C^{(r)}$ of C_1, from the representations C_f of C_2. Incidentally

Now there is a very important formula which, i.e. Clebsch-Gordan's relation between representations of C_2:

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$$C_f \times C_g = \sum_v C_v \qquad (4)$$

$$v = f+g,\ f+g-2,\ \cdots |f-g|\ ^*$$

It's generalization and application to q.m. is of utmost importance for the classification of atomic spectra and for the theory of valence (chemical binding).

Proof: If we can show

$$C_f \times C_g = C_{f+g} + (C_{f-1} \times C_{g-1}),$$

then (4) follows from

$$C_f \times C_0 = C_f,$$

by mathematical induction.

The substratum of the representation $C_f \times C_g$ is the totality of polynomials

$$\underset{\sim}{D} \sim \underset{\sim}{D}(x, y; \xi, \eta)$$

of orde f and g respectively. It consists of linear combinations of the $(f+1)\times(g+1)$ monomials

---

* $\sum_{v=|f-g|}^{f+g} v = \frac{(g+1)\{(f+g)+(f-g)+1\}}{2} = fg$

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$x^j y^k \cdot \xi^\mu \eta^\nu \qquad j+k=f \;\}\; \mu+\nu=g \;\}$

Now the problem is to reduce completely the space of the polynomials $\bar{\mathfrak{P}}$ into two sub-spaces $(\bar{\mathfrak{P}}_0)$ and $(\bar{\mathfrak{P}})$, which are the substrata of the representation $\mathfrak{C}_h$ $(h=f+g)$ and $\mathfrak{C}_{f-1} \times \mathfrak{C}_{g-1}$ respectively.

$(\bar{\mathfrak{P}})_0$:

$$(\alpha x+\beta y)^f(\alpha\xi+\beta\eta)^g = \alpha^h\phi_0 + \binom{h}{1}\alpha^{h-1}\beta\phi_1 + \cdots + \beta^h\phi_h$$

where $\binom{h}{n} = \dfrac{h(h-1)\cdots h!}{n!\,(h-n)!}$

and $\phi_j(xy;\xi\eta)$ are polynomials of type $\bar{\mathfrak{P}}$.

Now, if we carry out a transformation

$$\left.\begin{aligned} x' &= ax+by, & y' &= cx+dy \\ \xi' &= a\xi+b\eta, & \eta' &= c\xi+d\eta \end{aligned}\right\} (5)$$

then we have

$$\alpha x'+\beta y' = Ax+By$$

with $A = a\alpha+\beta c$, $B = \alpha b+\beta d$

hence

$$(\alpha x'+\beta y')^f(\alpha\xi'+\beta\eta')^g = (Ax+By)^f(A\xi+B\eta)^g$$

or

$$\sum_{j=0}^{h}\binom{h}{j}\alpha^{h-j}\beta^j\cdot\phi_j' = \sum_j\binom{h}{j}A^{h-j}B^j\phi_j$$

By writing A, B in terms of α, β and

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comparing the coefficients of $\alpha^{h-j}\beta^j$, we obtain (the both hand-sides must be equal for arbitrary values of α, β)

ϕ_j as a linear combination of ϕ_k.

Thus $\Phi = (\phi_0, \cdots \phi_h)$ constitute an invariant subspace under (5). This is the substratum for the representation C_h, being linearly indep, because, for

$$\xi = x, \quad \eta = y$$

ϕ_j becomes

$$\phi_j(xy; x y) = x^{h-j} y^j$$

Thus, $\phi_0 \cdots \phi_h$ are linearly independent and transforms like $x^{h-j} y^j$, because the ~~coefficient~~ matrix of linear transform is ~~was~~ already determined for $\xi = x$, $\eta = y$ and cannot be anything else.

Next, as $(\bar{\Phi})'$, take the substratum consisting of the polynomials

$$\Psi(xy; \xi\eta)$$

of the order $f-1$ in (x,y), and of the order $g-1$ in (ξ, η). If we construct

$$\bar{\Phi} = (x\eta - y\xi)\Psi$$

This give the substratum for the repres. $C_{f-1} \times C_{g-1}$, because $x\eta - y\xi$ is invariant with respect to the unimodular transformations (5).

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$$\therefore \begin{pmatrix} x' & z' \\ y' & \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & z \\ y & \eta \end{pmatrix}$$

$$\left\| \begin{matrix} x' & z' \\ y' & \eta' \end{matrix} \right\| = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\| \cdot \left\| \begin{matrix} x & z \\ y & \eta \end{matrix} \right\|$$

What is left to be prove is that

$$\Phi = (a_0\phi_0 + a_1\phi_1 + \cdots a_h\phi_h) + (x\eta - yz)\Psi \quad (6)$$

with unique constant coefficients a_j.
(In other words, any by $\phi_0 \cdots \phi_h$ and $(x\eta - yz)\Psi$, the whole $(f+1)(g+1)$-dimensional space is obtained)

$$(f+1)(g+1) = f+g+1+fg$$

Thus, there are enough number of linearly independent polynomials on the right-hand side, provided that they are actually linearly independent.
Now, for $(z\eta) = (xy)$,

$$\Phi = 0$$

becomes

$$a_0 x^h + a_1 x^{h-1} y + \cdots + a_h y^h = 0$$

identically in x and y. Hence $a_j = 0$.
Thus, Φ vanishes only if $a_j = 0$. ~~and Φ vanishes identically~~. For other sets of arbitrary values of (z, η) and (x, y), Φ can vanish only if, in addition, ~~the~~ identical $(x\eta - yz)\Psi = 0$

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is satisfied or

$$\Psi = 0$$

identically. Q.E.D.

Clebsch-Gordan's formula holds for the group G_2 of all linear transformations x, y with non-vanishing determinant, if we modify the interpretation: C_ν $\nu = k - 2l$ is representation whose substratum is the totality of homogeneous polynomials of order ν in x and y multiplied $(x\eta - y\xi)^l$. In other words, the new C_ν differs from the old that the transformation of the $\nu+1$-dimensional space is to be multiplied by the l^{th} power of the determinant $ad - bc$. (See the relation (6),

$$\Phi = (a_0\phi_0 + a_1\phi_1 + \cdots + a_k\phi_k) + (x\eta - y\xi)\Psi.$$

$$= (\quad) + (x\eta - y\xi)(\quad) + (x\eta - y\xi)^2\Psi' \cdots$$

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9. Unitary group in two dimensions

The group C_2 contains the subgroup $\mathfrak{U}_2$ of unitary transformations in two dimensions. Hence the representation C_f is also a representation of $\mathfrak{U}_2$, which becomes a ~~unitary representation~~ ~~matter of factor~~, but ~~need a slight modification~~ modifies it slightly: To a unitary transformation S of the coordinate (x, y), there corresponds a linear trans. C_f of $f+1$-variables

$$x_n = x^j y^k \quad (j+k=f)$$

of order f, where

$$n = j - k$$

which runs through the values $f, f-2, \ldots -f$.

We restrict ourselves to the subgroup of principal transformations

$$x \to \varepsilon x, \quad y \to \frac{1}{\varepsilon} y,$$

x_n is multiplied by ε^n. We now employ instead of x_n, the new definition of x_n:

$$x_n = \frac{x^j y^k}{\sqrt{j!\,k!}}$$

Then, the representation C_f of $\mathfrak{U}_2$ is unitary, because

$$\frac{1}{f!}(\tilde{x}x + \tilde{y}y)^f = \sum_{j+k=f} \frac{\tilde{x}^j \tilde{y}^k\, x^j y^k}{j!\,k!} = \sum_{n=-f}^{+f} \tilde{x}_n x_n$$

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We call C_f even or odd according as f is even or odd. The even representations associate the identity I with the reflection

$$x' = -x, \quad y' = -y$$

and the odd associate with it $= -I$.
C_f is irreducible also when it is considered as a representation of $\mathfrak{u}$ and on letting f assume the values $0, 1, 2, \ldots$ they constitute a complete system of inequivalent irreducible representation of $\mathfrak{u}$.
(Proof will be given in chapter V of Weyl's book)

The complete reduction of the product representation $C_f \times C_g$ is accomplished as in the case of C_2 and we have the Clebsch-Gordan's formula again:

$$C_f \times C_g = C_{f+g} + C_{f+g-2} + \ldots + C_{|f-g|}.$$

The proof is similar to the case of C_2.
One has to prove:

$$C_f \times C_g = C_{f+g} + (C_{f-1} \times C_{g-1})$$

a general polynomial Φ of x_n, ξ_ν:

$$\sum a_{n\nu} x_n \xi_\nu$$

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$$\left.\begin{aligned} n &= f, f-2, \cdots -f \\ \nu &= g, g-2, \cdots -g \end{aligned}\right\}$$

where

$$\xi_\nu = \frac{\xi^\lambda \eta^\lambda}{\sqrt{\lambda!\,\lambda!}} \qquad \left.\begin{aligned} n+\lambda &= g \\ n-\lambda &= \nu \end{aligned}\right\}$$

Two such polynomials P with coefficients $a_{n\nu}$, $b_{n\nu}$ are orthogonal if

$$\sum a_{n\nu}\, b_{n\nu} = 0$$

Now if we take a particular polynomial

$$x_f\, \xi_g \equiv C \cdot x^{f+g},$$

this is orthogonal to the polynomials $(\Phi)'$:

$$(x\eta - y\xi)\,\Psi$$

where Ψ is polynomials of order $f-1$ in (x, y) and $g-1$ in (ξ, η), so that the coefficient of x^{f+g} is zero.

Under the unitary transformation

$$S: \left.\begin{aligned} x' &= \alpha x + \beta y \\ y' &= -\bar\beta + \bar\alpha y \end{aligned}\right\} \qquad (*)$$

with

$$\alpha\bar\alpha + \beta\bar\beta = 1,$$

$x^f \xi^g$ goes into

$$(\alpha x + \beta y)^f (\alpha\xi + \beta\eta)^g \qquad (\dagger)$$

Now (Φ') is invariant under S. The orthogonality does is invariant under S (*) (or more generally under any unitary transformation, because

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$$\sum_{n,\nu} \tilde{a}'_{n\nu}\, b'_{n\nu} = \sum \tilde{S}_{n\nu,m\mu}\, a_{m\mu}\, S_{n\nu,l\lambda}\, b_{l\lambda}$$

$$= \sum a_{m\mu} S^*_{m\mu,n\nu} S_{n\nu,l\lambda}\, b_{l\lambda}$$

$$= \sum a_{m\mu}\, b_{m\mu},$$

because $S^* S = 1$.)

Thus, all polynomials $(\Phi)_0$ are orthogonal to those of (Φ'). They are invariant subspaces of $\mathfrak{C}_{f+g}$ and $\mathfrak{C}_{f-1} \times \mathfrak{C}_{g-1}$, which are mutually orthogonal to each other. ~~One can choose~~ One can proceed further and completely reduce the product representation, thereby introducing $(f+1)(g+1)$ coordinates

$\mathfrak{C}_{f+g}$: $z_1, z_2, \cdots z_{f+g+1}$

$\mathfrak{C}_{f+g-1}$: $z_{f+g+1}, \cdots z_{2f+2g}$

- - -

$\mathfrak{C}_{f-g}$: $z_{fg-|f-g|} \cdots z_{fg}$

where

$$\tilde{z}_1 z_1 + \cdots + \tilde{z}_{f+g+1} z_{f+g+1}$$

- - - ,

$$\tilde{z}_{fg-|f-g|} z_{fg-|f-g|} + \cdots + \tilde{z}_{fg} z_{fg}$$

are invariant under unitary transformations.

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Remarks:

1. If we expand (T), we have

$$(\alpha x+\beta y)^f(\alpha z+\cdots\beta\eta)^g$$
$$\equiv \alpha^h\Phi_0+\binom{h}{1}\alpha^{h-1}\beta\Phi_1+\cdots+\beta^h\Phi_h$$

which is orthogonal to any polynomial of $(\bar{P})$ for an arbitrary value of α,β satisfying $\bar{\alpha}\alpha+\bar{\beta}\beta=1$. This is only true, if only if

$$\Phi_0,\Phi_1,\cdots\Phi_h$$

are all orthogonal to $(\bar{\Phi})$. (For $\beta=0$, Φ_0 must be orthogonal. Then divide the orthogonality relation by β and put $\beta=0$, leaving Φ_1 orthogonal to $(\bar{P})$ etc.)

$$\sum_{n\nu}\tilde{b}_{n\nu}\left(\alpha^h a_{n\nu}^{(0)}+\binom{h}{1}\alpha^{h-1}\beta a_{n\nu}^{(1)}+\cdots\right)=0$$

or $$\alpha^h\sum\tilde{b}_{n\nu}a_{n\nu}^{(0)}+\binom{h}{1}\alpha^{h-1}\beta\sum\tilde{b}_{n\nu}a_{n\nu}^{(0)}+\cdots=0$$

2. If we have a ~~non~~ representation of $\mathfrak{u}_2$ in $h+1$-dimensions, it is equivalent to the ~~transformation~~ representation by monomials

$$x^h,\ x^{h-1}y,\ \cdots xy^h,$$

~~(which should be proved)~~, so that one can construct from it a unitary representation. Because the transformation laws for

~~$x^j y^k \xi^\mu \eta^\lambda$ $(\mu=f,f-2,\cdots -f$ $\nu=g,g-2,\cdots -g$ $j+k=f$, $\mu+\lambda=\nu$ f,g~~

$\Phi_0,\Phi_1,\cdots\Phi_h$ ~~must be the same as~~

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are determined for $x=\xi$, $y=\eta$.
In that case, they transforms as
x^k, $x^{k-1}y$, . . . y^k.

3. ~~Since~~ Thus, one can choose first $(f+1)(g+1)$ coordinates in such a way that
$C_{f+g}:(z_1, z_2, \cdots z_{f+g+1})$
$C_{f-1}\times C_{g-1}:(z_{f+g+2}, \cdots z_{fg})$
~~are multiply orthogonal to each other~~ Transform among themselves.
Then, one can go over to another coord. systems, in which the representations are unitary for each C_{f+g} and C_{f-1} x C_{g-1} separately. In this way, we arrive at the ~~reps~~ completely reduced representation in which
$\tilde{\zeta}_1\tilde{\zeta}'_1 + \cdots + \cdots \tilde{\zeta}_{f+g+1}\tilde{\zeta}_{f+g+1}$
and
$\tilde{\zeta}_{f+g+2}\zeta_{f+g+2} + \cdots + \tilde{\zeta}_{fg}\zeta_{fg}$
are invariant separately

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Problem

Show that the unitary transformation in two dimensions can be written as

$$\left.\begin{aligned} x' &= \alpha x + \beta y \\ y' &= e^{-i\omega}(-\tilde{\beta} x + \tilde{\alpha} y) \end{aligned}\right\}$$

with ~~$\tilde{\alpha}\alpha+\tilde{\beta}\beta=1$, $\tilde{\alpha}\gamma-\tilde{\gamma}\alpha=0$~~ $e^{i\omega} = \alpha\delta - \beta\gamma$

$$x' = \alpha x + \beta y \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$y' = \gamma y + \delta y$$

$$(\tilde{x}', \tilde{y}') = (\tilde{x}, \tilde{y}) \begin{pmatrix} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\beta} & \tilde{\delta} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\alpha}\alpha + \tilde{\gamma}\gamma & \tilde{\alpha}\beta + \tilde{\gamma}\delta \\ \tilde{\beta}\alpha + \tilde{\delta}\gamma & \tilde{\beta}\beta + \tilde{\delta}\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \tilde{\alpha} &= \frac{\delta}{\alpha\delta - \beta\gamma} & \tilde{\gamma} &= \frac{-\beta}{\alpha\delta - \beta\gamma} \\ \tilde{\beta} &= \frac{-\gamma}{\alpha\delta - \beta\gamma} & \tilde{\delta} &= \frac{\alpha}{\alpha\delta - \beta\gamma} \end{aligned}\right\}$$

$$\left.\begin{aligned} \tilde{\alpha}\alpha\cdot\delta - \tilde{\alpha}\beta\cdot\gamma &= \delta & (\tilde{\alpha}\alpha - 1)\delta &= \tilde{\alpha}\beta\gamma \\ \tilde{\beta}\alpha\delta - \tilde{\beta}\beta\gamma &= -\gamma & \tilde{\beta}\alpha\delta &= (\tilde{\beta}\beta - 1)\gamma \end{aligned}\right\}$$

$$(\tilde{\alpha}\delta - \tilde{\beta}\tilde{\gamma})(\alpha\delta - \beta\gamma) = 1. \qquad \alpha\delta - \beta\gamma = e^{i\omega}$$

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Problem:

By introducing the spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

one can write,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{\alpha+\delta}{2} + \frac{\beta+\gamma}{2}\sigma_1 + i\frac{\beta-\gamma}{2}\sigma_2 + \frac{\alpha-\delta}{2}\sigma_3$$

In particular, in the case of unimodular unitary transf.

$$\alpha = \varkappa + i\lambda, \qquad \beta = \mu + i\nu$$
$$\gamma = -\mu + i\nu, \qquad \delta = \varkappa - i\lambda$$

with $\varkappa^2 + \lambda^2 + \mu^2 + \nu^2 = 1$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \approx \varkappa + i\lambda\sigma_1 + i\mu\sigma_2 + i\nu\sigma_3$$

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} \approx \varkappa - i\lambda\sigma_1 - i\mu\sigma_2 - i\nu\sigma_3$$

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Problem

$$x' = \alpha x + \beta y$$
$$y' = \gamma x + \delta y$$

Prove that with arbitrary $\alpha, \beta, \ldots$ an arbitrary unitary transf. can be written as (37)

It should be noticed that (*) is the most general, unimodular unitary transformation, because the most general unitary transf. $\left.\begin{matrix} x' = \alpha x + \beta y \\ y' = \gamma x + \delta y \end{matrix}\right\}$ coefficients of must satisfy the requirements **

$$\left.\begin{aligned} \bar{\alpha}\alpha + \bar{\gamma}\gamma &= 1 \\ \bar{\beta}\beta + \bar{\delta}\delta &= 1 \\ \bar{\alpha}\beta + \bar{\gamma}\delta &= 0 \\ (\bar{\beta}\alpha + \bar{\delta}\gamma &= 0) \end{aligned}\right\}$$ one-rest. one-rest two-rest

Thus, among 8-available real parameters only 4-parameters are independent.

$$\alpha = x + i\lambda \qquad \beta = -\mu + i\nu$$
$$\gamma = x' + i\lambda' \mu' + i\nu' \qquad \delta = x' - i\lambda'$$
$$x^2 + \lambda^2 + \mu'^2 + \nu'^2 = 1$$
$$\mu^2 + \nu^2 + x'^2 + \lambda'^2 = 1$$
$$-x\mu + \lambda\nu + x'\mu' \underline{\;} \lambda'\nu' = x\nu + \lambda\mu + x'\nu' - \lambda'\mu' \quad = 0$$

** $x_j' = a_{jk} x_k$

$$\tilde{x}_j' = \tilde{a}_{jk} x_k = x_k a^*_{kj}$$

$$a^*_{kj}\, a_{jl} = \tilde{a}_{jk} a_{jl} = \delta_{kl}$$

$a_{11} = \alpha \qquad a_{12} = \beta$

$a_{21} = \gamma \qquad a_{22} = \delta$

$\tilde{\alpha}_{11} a_{11} + \tilde{\gamma}_{21} \alpha_{21} = \bar{\alpha}\alpha + \bar{\gamma}\gamma = 1$ etc

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~~Thus, or~~ Obviously, these relations are satisfied by taking

$$\left.\begin{aligned} \mu' &= \mu & \nu' &= \nu \\ \varkappa' &= \varkappa & \lambda' &= \lambda \end{aligned}\right\}$$

with four free parameter $\varkappa, \lambda, \mu, \nu$.
This is ~~the~~ must, thus, be the general solution, ~~of these equa covering~~
in other words, transformations

$$\left.\begin{aligned} x' &= \alpha x + \beta y \\ y' &= -\bar{\beta} x + \bar{\alpha} y \end{aligned}\right\}$$

cover the whole group of unitary transformations.

Another proof:

$$\tilde{a}_{jk} a_{jl} = \delta_{kl}$$

~~$\tilde{a}_{11} a_{11} + \tilde{a}_{21} a_{21} = 1$~~

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} \\ \tilde{a}_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{a}_{11} \\ \tilde{a}_{21} \end{pmatrix} = \frac{1}{\|a_{kl}\|} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\|a_{kl}\|} \begin{pmatrix} a_{22} \\ -a_{12} \end{pmatrix}$$

$\tilde{\beta} = -\gamma$

$a_{22} = \tilde{a}_{11}$, $\delta = \bar{\alpha}$; $\tilde{\gamma} = -\beta$.

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10. Rotation Group in Two dimensions d_2 (Rotations in the plane)

x: complex variable (for a point on a plane)

rotation:

$$x \to x' = \varepsilon x$$

$$\varepsilon = e^{i\phi}$$

ϕ: angle of rotation

$$x' \to x'' = \varepsilon' x \qquad \varepsilon' = e^{i\phi'}$$

then we have

$$x \to x'' = \varepsilon'' x \qquad \varepsilon'' = e^{i(\phi+\phi')}$$

Thus, we have one parameter continuous Abelian group.

One dimensional representation

$$D^{(m)}: \quad x \to x' = \varepsilon^m x = e^{im\phi} x$$

m: integer

If we take all integer values m, the totality $(D^{(m)})$ constitute a complete system of irreducible unitary representation of d_2. every

Proof: Since d_2 is Abelian, ~~a~~ irreducible representation must be one dimensional. (all matrices are mutually commutative, one can choose a suitable coordinate system, in which they are all diagonal.) (See chapt I, §6 of Weyl's book.)

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We associate with rotation ϕ a number $\chi(\phi)$ with $|\chi(\phi)| = 1$ and satisfying the ~~condition~~ relation

$$\chi(\phi+\phi') = \chi(\phi)\cdot\chi(\phi')$$

In order that the representation is continuous:

$$\chi(2\pi) = \chi(0) = 1$$

In general

$$\chi(\phi+2\pi) = \chi(\phi)$$

thus a periodic function of ϕ.

$$\chi(\phi) = e^{i\lambda}$$

$$\lambda(\phi+\phi') = \lambda(\phi)+\lambda(\phi')$$

From it we obtain:

$$\lambda(h\phi) = h\,\lambda(\phi), \quad \lambda(-\phi) = -\lambda(\phi)$$

h: integer

or $\phi \to \frac{\phi}{h}$:

$$\lambda\left(\frac{\phi}{h}\right) = \frac{1}{h}\lambda(\phi)$$

or

$$\lambda\left(\frac{k}{h}\phi\right) = \frac{k}{h}\lambda(\phi) \qquad (+)$$

for any rational k/h.

Since $\lambda(2\pi)$ must be an integral multiple of $2m\pi$ of 2π, we obtain from (+):

~~$\lambda(\phi) = \frac{k}{h}$~~

$$\chi\left(\frac{k}{h}2\pi\right) = \frac{k}{h}\lambda(2\pi) = \frac{k}{h}2m\pi$$

or

$$\lambda(\phi) = m\phi$$

for $\phi = \frac{k}{h}2\pi$. The continuity req. asserts that $\lambda(\phi) = m\phi$ for any real ϕ.

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~~Problem~~:
~~Prove that~~

(In this case, we have a very simple relation:

$$D^{(m)} \times D^{(m')} = D^{(m+m')} \qquad (f+g=2)$$

Transform of $x^f y^g$ can be completely reduced to one-dimensional representations.

~~$f=2$~~ x^2, xy, y^2

It is already one dimensional, however

$$x'y' = qx\, q'y$$
$$= e^{i(m+m')\phi}\, xy.$$

The relation to eigenfunctions in q.m. is very simple and obvious. We take ~~totality~~ period cont. function $\psi(\phi)$. for $\phi \in (0, 2\pi)$

$$\psi(\phi) = \sum_{m=-\infty}^{+\infty} c_m e^{im\phi}$$

By a rotation ϕ, ψ becomes

$$\psi(\omega+\phi) \equiv \psi'(\phi)$$
$$= \sum_{m=-\infty}^{+\infty} c_m e^{im\phi} e^{im\omega}$$
$$\equiv \sum_{m=-\infty}^{+\infty} c'_m e^{im\omega}$$

$$c'_m = e^{im\phi} c_m,$$

this is nothing but the totality of

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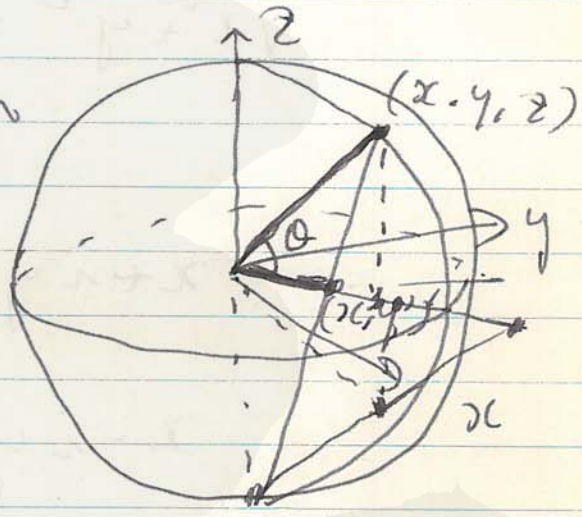
one-dimensional
irreducible representations of $\mathfrak{sl}_2$.

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11. O_3 : Group of rotations (in three dimensional space)

If we restrict ~~our~~ ourselves to proper rotations, one can find a complex two dimensional representation in the following way:

Let us consider a stereographic projection of a point on a unit sphere (x, y, z) on the equatorial plane with the south pole as the origin of projection.



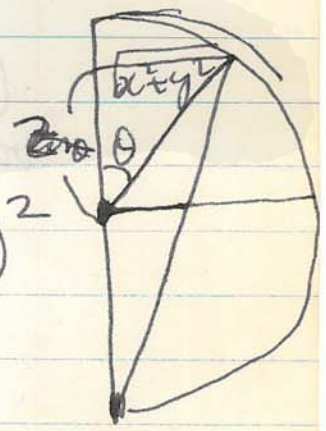
$$x' : y' = x : y$$

$$x' + iy' = \zeta$$

$$|\zeta|^2 + 1 - 2|\zeta| \cos\theta$$

$$= z^2 + (|\zeta| - \sqrt{x^2+y^2})^2$$

$$\cos\theta = \frac{\sqrt{x^2+y^2}}{1}$$



$$|\zeta|^2 + 1 - 2|\zeta|\sqrt{x^2+y^2}$$

$$= z^2 + |\zeta|^2 - 2|\zeta|\sqrt{x^2+y^2} + x^2 + y^2.$$

$$\sqrt{x'^2+y'^2} : \sqrt{x^2+y^2}$$

$$= 1 : 1+z$$

$$\sqrt{x'^2+y'^2} = \frac{\sqrt{x^2+y^2}}{1+z}$$

$$x'^2+y'^2 = \frac{1-z^2}{(1+z)^2}$$

$$= \frac{1-z}{1+z}$$

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$$\hat{\zeta}\hat{\zeta} = \frac{1-z}{1+z}$$

$$\hat{\zeta}\hat{\zeta} + z\hat{\zeta}\hat{\zeta} = 1 - z$$

$$z = \frac{1-\hat{\zeta}\hat{\zeta}}{1+\hat{\zeta}\hat{\zeta}} = \frac{\hat{\zeta}\hat{\zeta} - \hat{\eta}\hat{\eta}}{\hat{\zeta}\hat{\zeta} + \hat{\eta}\hat{\eta}}$$

$$\sqrt{x^2+y^2} = (1+z)\cdot\sqrt{x'^2+y'^2}$$

$$= \frac{2\hat{\zeta}\hat{\zeta}}{1+\hat{\zeta}\hat{\zeta}}\sqrt{x'^2+y'^2}$$

$$x + iy = \frac{2\zeta}{1+\hat{\zeta}\hat{\zeta}} = \frac{2\hat{\zeta}\eta}{\hat{\zeta}\hat{\zeta}+\hat{\eta}\eta}$$

$$x - iy = \frac{2\hat{\zeta}}{1+\hat{\zeta}\hat{\zeta}} = \frac{2\hat{\eta}\zeta}{\hat{\zeta}\hat{\zeta}+\hat{\eta}\eta},$$

One may use instead, two complex variables: $\zeta = \eta/\xi$

Then the south pole is included as

$$\xi : \eta = 0 : 1$$

$$x+iy : x-iy : z : 1$$

$$= 2\hat{\xi}\eta : 2\hat{\eta}\zeta : \hat{\zeta}\zeta - \hat{\eta}\eta : \hat{\zeta}\zeta + \hat{\eta}\eta$$

Then under by a unimodular unitary transformation

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$$\left.\begin{aligned} \xi' &= \alpha\xi + \beta\eta \\ \eta' &= -\tilde{\beta}\xi + \tilde{\alpha}\eta \end{aligned}\right\} \qquad (*)$$

with $\alpha\tilde{\alpha} + \beta\tilde{\beta} = 1$, (x, y, z) are transformed linearly as

$$\left.\begin{aligned} x &\sim \tilde{\xi}\eta + \tilde{\eta}\xi \quad \tilde{\eta}\xi + \tilde{\xi}\eta \\ y &\sim i(\tilde{\eta}\xi - \tilde{\xi}\eta) \\ z &\sim \tilde{\xi}\xi - \tilde{\eta}\eta \end{aligned}\right\}$$

without changing

$$x^2 + y^2 + z^2 \sim \tilde{\xi}\xi + \tilde{\eta}\eta .$$

The totality of unimodular unitary transformations of (ξ, η) cover the whole group of proper rotations d_3' because, (*) ~~can carry $(1,0,0)$; $(0,1,0)$, $(0,0,1)$ to arbitrary points on the unit sphere (mutual orthogonal, and right handsides)~~:

~~$(1,0,0)$~~ : (*) has 3-independent real parameters.

~~$x' \quad \tilde{\eta}'\xi' + \tilde{\xi}'\eta' \equiv (-\beta\tilde{\xi} + \alpha\tilde{\eta})$~~

$$\tilde{\eta}'\xi' + \tilde{\xi}'\eta' = \underbrace{(\alpha\xi + \beta\eta)}_{}\tilde{}\,(-\beta\tilde{\xi} + \alpha\tilde{\eta})(\tilde{\alpha}\tilde{\xi} + \tilde{\beta}\tilde{\eta})(-\beta\tilde{\xi} + \tilde{\alpha}\tilde{\eta})$$

$$= (-\alpha\beta - \tilde{\alpha}\tilde{\beta})\tilde{\xi}\xi - (\alpha^2 - \tilde{\beta}^2)\tilde{\eta}\xi + (\tilde{\alpha}^2 - \beta^2)\tilde{\xi}\eta + (\alpha\beta + \tilde{\alpha}\tilde{\beta})\tilde{\eta}\eta$$

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$$= -(\alpha\beta + \tilde{\alpha}\tilde{\beta})(\xi\bar{\xi} - \eta\bar{\eta})$$

$$-\tfrac{1}{2}(\alpha^2 - \beta^2 - \tilde{\alpha}^2 + \tilde{\beta}^2)(\eta\bar{\xi} + \xi\bar{\eta})$$

$$-\tfrac{1}{2}(\alpha^2 - \beta^2 + \tilde{\alpha}^2 - \tilde{\beta}^2)(\eta\bar{\xi} - \xi\bar{\eta})$$

etc.

Geometrically, the unimodular unitary transformation (*) or

$$\zeta' = \frac{-\beta\xi + \tilde{\alpha}\eta}{\alpha\xi + \tilde{\beta}\eta} = \frac{-\beta + \tilde{\alpha}\zeta}{\alpha + \tilde{\beta}\zeta}$$

can bring any point ζ into any other point ζ' with the same $\xi\bar{\xi} + \eta\bar{\eta}$ as

$$\eta' = (\xi\bar{\xi} + \eta\bar{\eta})$$

$$\left.\begin{aligned} \xi' &= \alpha\xi + \tilde{\beta}\eta = \xi\alpha + \eta\tilde{\beta} \\ \tilde{\eta}' &= -\beta\bar{\xi} + \tilde{\alpha}\bar{\eta} = \alpha\bar{\eta}\tilde{\alpha} - \bar{\xi}\beta \end{aligned}\right\}$$

$$\alpha = \frac{\xi\bar{\xi}' + \eta\bar{\eta}'}{\xi\bar{\xi} + \eta\bar{\eta}} \qquad \beta = \frac{\bar{\eta}\xi' - \xi\bar{\eta}'}{\xi\bar{\xi} + \eta\bar{\eta}}$$

$$\tilde{\alpha} = \frac{\xi\bar{\xi}' + \eta\bar{\eta}'}{\xi\bar{\xi} + \eta\bar{\eta}} \qquad \tilde{\beta} = \frac{\eta\bar{\xi}' - \xi\eta'}{\xi\bar{\xi} + \eta\bar{\eta}}$$

$$\tilde{\alpha}\alpha + \tilde{\beta}\beta = \frac{\xi\bar{\xi}\cdot\xi\bar{\xi}' + \tilde{\eta}\eta\cdot\tilde{\eta}\eta' + \eta\tilde{\eta}\cdot\tilde{\xi}\bar{\xi}' + \xi\bar{\xi}\cdot\tilde{\eta}'\eta'}{(\xi\bar{\xi} + \eta\tilde{\eta})^2}$$

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$$\left.\begin{aligned} \xi' &= \alpha\xi + \beta\eta \\ \eta' &= -\bar\beta\xi + \bar\alpha\eta \end{aligned}\right\} \quad (*)$$

with $\alpha\bar\alpha + \beta\bar\beta = 1$, the real variables

$$\left.\begin{aligned} x &= \bar\eta\xi + \bar\xi\eta \\ y &= i(\bar\eta\xi - \bar\xi\eta) \\ z &= \bar\xi\xi - \bar\eta\eta \end{aligned}\right\} \qquad \begin{aligned} x &\to \bar\psi\sigma_1\psi \\ y &\to \bar\psi\sigma_2\psi \\ z &\to \bar\psi\sigma_3\psi \end{aligned}$$

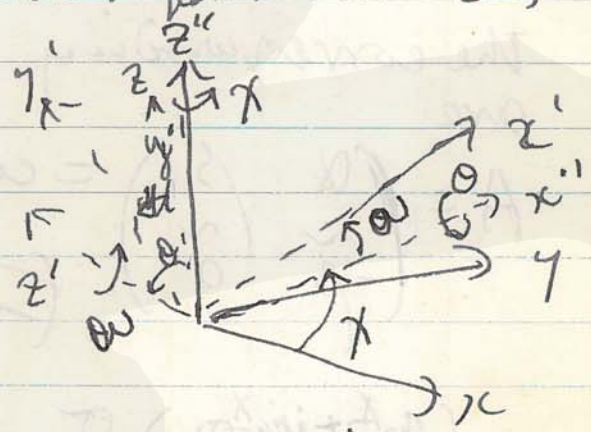
transform linearly and really without changing

$$x^2 + y^2 + z^2 = (\bar\xi\xi + \bar\eta\eta)^2$$

proper

all the rotations in the three dimensional space ~~find~~ its counterpart in (*)

$$\chi: \begin{pmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\theta: \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad \omega \doteq \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\omega\cdot\theta\cdot\chi: \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\chi & -\sin\omega & 0 \\ \cos\theta\sin\chi & \cos\theta\cos\chi & -\sin\theta \\ \sin\theta\cos\chi & \sin\theta\cos\chi & \cos\theta \end{pmatrix}$$

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$\omega \times \theta \times \chi$:

$$\begin{pmatrix} \cos\omega\cos\chi - \sin\omega\cos\theta\sin\chi & -\cos\omega\sin\chi - \sin\omega\cos\theta\cos\chi & \sin\chi\sin\theta \\ \sin\omega\cos\chi + \cos\omega\cos\theta\sin\chi & -\sin\omega\sin\chi + \cos\omega\cos\theta\cos\chi & \cos\omega\sin\theta \\ \sin\theta\cos\chi & \sin\theta\cos\chi & \cos\theta \end{pmatrix}$$

The corresponding unimod. unitary transf. are

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \cos\frac{\chi}{2} + i\sin\frac{\chi}{2}\sigma_3 = \begin{pmatrix} e^{\frac{i\chi}{2}} & \\ & e^{\frac{-i\chi}{2}} \end{pmatrix}$$

$$A^{-1} = \cos\frac{\chi}{2} + i\sin\frac{\chi}{2}\sigma_3$$

$$\left(\cos\tfrac{\chi}{2} + i\sin\tfrac{\chi}{2}\sigma_3\right)\sigma_1\left(\cos\tfrac{\chi}{2} + i\sin\tfrac{\chi}{2}\sigma_3\right) = \left(\cos^2\tfrac{\chi}{2} - \sin^2\tfrac{\chi}{2}\right)\sigma_1$$

$$+ 2\cos\tfrac{\chi}{2}\sin\tfrac{\chi}{2}\,\sigma_2$$

$$= \cos\chi\,\sigma_1 - \sin\chi\,\sigma_2$$

$$\left(\cos\tfrac{\chi}{2} + i\sin\tfrac{\chi}{2}\sigma_3\right)\sigma_2\left(\cos\tfrac{\chi}{2} + i\sin\tfrac{\chi}{2}\sigma_3\right) = \sin\chi\cdot\sigma_1 + \cos\chi\,\sigma_2$$

$$\beta_\theta = \left(\cos\frac{\theta}{2} \quad i\sin\frac{\theta}{2}\cdot\sigma_1\right)$$

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$(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\sigma_1)\sigma_\pm(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}) = \cos\theta\cdot\sigma_2 - \sin\theta\cdot\sigma_3$

with Hurwan arbitrary rotations (χ, θ, ω) is associated a u.u.t.:

$$(\cos\tfrac{\omega}{2} - i\sin\tfrac{\omega}{2}\sigma_3)(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\sigma_1)(\cos\tfrac{\chi}{2} + i\sin\tfrac{\chi}{2}\cdot\sigma_3)$$

$$= \begin{pmatrix} e^{-i\frac{\omega}{2}} & 0 \\ 0 & e^{i\frac{\omega}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\chi}{2}} & 0 \\ 0 & e^{i\frac{\chi}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\omega}{2}} & 0 \\ 0 & e^{i\frac{\omega}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\chi}{2}} & -i\sin\frac{\theta}{2}e^{i\frac{\chi}{2}} \\ -i\sin\frac{\theta}{2}e^{-i\frac{\chi}{2}} & \cos\theta\, e^{i\frac{\chi}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\omega}{2}}\cos\frac{\theta}{2}e^{-i\frac{\chi}{2}} & -i e^{-i\frac{\omega}{2}}\sin\frac{\theta}{2}e^{i\frac{\chi}{2}} \\ -i e^{i\frac{\omega}{2}}\sin\frac{\theta}{2}e^{-i\frac{\chi}{2}} & e^{i\frac{\omega}{2}}\cos\frac{\theta}{2}e^{i\frac{\chi}{2}} \end{pmatrix}$$

This is the most general u.u. transf. because arbitrary α, β are uniquely:

$$\alpha = \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\omega+\chi)} \qquad \beta = \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\chi-\omega)}$$

with $\tilde{\alpha}\alpha + \tilde{\beta}\beta = 1$

~~one can find θ, ω, χ there is a one-one correspondence (mod 2π)~~

can be obtained by choosing suitable χ, θ, ω.

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Problem: Prove that

However, it should be noticed that there are two ~~A's~~ l.u.t. corresponding to one rotation:

Namely, ~~if~~ A and -A give the same rotation. There are no more ~~the~~ general because all other matrices leave the form

~~$(x + i\lambda\sigma_1 - i\mu\sigma_2 - i\nu\sigma_3)\,\sigma_1\,(x + i\lambda\sigma_1 + i\mu\sigma_2 + i\nu\sigma_3)$~~

~~$= (x^2 + \lambda^2 - \mu^2 - \nu^2)\,\sigma_1$~~

~~$\lambda\mu\,(\mu^2 + \nu^2)\,\sigma_2$~~

$A = x_0 + i\lambda_1\sigma_1 + i\lambda_2\sigma_2 + i\lambda_3\sigma_3$

where λ's are real for unim. unitary transformations. If ~~some~~ all of λ's are not zero, A cannot be commutative with all of $\sigma_1, \sigma_2, \sigma_3$, so that $A^{-1}\sigma_j A$ cannot be identical with σ_j. Thus the only possibility is $A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ with $x^2 = 1$ or $x = \pm 1$.

~~$\sigma_1(x + i\lambda\sigma_1 + i\mu\sigma_2 + i\nu\sigma_3)(x + i\lambda\sigma_1 + i\mu\sigma_2 + i\nu\sigma_3)\,\sigma_1 = -i\mu\sigma_3 + i\nu\sigma_2$~~

$\therefore [\sigma_1, (\lambda_0 + i\lambda_1\sigma_1 + i\lambda_2\sigma_2 + i\lambda_3\sigma_3)]$

$= -\lambda_2\sigma_3 + \lambda_3\sigma_2 = -[\boldsymbol{\lambda} \times \boldsymbol{\sigma}]_1$

$[\boldsymbol{\sigma}, A] = -[\boldsymbol{\lambda} \times \boldsymbol{\sigma}]$

which is zero unless $\boldsymbol{\lambda} = 0$.

$A^{-1}\boldsymbol{\sigma}A - \boldsymbol{\sigma} = -A^{-1}[\boldsymbol{\lambda} \times \boldsymbol{\sigma}]$

$A^{-1}\sigma_1 A - \sigma_1 = (\lambda_0 - i\lambda_1\sigma_1 - i\lambda_2\sigma_2 - i\lambda_3\sigma_3)(-\lambda_2\sigma_3 + \lambda_3\sigma_2)$

$= -(\lambda_2^2 + \lambda_3^2)\,\sigma_1 + (\lambda_0\lambda_3 + \lambda_1\lambda_2)\sigma_2 + (\quad)\sigma_3$

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thus, if

$$A^{-1}\sigma A = B^{-1}\sigma B$$

for all of $\sigma_1, \sigma_2, \sigma_3$.

$$BA^{-1}\sigma AB^{-1} = \sigma$$

or

$$AB^{-1} = I \text{ or } -I.$$

$$AB = A \text{ or } -A.$$

~~Inclusion of Reflection~~
~~(Improper rotation)~~

Thus, $\mathfrak{A}_2'$ (unimodular unitary group of transformations in two dimensions) is two-valued representation of the group of proper rotations $\mathfrak{A}_3'$. Let us call this representation $\mathfrak{D}_{\frac{1}{2}}$, for the reasons which will be given in the following.

Then any representation $\mathfrak{C}_f$ of $\mathfrak{A}_2'$ in $f+1$ dimensional space becomes also the representation of $\mathfrak{A}_3'$. In this sense denote $\mathfrak{D}_l$ instead of $\mathfrak{C}_f$, where $l = f/2$. If f is odd or l is a half integer, the repres $\mathfrak{D}_l$ is double-valued, but if f is even or l is an integer, the repres $\mathfrak{D}_l$ is singled-valued. In particular, for

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$f=2$, $l=1$, $\mathfrak{D}_f$ is identical with the group of proper rotations d_3', so that it is ~~univocally~~ singled-valued. In other words, matrices A and $-A$ of $D_{1/2}$ induce the same matrix of D_1. Similarly, $D_{1/2}$ induces the same matrix of $\mathfrak{D}_l$. $A, -A$ of ~~two different~~ matrices differing only by sign of $\mathfrak{D}_l$ (l: half-integer). ~~In order that the representation $\mathfrak{D}_l$ of d_3' is continuous,~~

$$\left(\begin{array}{l} A \text{ of } u_2 \text{ or } D_{1/2} \\ -A \text{ of } u_2 \text{ or } D_{1/2} \end{array}\right) \text{ of } d_3' \text{ or } \mathfrak{D}_1$$

$$\ast) \left(\begin{array}{l} b \text{ of } \mathfrak{D}_{3/2} \\ -b \text{ of } \mathfrak{D}_{3/2} \end{array}\right) \to c \text{ of } \mathfrak{D}_2 \to (\quad)$$

~~Connect.~~ Relation to Spherical Harmonics ~~for integral l.~~ ~~If we take the case l integer,~~ We have a $j+1 = 2l+1$-dimensional representation $\mathfrak{D}_l$ of d_3'. We have $2l+1$ variables

$$\zeta_{l,m} = \frac{\xi_1^{j} \eta_2^{k}}{\sqrt{j!\,k!}}$$

$$m = j\,\frac{k-j}{2}$$

$$l = \frac{j+k}{2}$$

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which transform amongst themselves under rotations. m can take the values

$$m = \frac{-j+k}{2} + k \qquad k = 0, 1, 2, \cdots 2l$$
$$= l - k \qquad k - l$$

or

$$m = -l, -l+1, \cdots l-1, +l$$

which are half-integer or integer according as l is integer or half-integer. Now if we restrict ourselves to the subgroup of d_3 consisting of rotations around z-axis. Then, it induces

$$\left.\begin{aligned} \zeta' &= e^{-i\frac{w}{2}} \zeta \\ \eta' &= e^{i\frac{w}{2}} \eta \end{aligned}\right\}$$

$$\zeta'_{l,m} = e^{+imw} \zeta_{l,m}$$

for integer l and m,

Thus, the behavior of $\zeta_{l,m}$ is similar to that of the eigenfunctions

$$e^{im\varphi}$$

in q.m. where φ is the azymuth, ~~(i.e. under a rotation of angle w around z-axis, $\varphi \to \varphi' = \varphi - w$.)~~

If we take the total eigenfunctions in polar coord. (r, θ, φ) in the central field, the radial part does not change under rotation and the angular part

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$Y_{l,m}(\theta,\varphi)$

transforms linearly among themselves for a fixed $l (= 0, 1, 2, \ldots)$. Thus, the spherical harmonics of order l can be chosen as the substitutum of the $2l+1$-dimensional representation of the rotation group. We have a unitary representation if we normalize them:

$$\iint \tilde{Y}_{l,m} Y_{l',m'} \sin\theta \, d\theta \, d\varphi = \delta_{ll'} \delta_{mm'},$$

If we accept the theorem (without proof, yet) that all of the non-equivalent representations are obtained by the representations $D_{1/2}$ D_1 $D_{3/2} \cdots$, then the representation by using $Y_{l,m}$ must be equivalent to D_l.

In the case of half-integral l, there ~~If we construct a direct product~~ ~~$D_l \times D_{1/2} = D_{l+\frac{1}{2}} + D_{l-\frac{1}{2}}$~~ is not very simple relation to $Y_{l,m}$. However, for the case $l = \frac{1}{2}$, we have shown the connection with the spin. ~~In general~~

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Now ~~to note~~ we have ~~obtain~~ the important formula

$$D_l \times D_{l'} = D_{l+l'} + D_{l+l'-1} + \cdots + D_{|l-l'|},$$

which is the direct consequence of Clebsch-Gordan ~~theorem~~ formula between the representations of O_3.

~~Problem:~~ We above ~~can verify~~ Prove the relation directly by making use of the behavior of the monomials

$$\zeta_{l,m} = \frac{3^i \eta^k}{\sqrt{j!\,k!}}$$

with respect to the rotation around the ~~z~~ z-axis,

$$\left.\begin{aligned} \zeta'_{l,m} &= e^{-im\omega}\zeta_{l,m} \\ z'_{l',m'} &= e^{-im'\omega} z_{l,m} \end{aligned}\right\}$$

$$\zeta'_{lm} z'_{l'm'} = e^{-i(m+m')\omega} \zeta_{l,m} z_{l,m}$$

Now, suppose that we have completely reduced the ~~matr~~ representation $D_l \times D_{l'}$ into the sum of representations $D_{\cdots}$ etc.

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(~~they~~ L are integers or half-integers. Some L may appear more than once.)

D_L : $u_{L,M}$

$M = L, L-1, \ldots -L$.

$u_{L,M}$ transforms under the rotation around the z-axis of angle α:

$$u'_{L,M} = e^{-iM\alpha} u_{L,M}$$

Now among all linear combinations of $\xi_{l,m}\xi'_{l',m'}$, there is only one polynomial which transforms as $u_{L,M}$, ~~that is~~ if we take $M = m+m'$. $L = l+l'$. There is none for $M > l+l'$. Thus, there must be contained in $D_l \times D_{l'}$ ~~one representation~~ the representation $D_{l+l'}$ once and only once. This implies that ~~the~~ a certain polynomial which behaves like $u_{l+l',\,l+l'}$ is needs for $D_{l+l'}$. There are two linearly independent polynomials $\xi_{l,l}\xi'_{l',l'-1}$ and $\xi_{l,l-1}\xi'_{l',l'}$. Thus we are left with only one polynomial which behaves like $u_{l+l'-1,\,l+l'-1}$. Thus, $D_{l+l'-1}$ must be included once. We can go further similarly. We can see the

$\xi_{l,l}\xi'_{l',l'}$

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whole situation clearly by ~~work~~ drawing the scheme;

$D_l \times D_{l'}$, (m, m')

$M = l + l'$ $\quad l+l'-1 \quad l+l'-2 \cdots \cdots$

$l+l'-1 \quad l+l'-2 \cdots$

$l+l'-2 \cdots$

$\cdots$

$|l-l'| \cdots$

$(l, l'-1) \quad (l-1, l')$

$(l, l'-2) \quad (l-1, l'-1) \quad (l-2, l')$

Problem: Show that the representation D_1 is ~~identical with~~ equivalent to the group of three dimensional ~~transformation~~ rotation of (x, y, z); by making use of the fact that (ξ, η) transform like $(\eta, -\xi)$

$$\left.\begin{aligned} \xi' &= \alpha \xi + \beta \eta \\ \eta' &= -\beta \xi + \alpha \eta \end{aligned}\right\}$$

~~so that~~

$$\left.\begin{aligned} \eta' &= \alpha \eta + \beta(-\xi) \\ \xi' &= \beta \eta + \alpha \xi \end{aligned}\right\}$$

thus

$$\left.\begin{aligned} x &\sim -\xi^2 + \eta^2 \\ y &\sim -i(\xi^2 + \eta^2) \\ z &\sim 2\xi\eta \end{aligned}\right\} \qquad \left.\begin{aligned} x + iy &\sim 2\eta^2 \\ x - iy &\sim -2\xi^2 \\ z &\sim 2\xi\eta \end{aligned}\right.$$

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$$\begin{pmatrix} x+iy \\ z \\ x-iy \end{pmatrix} = \begin{pmatrix} r\cdot\sin\theta e^{i\varphi} \\ r\cos\theta \\ r\sin\theta e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \eta^2 \\ \sqrt{2}\eta\xi \\ \xi^2 \end{pmatrix}$$

$Y_{l,m}(\theta,\varphi)$:

$l=1$,

$$Y_{1,1} = \frac{1}{2\pi}\frac{3}{4}\sin\theta e^{i\varphi} \qquad Y_{1,0} = \frac{1}{2\pi}\frac{3}{2}\cos\theta$$

$$Y_{1,-1} = \frac{1}{2\pi}\frac{3}{4}\sin\theta e^{-i\varphi}$$

$$\iint \sin^2\theta\cdot\sin\theta\, d\theta d\varphi = 2\pi\cdot\int_{-1}^{+1}(1-x^2)dx$$

$$= (2\pi)\left(2-\frac{2}{3}\right) = (2\pi)\frac{4}{3}$$

$$\iint \cos^2\theta\sin\theta\, d\theta d\varphi = 2\pi\cdot\frac{2}{3}$$

$\rightarrow (Y_{1,1}, Y_{1,0}\ Y_{1,-1})$

$\rightarrow \left(\frac{\eta^2}{\sqrt{2}}, \xi\eta, -\frac{\xi^2}{\sqrt{2}}\right)$

$$\left(\frac{\xi^j\eta^k}{\sqrt{j!\,k!}}\right)$$

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Problem:

Show explicitly for $l=1$ and $l=2$ that

$$Y_{l\,m}(\theta,\varphi) \qquad m=-l,\ \cdots +l$$

transforms like

$$\frac{\xi^j \eta^k}{\sqrt{j!\,k!}}$$

$$j+k=2l$$

$$k-j=2m$$

if we normalize $Y_{l\,m}(\theta,\varphi)$:

$$\int \tilde{Y}_{l,m}(\theta,\varphi) Y_{l\,m'}(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = \delta_{ll'}\delta_{mm'}$$

~~It is, in general, true that~~ We have seen that by a rotation around the z-axis they transform similarly. If we can prove that they transform similarly for the rotation around x-axis, we are sure that they transform similarly for any rotation.

$$A_x^{(\chi)} = \begin{pmatrix} \cos\frac{\chi}{2} & -i\sin\frac{\chi}{2} \\ -i\sin\frac{\chi}{2} & \cos\frac{\chi}{2} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\frac{\xi^j \eta^k}{\sqrt{j!\,k!}} = \left(\cos\frac{\chi}{2}\cdot\xi - i\sin\frac{\chi}{2}\cdot\eta\right)^j \left(-i\sin\frac{\chi}{2}\xi + \cos\frac{\chi}{2}\eta\right)^k$$

$$= \sum_{p+q=2l} \frac{\xi^p \eta^q}{\sqrt{p'!\,q'!}} \phi_{p,q}(\chi) \qquad \begin{matrix} p'+q'=p \\ (j-p')+(k-q')=q \end{matrix}$$

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$$\left.\begin{aligned} x' &= x \\ y' &= y\cos\chi - z\sin\chi \\ z' &= y\sin\chi + z\cos\chi \end{aligned}\right\}$$

$$x' = r'\sin\theta'\cos\varphi' = r\sin\theta\cos\varphi$$

$$y' = r'\sin\theta'\sin\varphi' = r(\sin\theta\sin\varphi\cos\chi - \cos\theta\sin\chi)$$

$$z' = r'\cos\theta' = r(\sin\theta\sin\varphi\sin\chi + \cos\theta\cos\chi)$$

$$\sin\theta'\cos\varphi' = \sin\theta\cos\varphi$$

$$\sin\theta'\sin\varphi' = \sin\theta\sin\varphi\cos\chi - \cos\theta\sin\chi$$

$$\cos\theta' = \sin\theta\sin\varphi\sin\chi + \cos\theta\cos\chi$$

$$Y_{l,m}(\theta',\varphi') = P_{l,m}(\cos\theta')e^{im\varphi'}$$

$$= P_{l,m}(\sin\theta\sin\varphi\sin\chi + \cos\theta\cos\chi) \times e^{im\varphi'}$$

$$\sum_{p'+q'=p} \frac{\left(\cos\frac{\chi}{2}\right)^{p'}\left(-i\sin\frac{\chi}{2}\right)^{j-p'}}{p'!\,(j-p')!} \cdot \frac{\left(-i\sin\frac{\chi}{2}\right)^{k-q'}\left(\cos\frac{\chi}{2}\right)^{k-q'+p'}}{q'!\,(k-q')!}$$

$$= \sum_{p'+q'=p} \frac{\left(\cos\frac{\chi}{2}\right)^{k+p'-q'}\left(-i\sin\frac{\chi}{2}\right)^{j-p'+q'}}{p'!\,(j-p')!}$$

$$p'=q'=2m' \qquad = \sum \frac{\left(\cos\frac{\chi}{2}\right)^{k+2m'}\left(-i\sin\frac{\chi}{2}\right)^{j-2m'}}{p''!\,(j-p')!}$$

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whole group d_3 of proper and improper rotations.

The above considerations are restricted to the proper rotations

$$x_j' = a_{jk} x_k$$

with $\| a_{jk} \| = 1$.

There are improper rotations with

$$\| a_{jk} \| = -1,$$

because the conditions

$$\sum a_{jk} a_{jl} = \delta_{kl}$$

implies that

$$\| a^\dagger \cdot a \| = 1,$$

$$\| a^\dagger \| \| a \| = \| 1 \|$$

or

$$(\| a_{jk} \|)^2 = 1.$$

or

$$\| a_{jk} \| = \pm 1.$$

~~There are~~ The case with -1 was not included ~~Now~~ in the above arguments. Namely, the reflection R with

$$R: \quad a_{jk} = -\delta_{jk}$$

was not there, because

$$x = \bar{\psi} \sigma_1 \psi \qquad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \bar{\psi} = (\bar{\xi}, \bar{\eta})$$

$$y = \bar{\psi} \sigma_2 \psi$$

$$z = \bar{\psi} \sigma_3 \psi$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$x' = -x, \quad y' = y, \quad z' = z$

~~implies that~~ $A = \lambda_0 + i\lambda_1 \sigma_1 + i\lambda_2 \sigma_2 + i\lambda_3 \sigma_3$ }

~~$A^{-1} \sigma_1 A =$~~ $A^{-1} = \lambda_0 - i\lambda_1 \sigma_1 - i\lambda_2 \sigma_2 - i\lambda_3 \sigma_3$ }

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$$x'_j = a_{jk} x_k$$

$$\rightarrow \quad \bar{\psi}' \sigma_j \bar{\psi}' = \bar{\psi} A^{-1} \sigma_j A \psi$$
$$= a_{jk} \bar{\psi} \sigma_k \psi$$

or $$A^{-1} \sigma_j A = a_{jk} \sigma_k$$

Now $x' = -x$, $y' = -y$, $z' = -z$

$$\rightarrow \quad A^{-1} \sigma_j A = -\sigma_j$$
$$\sigma_j A = -A \sigma_j$$

Obviously there's not such matrix A which anticommutes with all $\sigma_1, \sigma_2, \sigma_3$. Thus, ~~and~~ $x_j \rightarrow -x_j$ cannot be included in the representation D_+, so are any other improper rotations cannot, ~~(If they can,~~ (Because any improper rotation A is a product of reflection R and a proper rotation A', which has the inverse A'^{-1}. Since A'^{-1} is in D_+, so is $A^{-1} = A'^{-1} A' R A = R$, provided that A is in D_+. This is a contradiction)

Now the group of proper rotations

$$d_3 : \quad A, B, C, \ldots$$

can be extended to the whole group

$$d_3 : \quad A, B, C, \ldots$$
$$RA, RB, RC, \ldots$$

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(48)

R is commutative with all ~~elem~~ rotations and $R^2 = I$ where I is identity. ~~There is two~~ Now if we go back to the group v_2, one can extend it by introducing ~~the~~ new element r in addition to

$$v_2: \quad a, b, c, \ldots$$

and corresponding new elements

$$ra, rb, rc, \ldots$$

We assume that

$$ra = ar$$

$$r^2 = 1.$$

~~Then~~ Let us call this new group v_2. Now ~~for~~ any representation of C_f of v_2 can be extend to the representation of v_2 either by associating ~~the~~ ~~number~~ the unit matrix $+1$ or -1 to r. We obtain $C_f^{(+)}$, $C_f^{(-)}$. We call this ± signature of the representation. If we consider the connection with the rotation groups, the choice of signature is not arbitrary: the rotation group itself is ~~the~~

$$C_2^{(-)} = D_1^{(-)},$$

because the reflection R ~~who~~ of d_3 which corresponds to r of v_2 has the matrix -1. If we ignore the spins of the particles,

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the eigenfunctions with the ~~total~~ resultant azymuthal quantum number l ~~even~~ & change or do not change sign by ~~R~~flection of coordinates of particles according as l is ~~odd~~ or even. So we have only the representations

$$D_0^{(+)}, D_1^{(-)}, D_2^{(+)}, D_3^{(-)} \cdots\cdots$$

Thus, we have the even or odd parity coincides with the even or odd l. But if one included the spins, the situation is more complicated, ~~There are other~~ We may well have $D_0^{(-)}, D_1^{(+)}$ ..... as well, because the spin variables ~~are~~ must behave like the angular momentum

$$[r \times p].$$

which does not change by reflection in contrast to r vector itself.

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11. Lorentz Group

$$\left.\begin{aligned} \xi' &= a\xi + b\eta \\ \eta' &= c\xi + d\eta \end{aligned}\right\} \quad ad - bc \neq 0$$

Real variables

$$\left.\begin{aligned} x_0 &= \bar{\xi}\xi + \bar{\eta}\eta \\ x_1 &= \bar{\eta}\xi + \bar{\xi}\eta \\ x_2 &= i(\bar{\eta}\xi - \bar{\xi}\eta) \\ x_3 &= \bar{\xi}\xi - \bar{\eta}\eta \end{aligned}\right\}$$

$$(-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0)$$

transform linearly (with real coefficients) and leaves the equation

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

invariant.

~~$\psi' = A\psi$~~ $\sigma_0 = 1$.

~~$\bar{\psi}' = \bar{\psi} A^*$~~ $\zeta = \lambda_\mu \sigma_\mu$

~~$A = \lambda_0 + \sum_j \lambda_j \sigma_j$~~ ~~$\lambda_0, \lambda_j$: complex~~

~~$A^* = \tilde{\lambda}_0 + \sum \tilde{\lambda}_j \sigma_j = \tilde{\lambda}_\mu \sigma_\mu$~~

~~$A^* \sigma_\mu A = \sum a_{\mu\nu} \sigma_\nu$~~ ($\xi, \eta$) and x_μ correspond. between

~~through~~ modify the above cond. and

Now let us consider a quadratic form according to Van der Waerden

$$c_{11}\bar{\xi}\xi + c_{21}\bar{\eta}\xi + c_{12}\bar{\xi}\eta + c_{22}\bar{\eta}\eta$$

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By a linear transformation of (ξ, η) $\to (\xi', \eta')$, it will become

$$C'_{11}\,\tilde{\xi}'\xi' + C'_{21}\,\tilde{\eta}'\xi' + C'_{12}\,\tilde{\xi}'\eta' + C'_{22}\,\tilde{\eta}'\eta'.$$

Now the above form is

$$(\tilde{\xi}, \tilde{\eta}) \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = (\tilde{\xi}', \tilde{\eta}') \begin{pmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$

C $\qquad$ C'

or

~~$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = A^* \begin{pmatrix} C_{12} \end{pmatrix}$~~

$$C = A^* C' A$$

or

$$C' = (A^*)^{-1} C^* A^{-1}.$$

Thus, if we restrict ourselves to unimodular transformations with

$$ad - bc = 1.$$

$$\|C'\| = \|C\|$$

$$C'_{11}C'_{22} - C'_{12}C'_{21} = C_{11}C_{22} - C_{12}C_{21}$$

is invariant.

~~Thus, if we define~~ ~~take~~

x $C_{11} = x_0 + x_3$

~~$C_{22} = x_0 - x_3$~~

~~$C_{12} =$~~

~~$C_{21} =$~~

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If we take C: Hermitian, C' will be Hermitian, too. Then one finds

$$\left.\begin{aligned} C_{11} &= x_0 + x_3 \\ C_{22} &= x_0 - x_3 \\ C_{12} &= x_1 - i x_2 \\ C_{21} &= x_1 + i x_2 \end{aligned}\right\},$$

the real variables (x_0, x_1, x_2, x_3) transform linearly (with real coeff.) and

$$C_{11}C_{22} - C_{12}C_{21} = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

is unchanged. Thus we have the representation of the Lorentz group.

two dimensional

$$C = \sum_{\mu=0}^{3} x_\mu \sigma_\mu$$

$$x_\mu \sigma_\mu = A^*(x'_\mu \sigma'_\mu) A$$

$$= x'_\mu \cdot (A^* \sigma'_\mu A)$$

$$x_\mu \left((A^*)^{-1} \sigma_\mu A^{-1}\right) = x'_\mu \cdot \sigma_\mu$$

$$= a_{\mu\lambda} x_\lambda \sigma_\mu$$

$$(A^*)^{-1} \sigma_\lambda A^{-1} = a_{\mu\lambda} \sigma_\mu$$

$$(A^*)^{-1} (a_{\nu\lambda} \sigma_\lambda) A^{-1} = \sigma_\nu$$

$$(A^*)^{-1} \sigma_\nu A^{-1} = \sigma_\mu a_{\mu\nu}$$

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Or if we write

$$A^{-1} = S^{*}$$

$$S^{*} \sigma_\nu S = \sigma_\mu a_{\mu\nu}$$

which is the well known relation is Dirac's theory of spin, ~~however~~ except that we are considering two-two matrices σ's instead of γ's which are four-four matrices.

This difference comes from the fact that we are dealing with the proper Lorentz transformations alone.

Namely, $a_{jk} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

is not there;

---

* $C' = (A^*)^{-1} C A^{-1}$

$C'' = (B^*)^{-1} C' B^{-1}$

$= (B^*)(A^*)^{-1} C' A^{-1} B^{-1}$

$A^{-1} = S$
$B^{-1} = T$

$= ((BA)^*)^{-1} \cdot C' \cdot (BA)^{-1}$

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~~$(A^{*}A^{-1})$~~ $\sigma_\nu A^{-1} = \sigma_\mu A^{*} a_{\mu\nu}$

$\nu=0$; $A^{-1} = A^{*}$

Thus, A must be unimodular and unitary (as in the case of rotations in three dimensions). We have shown already that there is no matrix A^{-1} satisfying

$\sigma_\nu A^{-1} = -A^{-1}\sigma_\nu$ for $\nu=1,2,3$.

So improper rotations are no more.

Also. $a_{jk} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ is not more

$\nu=0$; $A^{-1} = -A^{*}$

~~$A = iB$; $-iB^{-1} = +iB^{*}$~~

$\nu=1,2,3$;

$\sigma_\nu A^{-1} = -A^{-1}\sigma_\nu$

Again there's no such matrix A^{-1}.

There's no

$a_{jk} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$

either because

$A^{-1} = -A^{*}$, $\sigma_\nu A^{-1} = A^{-1}\sigma_\nu$ $\nu=1,2,3$

Both of

$A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

do not satisfy $A^{-1} = -A^{*}$.

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~~Thus~~ Now, a most general Lorentz transformations which leaves

$$x^2+y^2=x_0^2+x_1^2+x_2^2+x_3^2$$

invariant can be ~~determined~~ first classified into 2 parts:

1 class: $a_{00}\geq 1$,

2 class: $a_{00}\leq 1$.

~~The latter~~ If ~~we choose~~ the matrix $(a_{\mu\nu})$ belongs to the class (2), ~~other~~ there ~~could be~~ exists a matrix $(-a_{\mu\nu})$ belonging to the class (1). Conversely, any matrix of class (2) can be constructed from a certain matrix of (1) by multiplying by

$$-1=\begin{pmatrix}-1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$\sum_\mu a_{\lambda\mu}x_\mu \cdot a_{\lambda\nu}x_\nu = \sum_\mu \delta_{\mu\nu}x_\mu x_\nu$$

$$\sum_\lambda a_{\lambda\mu}a_{\lambda\nu}=\delta_{\mu\nu}$$

$$-a_{00}^2+a_{10}^2+a_{20}^2+a_{30}^2\equiv -1$$

$$a_{00}^2\geq 1,$$

$$-a_{01}^2+a_{11}^2+a_{21}^2+a_{31}^2=1$$

$$a_{11}^2+a_{21}^2+a_{31}^2\geq 1$$

$$a_{00}a_{01}+a_{10}a_{11}+a_{20}a_{21}+a_{30}a_{31}=0 \quad \text{etc}$$

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~~Since~~ By the transf. of (2) the direction of time is reversed, whereas by the transf. of (1) it is not. ~~All~~ the transf. of class (1) constitute the subgroup of whole Lorentz transformation. ~~which is~~ ~~Now in this subgroup~~ We have shown that I_1 was not there in the expr. of C_2', so that any (4-dim.) ~~class~~ transf. of (2) must be absent, too, ~~otherwise~~ provided that all the transf. of (1) are present. (which is not exactly true, but we can easily modify the proof argument as will be shown immediately.)

Now, ~~another~~ any of the transformations of class (1) can be constructed as ~~from~~ a product of

(i) ~~rotation~~ proper

(ii) reflection (in 3-space)

(iii) Lorentz transformation in the narrow sense of the type

$$x_0' = \frac{x_0 + \beta x_1}{\sqrt{1-\beta^2}} \qquad x_2' = x_2$$

$$x_1' = \frac{\beta x_0 + x_1}{\sqrt{1-\beta^2}} \qquad x_3' = x_3$$

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~~More~~ It is ~~more~~ convincing to see the following situation.

$$x'_\mu = a_{\mu\nu} x_\nu$$

$$x_\lambda = b_{\lambda\mu} x'_\mu$$

$$b_{\lambda\mu} a_{\mu\nu} = \delta_{\lambda\nu}$$

$$\varepsilon_\mu a_{\mu\lambda} \cdot a_{\mu\nu} = \varepsilon_\lambda \delta_{\lambda\nu}$$

$$\varepsilon_\mu \varepsilon_\lambda a_{\mu\lambda} \cdot a_{\mu\nu} = \delta_{\lambda\nu}$$

$$b_{\lambda\mu} = \varepsilon_\mu \varepsilon_\lambda a_{\mu\lambda}$$

~~On the~~ Thus, $\| b_{00} \| \geq 1$

On the other hand, the determinant

$$\| a_{\mu\nu} \| = a_{00} \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + a_{10} |(1)| \Big|$$

$$+ a_{20} |(2)| + a_{30} |(3)| \Big| = \pm 1$$

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & \\ a_{10} & a_{11} & \cdots & \\ a_{20} & a_{21} & & \\ a_{30} & a_{31} & \cdots & \end{vmatrix} = a_{00} \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + a_{11} |(1)| \Big|$$

$$+ a_{21} |(2)| + a_{31} |(3)| \Big| = 0$$

$$\cdots$$

$$\| b_{00} \| = \pm \| \quad \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} = \pm \| a_{jk} \| \quad \left(\approx \frac{\| a_{jk} \|}{\| a_{\mu\nu} \|} \right)$$

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$\| a_{jk} \| \geq 1$ or ≤ -1.

Thus, one can classify the general (must) homogeneous transf. into four classes:

(1) $a_{00} \geq 1$ $\quad \| a_{jk} \| \geq +1$ ($\| a_{jk} \| = +1$)

(2) $a_{00} \geq 1$ $\quad \| a_{jk} \| \leq -1$ ($\| a_{jk} \| = -1$)

(3) $a_{00} \leq 1$ $\quad \| a_{jk} \| \geq +1$

(4) $a_{00} \leq 1$ $\quad \| a_{jk} \| \leq -1$

~~(1),(2)~~ The (1) class (1) constitute a subgroup of the whole h.g. as well as of the larger subgroup (1), because from the above argument, it is clear that, if $\| a_{\mu\nu} \| = +1$, $\| b_{\mu\nu} \| = +1$, $\| a_{\mu\nu} b_{\nu\pi} \| = +1$ and if $a_{00}, b_{00} \geq 1$, then $a_{0\mu} \cdot b_{\mu 0} \geq 1$,

Problem !. because $\| a_{jk} \|, \| b_{jk} \| \geq 1$ and

Prove that $a_{0\mu} b_{\mu} \geq 1$ if $a_{00}, b_{00} \geq 1$

$$(a_{0j} b_{j0})^2 \leq (a_{10}^2 + a_{20}^2 + a_{30}^2) \times (b_{10}^2 + b_{20}^2 + b_{30}^2)$$

$$= (a_{00}^2 - 1)(b_{00}^2 - 1)$$

$$(a_{0j} b_{j0})^2 \leq a_{00}^2 b_{00}^2$$

$$|a_{0j} b_{j0}| \leq |a_{00} b_{00}| = a_{00} b_{00}$$

$a_{0\mu} b_{\mu 0} \geq 0$ or $a_{0\mu} b_{\mu 0} \geq 1$,

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Problem 2. Prove that, if $a_{00} \geq 1$, $\|a_{\mu\nu}\| = 1$ then $\|a_{jk}\| \geq 1$.

~~Conversely~~ From the conditions

$$a_{00} \geq 1 \qquad \|a_{\mu\nu}\| = 1$$

we obtain the relation

$$\|a_{jk}\| \geq 1 \quad (\equiv b_{00}\,\|a_{\mu\nu}\|)$$

(If $a_{00} \geq 1$ then $b_{00} = (a^{-1})_{00} \geq 1$, must because, ~~if~~

$$a_{0\mu} b_{\mu 0} \neq 1$$

$$\therefore |a_{00} b_{00}| \geq |a_{0j} b_{j0}|$$

Thus, $a_{00} b_{00}$ the sign of $a_{0\mu} b_{\mu 0}$ must be the same as that of $a_{00} b_{00}$.)

Now the elements of the subgroups $(1)_a$ can be constructed as the product of

(i) proper rotations

(ii) Lorentz transformations in the narrow sense.

On the other hand, one can find the corresponding linear transformation A of ξ, η for any element of $(1)_a$; as for the proper rotation, the situation is the same as before:

$$A_x^{-1} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\theta \end{pmatrix}$$

$$= \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} \cdot \sigma_1 .$$

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etc.

The Lorentz transf.

$$x_0' = \frac{x_0 + \beta x_1}{\sqrt{1-\beta^2}} \qquad x_2' = x_2$$

$$x_1' = \frac{\beta x_0 + x_1}{\sqrt{1-\beta^2}} \qquad x_3' = x_3$$

is obtained from

$$A^{-1} = \cosh\frac{\chi}{2}\sigma_0 + \sinh\frac{\chi}{2}\sigma_1$$

$$(A^{-1})^* = \cosh\frac{\chi}{2}\sigma_0 + \sinh\frac{\chi}{2}\sigma_1$$

$$(A^{-1})^*\begin{pmatrix}\sigma_2\\ \sigma_3\end{pmatrix}A^{-1} = \left\{\left(\cosh\frac{\chi}{2}\right)\sigma_0 + \left(\sinh\frac{\chi}{2}\right)\sigma_1\right\}\begin{pmatrix}\sigma_2\\ \sigma_3\end{pmatrix}\left\{\left(\cosh\frac{\chi}{2}\right)\sigma_0 + \left(\sinh\frac{\chi}{2}\right)\sigma_1\right\}$$

$$= \begin{pmatrix}\sigma_2\\ \sigma_3\end{pmatrix}$$

$$\cosh\frac{\chi}{2} = \frac{e^{\frac{\chi}{2}} + e^{-\frac{\chi}{2}}}{2} \qquad \sinh\frac{\chi}{2} = \frac{e^{\frac{\chi}{2}} - e^{-\frac{\chi}{2}}}{2}$$

$$(A^{-1})^*\sigma_0 A^{-1} = \left(\cosh^2\frac{\chi}{2} + \sinh^2\frac{\chi}{2}\right)\sigma_0 + 2\cosh\frac{\chi}{2}\sinh\frac{\chi}{2}\cdot\sigma_1$$

$$= \cosh\chi\cdot\sigma_0 + \sinh\chi\cdot\sigma_1$$

$$\cosh\chi = \frac{1}{\sqrt{1-\beta^2}} \qquad \sinh\chi = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\tanh\chi = \beta \qquad \chi = \operatorname{arctanh}\beta$$

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Thus, the subgroup (1)a is covered but covered twice, because there corresp. to $(a_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$,
two transformations

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

because

$$(A^{*-1})A^{-1} = 1$$

$$(A^{*-1})\sigma_j A^{-1} = \sigma_j$$

$A^{-1} = A^*$ unitary unimodular

$$\sigma_j A^{-1} = A^* \sigma_j$$

$$= A^{-1}\sigma_j$$

$A^{-1} = 1$ or -1,

Thus, C_2' is the two-valued ~~Now that~~ represent. of (1)a.
There's no element of (1)b.
∴ Otherwise, any element of (1)b can be obtained by multiplying

$$R = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

to a certain element of (1)a
(∵ ~~can~~ any element of (1)b multiplied

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by R must belong to the class (1)a.
But it was already shown that ~~R~~ was not there.

Similarly, (2)a(2)b cannot find corresponding elements in $\mathfrak{C}_2'$.

In order to have a representation of the whole group of L.T., it is necessary to consider something more complicated than the simple linear transf. in two dimensions, or, at least, some modification of the latter. will be shown also

It is ~~also~~ clear that the relativistic equation for the electron (with non-zero mass) can be obtained only by modifying the above formalism.
(Only possible relation between (ξ_1', ξ_2') linear with mass parameter $\varkappa$)
like

$$\left.\begin{aligned} c^{i1}\xi_1 + c^{i2}\xi_2 &= \varkappa \xi_1' \\ c^{i1}\xi_1 + c^{i2}\xi_2 &= \varkappa \xi_2' \end{aligned}\right\}$$

can not be conform to the relation

$$(p_0^2 - p_j^2 + m^2 c^2)\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = 0$$

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12. Spinor Analysis

$$\left.\begin{aligned}\xi_1' &= \alpha_{11}\xi_1 + \alpha_{12}\xi_2\\ \xi_2' &= \alpha_{21}\xi_1 + \alpha_{22}\xi_2\end{aligned}\right\}$$

$$\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix} = 1$$

$$\left.\begin{aligned}\tilde{\xi}_1' &= \tilde{\alpha}_{11}\tilde{\xi}_1 + \tilde{\alpha}_{12}\tilde{\xi}_2\\ \tilde{\xi}_2' &= \tilde{\alpha}_{21}\tilde{\xi}_1 + \tilde{\alpha}_{22}\tilde{\xi}_2\end{aligned}\right\}$$

~~$\xi_i' = \tilde{\alpha}_{11}\xi_i + \alpha_{12}$~~

$$\eta_i' = \tilde{\alpha}_{11}\eta_i + \tilde{\alpha}_{12}\eta_i$$

$$\eta_i' = \tilde{\alpha}_{21}\eta_i + \tilde{\alpha}_{22}\eta_i$$

(ξ_1, ξ_2), (η_1, η_2) : covariant spinors

$\xi' \equiv \xi_2$, $\xi^2 \equiv -\xi_1$

$\xi^1\eta_1 + \xi^2\eta_2 = \xi_2\eta_1 - \xi_1\eta_2$: invariant

(ξ^1, ξ^2), $(\xi^1, \xi^2) \equiv (\xi_2, -\xi_1)$:
contravariant spinors

$$C^{11}\xi_1\eta_1 + C^{21}\xi_1\eta_2 + C^{12}\xi_2\eta_1 + C^{22}\xi_2\eta_2$$

$$= C'^{11}\xi_1'\eta_1' + \cdots$$

$$\xi_i C^{is}\eta_s = \xi'_p C'^{pq}\eta'_q$$

$$= (\xi_1', \xi_2')\begin{pmatrix}\tilde{\alpha}_{11} & \tilde{\alpha}_{21}\\ \tilde{\alpha}_{12} & \tilde{\alpha}_{22}\end{pmatrix}\begin{pmatrix}C'^{11} & C'^{12}\\ C'^{21} & C'^{22}\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\eta_1'\\ \eta_2'\end{pmatrix}$$

$$C'' = \alpha^* C' \alpha$$

$\|C''\| = \|C'\|$. is invariant

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$$\begin{pmatrix} C^{i_1} & C^{i_2} \\ C^{\dot{i}_1} & C^{\dot{i}_2} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$$

$$\| C^{\dot{r}s} \| = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

invariant.

If x_μ are real $(C^{\dot{r}s})$ is Hermitian and hence (C') is Hermitian, so that x'_μ are real, too.

$$(C') = \sum_\mu x_\mu \sigma_\mu = x_0 + \sum_j x_j \sigma_j$$

Contravariant (mixed) spinor of the second rank $C^{\dot{r}s} \longrightarrow$ contravariant vector. (x_0, x_1, x_2, x_3).

Covariant (mixed) spinor of the second rank

$$\begin{pmatrix} C_{i_1} & C_{i_2} \\ C_{\dot{i}_1} & C_{\dot{i}_2} \end{pmatrix} = \begin{pmatrix} +x_0 + x_3 & -x_1 - i x_2 \\ -x_1 + i x_2 & x_0 + x_3 \end{pmatrix}$$

$$C_{\dot{r}s} C^{\dot{r}s} = 2(x_0^2 - x_1^2 - x_2^2 - x_3^2)$$

is invariant.

~~$(C^*_{\dot{r}s})$~~ Interchanging rows and columns

$$(C_{s\dot{r}}) = \begin{pmatrix} x_0 - x_3 & -x_1 + i x_2 \\ -x_1 - i x_2 & x_0 + x_3 \end{pmatrix}$$

$$= x_0 - \sum_j x_j \sigma_j$$

Thus, to the covariant spinor of 2nd

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rank, where corresponds the covariant vector $(x_0, -x_1, -x_2, -x_3)$ or $(-x_0, x_1, x_2, x_3)$ $(\sigma_{s\dot{r}})$

let us define the displacement spinor of the second rank by

$$(\partial^{\dot{r}s}) = \begin{pmatrix} -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3} \end{pmatrix}$$

$$= -\frac{\partial}{\partial x_0} + \sigma_j \frac{\partial}{\partial x_j}$$

and

$$(\sigma_{s\dot{r}}) = \begin{pmatrix} -\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_3} \end{pmatrix}$$

$$= -\frac{\partial}{\partial x_0} - \sigma_j \frac{\partial}{\partial x_j}$$

$$(\varphi_s) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad (\chi^{\dot{r}}) = \begin{pmatrix} \chi^{\dot{1}} \\ \chi^{\dot{2}} \end{pmatrix}$$

$$\partial^{\dot{r}t}\varphi_t = \varkappa \chi^{\dot{r}}$$

$$\partial_{s\dot{r}}\chi^{\dot{r}} = \varkappa \varphi_s$$

$$\partial_{s\dot{r}}\partial^{\dot{r}t}\varphi_t = \varkappa^2 \varphi_s$$

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)\begin{pmatrix} \varphi_s \\ \chi^{\dot{r}} \end{pmatrix} = -\varkappa^2 \begin{pmatrix} \varphi_s \\ \chi^{\dot{r}} \end{pmatrix}$$

$$(-p_0^2 + p_1^2 + p_2^2 + p_3^2)\psi = -m^2c^2\psi$$

$$\varkappa = \frac{mc}{\hbar}, \quad p_0 = -i\hbar\frac{\partial}{\partial x_0}, \quad p_j = -i\hbar\frac{\partial}{\partial x_j}$$

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$$\psi = \begin{pmatrix} \varphi_s \\ \chi^{\dot r} \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with: $\left(-\frac{\partial}{\partial x_0} + \rho_3 \sigma_j \frac{\partial}{\partial x_j} - i\rho_1 \varkappa\right)\psi = 0$ (+)

$$\left(-p_0 + \rho_3 \sigma_j p_j - \rho_1 mc\right)\psi = 0$$

or $\quad p_0 = -\frac{W}{c} \qquad W = ihc\frac{\partial}{\partial x_0}$

$$\left(\frac{W}{c} + \rho_3 \sigma_j p_j - \rho_1 mc\right)\psi = 0$$

Problem: This is slightly different from Dirac's wave eq. We need only a transf.

$$\psi' = A\psi \qquad \psi = A\psi'$$

$$A = \frac{1}{\sqrt{2}}(1 - i\rho_2)$$

$$A^* = \frac{1}{\sqrt{2}}(1 + i\rho_2)$$

$$A^*\rho_1 A = \tfrac{1}{2}(1+i\rho_2)\rho_1(1-i\rho_2)$$
$$= \rho_1 \cdot (-i\rho_2) = \rho_3$$

$$A^*\rho_3 A = \tfrac{1}{2}(1+i\rho_2)\rho_3(1-i\rho_2)$$
$$= \rho_3(-i\rho_2) = -\rho_1$$

Thus, we have

$$\left(\frac{W}{c} - \rho_1 \sigma_j p_j - \rho_3 mc\right)\psi' = 0.$$

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By multiplying (†) by $-i\rho_2$ from the left, we have

~~$-i\rho_1 \frac{\partial}{\partial x_0} + \rho_2 \sigma_j \frac{\partial}{\partial x_j} + \varkappa)$~~

$$\left(\rho_2 \frac{\partial}{\partial x_0} - i\rho_1 \sigma_j \frac{\partial}{\partial x_j} + \rho_3 \varkappa\right) \psi = 0 \quad (*)$$

Thus, we can easily see how one can include the transformations of class (1)b, (2)a, (2)b, in such a way that they leave (*) invariant:

(i) Space reflection of class (1)b

$x_0' = x_0$, $x_1' = -x_1$, $x_2' = -x_2$, $x_3' = -x_3$

$$\frac{\partial}{\partial x_0'} = \frac{\partial}{\partial x_0} \qquad \frac{\partial}{\partial x_j'} = -\frac{\partial}{\partial x_j}$$

$$\underline{\psi' = \rho_1 \psi} \quad (\text{and/or } \underline{\psi' = -\rho_1 \psi})$$

$$\left(\rho_2 \frac{\partial}{\partial x_0'} - i\rho_1 \sigma_j \frac{\partial}{\partial x_j'} + \rho_3 \varkappa\right) \rho_1 \psi = 0$$

or

$$-\rho_1 \left(\rho_2 \frac{\partial}{\partial x_0} + i\rho_1 \sigma_j \frac{\partial}{\partial x_j} - \rho_3 \varkappa\right) \psi = 0 .$$

(ii) Time reversal class (2)a, b

$x_0' = -x_0$, $x_j' = x_j$

$$\underline{\psi' = i\rho_2 \psi} \quad (\text{and/or } \psi' = -i\rho_2 \psi)$$

$$-i\rho_2 \left(-\rho_2 \frac{\partial}{\partial x_0'} + i\rho_1 \sigma_j \frac{\partial}{\partial x_j'} + \rho_3 \varkappa\right) \psi = 0$$

or $-i\rho_2 \left(+\rho_2 \frac{\partial}{\partial x_0} - i\rho_1 \sigma_j \frac{\partial}{\partial x_j} + \rho_3 \varkappa\right) \psi = 0$

(iii) S. T. Reflection: $-x_\mu' = x_\mu$

$\psi' = \rho_3 \psi$ (and/or $\psi' = -\rho_3 \psi$)

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(iv) Transformations of group $(1)_c$,

$$\varphi'_r = \alpha_{rs}\varphi_s \,, \quad \|\alpha_{rs}\| = 1$$

$$\left.\begin{aligned} \chi'_{i1} &= \tilde{\alpha}_{11}\chi_{i1} + \tilde{\alpha}_{12}\chi_{i2} \\ \chi'_{i2} &= \tilde{\alpha}_{21}\chi_{i1} + \tilde{\alpha}_{22}\chi_{i2} \end{aligned}\right\}$$

$$\chi^{i1} = \chi_{i2} \,, \quad \chi^{i2} = -\chi_{i1}$$

$$\left.\begin{aligned} \chi^{i1'} &= \tilde{\alpha}_{22}\chi^{i1} - \tilde{\alpha}_{21}\chi^{i2} \\ \chi^{i2'} &= -\tilde{\alpha}_{12}\chi^{i1} + \tilde{\alpha}_{11}\chi^{i2} \end{aligned}\right\}$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \lambda_0 + i\lambda_j\sigma_j$$

$$= \begin{pmatrix} \lambda_0 + i\lambda_3 & i\lambda_1 + \lambda_2 \\ i\lambda_1 - \lambda_2 & \lambda_0 - i\lambda_3 \end{pmatrix}$$

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

λ's : complex

$$\psi' = \begin{pmatrix} (\lambda_0 + i\lambda_j\sigma_j) & 0 \\ 0 & (\tilde{\lambda}_0 + i\tilde{\lambda}_j\sigma_j) \end{pmatrix}\psi$$

$$= \left\{(\lambda_0 + i\lambda_j\sigma_j)\frac{1+\beta_3}{2} + (\tilde{\lambda}_0 + i\tilde{\lambda}_j\sigma_j)\frac{1-\beta_3}{2}\right\}\psi$$

$$\left.\begin{aligned} \lambda_0 &= \mu_0 + i\nu_0 \\ \lambda_j &= \mu_j + i\nu_j \end{aligned}\right\} \quad \underline{\mu, \nu : \text{real}}$$

$$\psi' = \left\{(\mu_0 + i\mu_j\sigma_j) + (\nu_0 + i\nu_j\sigma_j)(i\beta_3)\right\}\psi$$

unimodular: $\|(\mu_0 + i\mu_j\sigma_j) + (\quad i\beta_3)\|$

$$\left.\begin{aligned} &(\mu_0^2 - \nu_0^2 + \mu_j^2 - \nu_j^2) = 1 \\ &\mu_0\nu_0 + \mu_j\nu_j = 0 \end{aligned}\right\}$$

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Thus, the whole group can be constructed from C_2 by adding

$+$ or $-$ ρ_1, $+$ or $-$ $i\rho_2$, $+$ or $-$ ρ_3

The matrices are all unimodular 4-4 matrices. This is a two-valued repr. of the entire l.g.

Problem: Show that the product of any of two of the ~~or~~ class consisting of

$$(\mu_0 + i\mu_j\sigma_j) + (\nu_0 + i\nu_j\sigma_j)\chi(i\rho_3)$$

$\pm\rho_1$, $\pm i\rho_2$, $\pm\rho_3$

again belong to the same class.

Relation to γ-formalism:

By multiplying $i\rho_1$ from the left, (1) becomes

$$\left(-i\rho_1\frac{\partial}{\partial x_0} + \rho_2\sigma_j\frac{\partial}{\partial x_j} + \varkappa\right)\psi = 0$$

$x_4 = ix_0$ $\qquad \frac{\partial}{\partial x_4} = -i\frac{\partial}{\partial x_0}$

$$\gamma_4 = \rho_1, \quad \gamma_1 = \rho_2\sigma_1, \quad \gamma_2 = \rho_2\sigma_2, \quad \gamma_3 = \rho_2\sigma_3$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \varkappa\right)\psi = 0$$

This, together with the adjoint equation

$$\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu - \varkappa\bar{\psi} = 0$$

can be deduced from the Lagrangian action principle

$$\delta\bar{L} = \int L\,(dx)^4$$

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where

$$L = \bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa\right)\psi$$

which must be invariant.

Thus,

$$\bar{\psi}\gamma_\mu\psi$$

must transform like a vector.

If we introduce the canonical variables

$$\psi_\alpha \to \pi_\alpha = \frac{\partial L}{\partial \frac{\partial \psi}{\partial x_4}} = \frac{\partial L}{i\frac{\partial \psi}{\partial x_4}} = -i(\bar{\psi}\gamma_4)_\alpha$$

$$H = \pi_\alpha \frac{\partial \psi_\alpha}{\partial x_0} - L$$

$$= -i\bar{\psi}\gamma_4\left(i\frac{\partial \psi}{\partial x_4}\right) - L$$

$$= \bar{\psi}\left(\gamma_j \frac{\partial}{\partial x_j} - \kappa\right)\psi$$

If we put in $i\pi_\alpha = \psi^*_\alpha = (\bar{\psi}\gamma_4)_\alpha$,

we have

$$H = \psi^*\left(-\rho_1\rho_2\sigma_j\frac{\partial}{\partial x_j} - \rho_1\kappa\right)\psi$$

$$= \psi^*\left(-i\rho_3\sigma_j\frac{\partial}{\partial x_j} - \rho_1\kappa\right)\psi$$

Thus, we have

$$\bar{\psi}\gamma_\mu\psi = \psi^*\gamma_4\gamma_\mu\psi$$

transform like a vector

If we use this notion

$$\gamma_4 = \rho_1, \quad \gamma_j = \rho_2\sigma_j$$

$$\gamma_4\gamma_j = -\gamma_j\gamma_4 = i\rho_3\sigma_j$$

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Thus, the whole group can be constructed from $\mathfrak{C}_2$ by adding

$+$ or $-$ ρ_1, $+$ or $-$ $i\rho_2$, $+$ or $-$ ρ_3

The matrices are all unimodular 4-4 matrices. this is a two-valued repr. of the entire l. g.

Problem: Show that the product of any of two of the ~~or~~ class consisting of

$$(\mu_0 + i\mu_j\sigma_j) + (\nu_0 + i\nu_j\sigma_j)\chi(i\rho_3)$$

$\pm\rho_1$, $\pm i\rho_2$, $\pm\rho_3$ same

again belong to the class.

Relation to γ-formalism:

By multiplying $i\rho_1$ from the left, (1) becomes

$$\left(-i\rho_1\frac{\partial}{\partial x_0} + \rho_2\sigma_j\frac{\partial}{\partial x_j} + \varkappa\right)\psi = 0$$

$x_4 = ix_0$ $\qquad \frac{\partial}{\partial x_4} = -i\frac{\partial}{\partial x_0}$

$$\gamma_4 = \rho_1, \quad \gamma_1 = \rho_2\sigma_1, \quad \gamma_2 = \rho_2\sigma_2, \quad \gamma_3 = \rho_2\sigma_3$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \varkappa\right)\psi = 0$$

This, together with the adjoint equation

$$\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu - \varkappa\bar{\psi} = 0$$

can be deduced from the lagrangian action principle

$$\delta\bar{L} = \int L\,(dx)^4$$

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where

$$L = \bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa\right)\psi$$

which must be invariant.

Thus,

$$\bar{\psi}\gamma_n\psi$$

must be transform like a vector.

If we introduce the canonical variables

$$\psi_\alpha \to \pi_\alpha = \frac{\partial L}{\partial \frac{\partial \psi}{\partial x_4}} = \frac{\partial L}{i\frac{\partial \psi}{\partial x_4}} = -i(\bar{\psi}\gamma_4)_\alpha$$

$$H = \pi_\alpha \frac{\partial \psi_\alpha}{\partial x_0} - L$$

$$= -i\bar{\psi}\gamma_4\left(i\frac{\partial \psi}{\partial x_4}\right) - L$$

$$= \bar{\psi}\left(-\gamma_j \frac{\partial}{\partial x_j} - \kappa\right)\psi$$

If we put in $i\pi_\alpha = \psi^*_\alpha = (\bar{\psi}\gamma_4)_\alpha$,

we have

$$H = \psi^*\left(-\rho_1\rho_2\sigma_j\frac{\partial}{\partial x_j} - \rho_1\kappa\right)\psi$$

$$= \psi^*\left(-i\rho_3\sigma_j\frac{\partial}{\partial x_j} - \rho_1\kappa\right)\psi$$

Thus, we have

$$\bar{\psi}\gamma_\mu\psi = \psi^*\gamma_4\gamma_\mu\psi$$

transform like a vector

If we use this notion

$$\gamma_4 = \rho_1, \quad \gamma_j = \rho_2\sigma_j$$

$$\gamma_4\gamma_j = -\gamma_j\gamma_4 = i\rho_3\sigma_j$$

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$$\left.\begin{aligned}\gamma_1\gamma_2 &= -\gamma_2\gamma_1 = i\sigma_3\\ \gamma_3\gamma_1 &= -\gamma_1\gamma_3 = i\sigma_2\\ \gamma_2\gamma_3 &= -\gamma_3\gamma_2 = i\sigma_1\end{aligned}\right\}$$

$$\gamma_1\gamma_2\gamma_3 = i\sigma_3\cdot\rho_2\sigma_3 = i\rho_2$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\rho_2\rho_1 = \rho_3 \quad \text{or} \quad \gamma_5 = \rho_3$$

Problem: ~~Show that~~ the elements of enlarged $\mathfrak{C}_2'$ are

$$(\mu_0 + i\mu_j\sigma_j) + (\nu_0 + i\nu_j\sigma_j)(i\rho_3)$$

$$= \mu_0 + \mu_j\gamma_k\gamma_l + (\nu_0 + \nu_j\gamma_k\gamma_l)(+i\gamma_5)$$

$$\sim \mu_0 + \lambda_{kl}\gamma_k\gamma_l + i\lambda_{4l}\gamma_4\gamma_l + \lambda_{\mu\nu}\gamma_\mu\gamma_\nu$$

$$+ i\nu_0\gamma_1\gamma_2\gamma_3\gamma_4 \left[\because (\mu_0^2 - \nu_0^2 + \mu_j^2 - \nu_j^2) \geq 1 \quad \mu_0^2 + \mu_j^2 > 1 + \nu_0^2 + \nu_j^2 \quad \mu_0\nu_0 + \mu_j\nu_j = 0\right]$$

S. R.

$$\pm\rho_1 = \pm\gamma_4 \quad \text{or} \quad +\gamma_4$$

T. R.

$$\pm i\rho_2 = \pm\gamma_1\gamma_2\gamma_3 \quad \text{or} \quad +\gamma_1\gamma_2\gamma_3$$

S. T. R.

$$\pm\rho_3 = \pm\gamma_5 = \pm\gamma_1\gamma_2\gamma_3\gamma_4 \quad \text{or} \quad +\gamma_1\gamma_2\gamma_3\gamma_4$$

$$\psi' = A\psi$$
$$\bar{\psi}' = \bar{\psi}A^{-1}$$
$$A^{-1}\gamma_\mu A = a_{\mu\nu}\gamma_\nu$$

S. T. R.

$$(\pm\gamma_4\gamma_2\gamma_3\gamma_4)\cdot\gamma_\mu\cdot(\pm\gamma_1\gamma_2\gamma_3\gamma_4) = -\gamma_\mu$$

(A) corresponding to ~~even~~ proper or improp. h. T. is even or odd with respect to γ_μ.

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Pseudospinor
(Generalization of the spinor)

S.R. $\begin{cases} \psi' = \pm \rho_1 \psi & (1)(2) \\ \psi' = \pm i \rho_1 \psi & (3)(4) \end{cases}$

(all of them ~~have~~ are unimodular)

~~S~~.R. $\begin{cases} \psi' = \pm i \rho_2 \psi & (a)(b) \\ \psi' = \pm \rho_2 \psi & (c)(d) \end{cases}$

S.T.R $\begin{cases} \psi' = \pm \rho_3 \psi & (\alpha)(\beta) \\ \psi' = \pm i \rho_3 \psi & (\gamma)(\delta) \end{cases}$

(1), (a), (α) : regular spinor ψ

(2), (b), (β) : pseudo spinor φ

$\bar{\psi} \gamma_\mu \psi$, $\bar{\varphi} \gamma_\mu \varphi$: regular vector.

	S.R.	T.R.	S.T.R
regular vector	+ − − −	− + + +	− − − −
$\bar{\psi} \gamma_\mu \psi$			
pseudo vector	− + + +	+ − − −	+ + + +

etc.

φ: (2), (b), (α)

− + + +	+ ~~+ + +~~	− − − −

Wataccake, Phys. Rev. 84 (1951), 1008.

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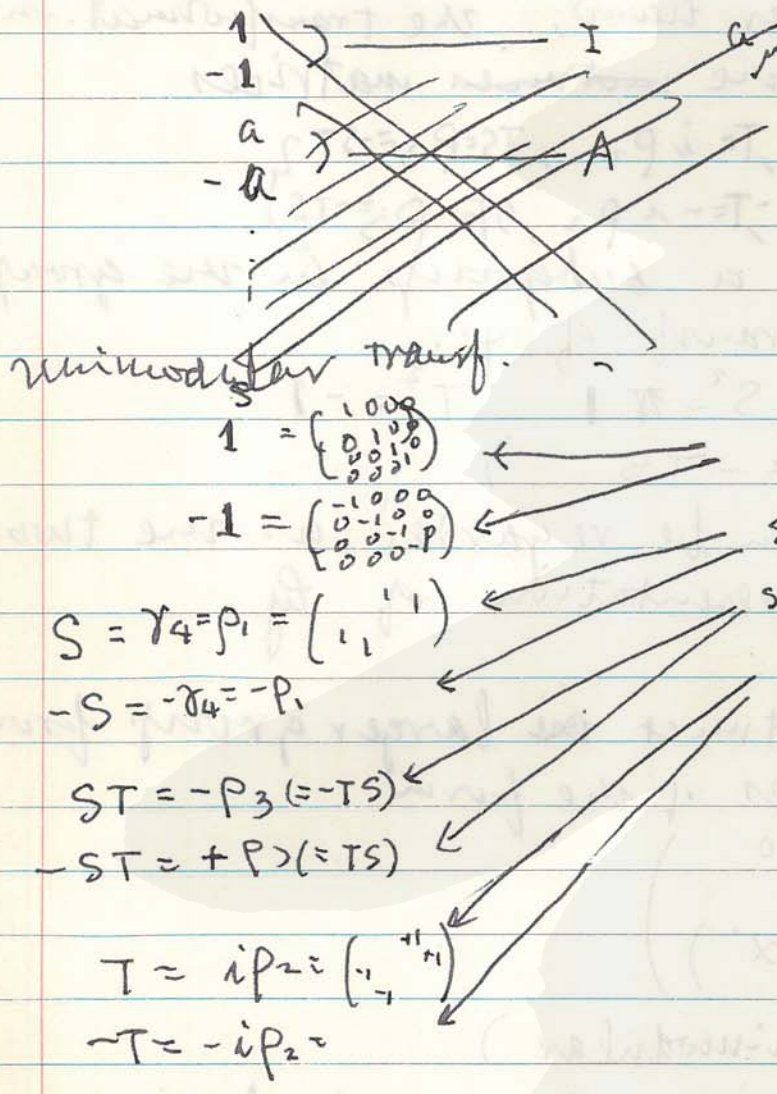
Structure of Whole Lorentz Group and its two valued representation

Class 1. $|a_{00}| \geq +1$ | $\|a_{\mu\nu}\| = +1$ | $1, \cancel{\quad}, a, b, c \cdots$ (subgroup)

Class 2. $|a_{00}| \geq +1$ | -1 | $s1, \cancel{\quad}, sa, sb, sc \cdots$

Class 3. $|a_{00}| \leq -1$ | $+1$ | $-1 = st1, \cancel{\quad}, sta, \cdots$

Class 4. $|a_{00}| \leq -1$ | -1 | $t1 \ \cancel{\quad}, ta, \cdots$



In the whole group of L. T., Of Class 1 is an invariant subgroup and one can construct a factor Of/g' consisting of four elements. ~~To obtain~~ Conversely,

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four elements

$$1,\quad s,\quad t,\quad st=ts$$

constitute a subgroup h of G. ~~h is~~ the whole h.g. is constructed from G' and h. ($s^2=t^2=1$) h ~~has the same~~ ~~On the other hand~~ is isomorphic to G/G'.

On the other hand, the transformations of ψ in the ~~coordinates~~ matrices

$$1,\ S=\rho_1,\ T=i\rho_2,\ TS=\rho_3=-ST$$
$$-1,\ -S=-\rho_1,\ -T=-i\rho_2,\ -ST=-\rho_3=TS$$

constitute a subgroup in the group of linear transf. of ψ.

(~~ρ_1~~ $S^2=1$ $T^2=-1$
$ST=-TS$)

The latter can be regarded as the two-valued representation of h.

If we construct the larger group from the matrices of the form

$$\begin{pmatrix} (\alpha) & 0 \\ 0 & (\alpha') \end{pmatrix}$$

(α, α' : unimodular)

and $\rho_1, i\rho_2, \rho_3$, one obtains a two-valued representation of G.

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Spinors of the higher rank

Just as in tensor analysis, one can ~~construct~~ define a great variety of spinors not the complex components.

(i) 1st rank: (ξ_1, ξ_2) $(\xi_{\dot 1}, \xi_{\dot 2})$

$(\xi^1, \xi^2) = (\xi_2, -\xi_1)$, $(\xi^{\dot 1}, \xi^{\dot 2})$

(ii) 2nd rank:

(mixed spinors) $C_{\dot r s} = C_{s \dot r} = (C_{11}, C_{12}, C_{\dot 2 1}, C_{\dot 2 2})$

$C^{\dot r s} = C^{s \dot r} =$ ~~$C_{\dot 2 2}$~~ C

$C^{11} = C_{\dot 2 2}$, $C^{12} = -C_{\dot 2 1}$

$C^{\dot 2 1} = C_{12}$, $C^{\dot 2 2} = C_{11}$

$C^{\dot r}{}_s$: $C^{\dot 1}{}_1 = C_{\dot 2 1}$ $C^{\dot 1}{}_2 = C_{\dot 2 2}$

$C^{\dot 2}{}_1 = -C_{11}$ $C^{\dot 2}{}_2 = -C_{12}$

$C^s{}_{\dot r}$: . . .

Any of these mixed spinors of the 2nd rank ~~can~~ a vector corresponds. For instance

$$C^{\dot r s} = \begin{pmatrix} C^{\dot 1 1} & C^{\dot 1 2} \\ C^{\dot 2 1} & C^{\dot 2 2} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$$

In general, (x_0, x_1, x_2, x_3) can be a complex vector. It is a real vector, if $C^{\dot 1 1}, C^{\dot 2 2}$ are real and $C^{\dot 1 2} = C^{\dot 2 1}$ or $(C^{\dot i j})$ is a hermitian matrix.

(iii) Contraction of two spinors results in a spinor of lower rank:

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For instance, $C^{\dot r s}\xi_s$ transforms like a spinor $\eta^{\dot r}$.

$$\left.\begin{aligned}\xi_1' &= \alpha_{11}\xi_1 + \alpha_{12}\xi_2\\ \xi_2' &= \alpha_{21}\xi_1 + \alpha_{22}\xi_2\end{aligned}\right\}$$

$$C'^{\dot r s}\xi_s' = (\alpha^*)^{-1}(C)(\alpha^{-1})(\alpha)\xi$$
$$= (\alpha^*)^{-1} C \xi$$
$$\eta'^{\dot r} = \eta_{\dot t}(\alpha^*)_{\dot t \dot r}$$
$$\eta'^{\dot r} = \eta_{\dot t}(\alpha^*)^{-1}_{\dot t s}\eta^s$$

(iv) Spinor of rank 2nd rank

$$C'_{rs} = \alpha_{rt}\alpha_{su}C_{tu} = \alpha_{rt}C_{tu}\alpha^{\dagger}_{us}$$

$$\begin{pmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22}\end{pmatrix} = \begin{pmatrix} \alpha_{11}\alpha_{11}C_{11} + \alpha_{11}\alpha_{12}C_{12} + \alpha_{12}\alpha_{12}C_{21} + \alpha_{12}\alpha_{12}C_{22} & \alpha_{11}\alpha_{21}C_{11} + \alpha_{12}\alpha_{22}C_{12} + \cdots \\ \alpha_{21}\alpha_{11}C + \cdots & \cdots \end{pmatrix}$$

$$C_{rs} = f_{rs} + g_{rs}$$

$$f_{rs} = \tfrac{1}{2}(C_{rs} + C_{sr}) = \begin{pmatrix} -f_2 - if_1 & if_3 \\ if_3 & -f_2 + if_1\end{pmatrix}$$

$$g_{rs} = \tfrac{1}{2}(C_{rs} - C_{sr}) = \begin{pmatrix} 0 & -g \\ +g & 0\end{pmatrix}$$

where f's and g are complex numbers.

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Ci sym. and anti. spinors transform into each of them, because

$$C'_{rs} \pm C'_{sr} = \alpha_{rt}\alpha_{su} C_{tu} \pm \alpha_{st}\alpha_{ru} C_{ut}$$
$$= \alpha_{rt}\alpha_{su}(C_{tu} \pm C_{ut})$$

In particular, an antisym. spinor transforms like

$$g' = \tfrac{1}{2}(C'_{21} - C'_{12}) = \tfrac{1}{2}\alpha_{2t}\alpha_{1u}(C_{21} - C_{12})$$
$$+ \tfrac{1}{2}\alpha_{2t}\alpha_{1u}(C_{12} - C_{21})$$
$$= (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\, g .$$

or $$g' = g$$

Thus, an antisymmetric spinor

$$C_{rs} = \begin{pmatrix} 0 & -g \\ +g & 0 \end{pmatrix}$$

an invariant g is associated. g is an invariant with respect to any h.T. in the group (I)a. Thus, g can be a scalar or a pseudo-scalar. This can be decided by taking into account $C^{\dot r\dot s}$ together with C_{rs}.:

$$C_{\dot r\dot s} = \begin{pmatrix} 0 & -h \\ +h & 0 \end{pmatrix} = \begin{pmatrix} d_{\dot 1\dot 1} & d_{\dot 1\dot 2} \\ d_{\dot 2\dot 1} & d_{\dot 2\dot 2} \end{pmatrix}$$

$$d^{\dot r\dot s} = \begin{pmatrix} 0 & -h \\ +h & 0 \end{pmatrix} = \begin{pmatrix} d_{\dot 2\dot 2} & -d_{\dot 2\dot 1} \\ -d_{\dot 1\dot 2} & d_{\dot 1\dot 1} \end{pmatrix}$$

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$$(\rho_0,\rho_1,\rho_2)\begin{pmatrix}0&0&0\\0&0&-h\\0&h&0\end{pmatrix}\begin{pmatrix}0&0&-g\\0&0&0\\g&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=\begin{pmatrix}0&-h&0\\h&0&0\\0&0&-g\\0&g&0\end{pmatrix}$$

Problem:

S.R.: $S \to \rho_1 \cdot \begin{pmatrix} C_{rs} & d^{irs} \end{pmatrix} \cdot \rho_1$

$$C_{rs} = \begin{pmatrix} 0 & -g \\ +g & 0 \end{pmatrix} \to C'_{rs} = \begin{pmatrix} 0 & -g' \\ +g' & 0 \end{pmatrix}$$

If $h = g$

$$\therefore d^{irs} = \begin{pmatrix} 0 & -h \\ +h & 0 \end{pmatrix}$$

$$d^{irs} = \begin{pmatrix} 0 & -h \\ +g & 0 \end{pmatrix} \to d'^{irs} = \begin{pmatrix} 0 & -g' \\ +g' & 0 \end{pmatrix}$$

$$\therefore C_{rs} = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}$$

Thus, we have

$$g' = h, \quad h' = g$$

for space reflection.

For particular, if we took

$g = h$, (C_{rs}, d^{irs}) defines a scalar g.

On the contrary, if we took

$-g = h$, g, a pseudoscalar, becomes

in the sense that it changes sign by S.R.

T.R.: $(i\rho_2)\begin{pmatrix} C_{rs} & d^{irs} \end{pmatrix}(-i\rho_2)$

$h = g$: $C'_{rs} = \begin{pmatrix} 0 & -g \\ g' & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g \\ +g & 0 \end{pmatrix}$

$$d'^{irs} = \begin{pmatrix} 0 & -g' \\ g' & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g \\ +g & 0 \end{pmatrix}$$

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Thus, if $h = g$: $g' = g$
if $h = -g$: $g' = -g$

S. T. R. P_3:
Both for $h = g, h = -g$; $g' = g$.

Thus $\begin{pmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & -g \\ 0 & 0 & g & 0 \end{pmatrix} = -ig\sigma_2$ behaves as a scalar which does not change sign under S.R., T.R., S.T.R.

$\begin{pmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & -g & 0 \end{pmatrix} = -ig\rho_3\sigma_2$ behaves as a pseudoscalar in the sense that it changes sign under S.R. and T.R. but not under S.T.R.

For a ~~general~~ symmetric spinor one can associate by putting $C_{rs} = \begin{pmatrix} -f_1 + if_2 & -f_3 \\ f_3 & f_1 + if_2 \end{pmatrix}$
a complex antisymmetric tensor of the second rank which is self-dual.
This can be most easily seen from the following situation. C_{rs} transforms just like

$$\partial^{rt} a_{ts} = \begin{pmatrix} -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_3}, & \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}, & -\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} A_0 - A_3 & -A_1 + iA_2 \\ -A_1 + iA_2 & A_0 + A_3 \end{pmatrix}$$

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$$= \begin{pmatrix} -\left(\frac{\partial A_0}{\partial x_0} + \sum \frac{\partial A_j}{\partial x_j}\right) + \left(\frac{\partial A_0}{\partial x_3} + \frac{\partial A_3}{\partial x_0}\right) & \left(\frac{\partial A_0}{\partial x_1} + \frac{\partial A_1}{\partial x_0}\right) + i\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) \\ + i\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) & + i\left(\frac{\partial A_0}{\partial x_2} + \frac{\partial A_2}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) \\ \hline \left(\frac{\partial A_0}{\partial x_1} + \frac{\partial A_1}{\partial x_0}\right) + i\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) & \left(\frac{\partial A_0}{\partial x_0} + \sum \frac{\partial A_j}{\partial x_j}\right) - \left(\frac{\partial A_0}{\partial x_3} + \frac{\partial A_3}{\partial x_0}\right) \\ - i\left(\frac{\partial A_0}{\partial x_2} + \frac{\partial A_2}{\partial x_0}\right) + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) & - i\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \end{pmatrix}$$

Thus, if we put $G = \frac{\partial A_0}{\partial x_0} + \sum_j \frac{\partial A_j}{\partial x_j}$ and

$$E_1 = -\frac{\partial A_0}{\partial x_1} - \frac{\partial A_1}{\partial x_0}, \qquad H_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}.$$

$$E_2 = -\frac{\partial A_0}{\partial x_2} - \frac{\partial A_2}{\partial x_0}, \qquad H_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1},$$

$$E_3 = -\frac{\partial A_0}{\partial x_3} - \frac{\partial A_3}{\partial x_0}, \qquad H_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2},$$

we obtain by putting $\partial^r t a_{ts} = C^r{}_s$

$$(C^r{}_s) = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} = \begin{pmatrix} -G + E_3 + iH_3 & -E_1 + iH_1 \\ & +i(E_2 + iH_2) \\ -E_1 + iH_1 & -G + E_3 - iH_3 \\ -i(E_2 + iH_2) & \end{pmatrix}$$

$$= \begin{pmatrix} -i(H_1 + iE_1) - (H_2 + iE_2) & i(H_3 + iE_3) \\ i(H_3 + iE_3) & i(H_1 + iE_1) - (H_2 + iE_2) \end{pmatrix}$$

$$(C_{rs}) = \begin{pmatrix} C_{11} = C_{12} \\ C_{21} \quad C_{22} \end{pmatrix} = \begin{pmatrix} -C^2{}_1 & -C^2{}_2 \\ C^1{}_1 & C^1{}_2 \end{pmatrix} = \left(\begin{array}{c|c} (E_1 + iH_1) & +G + E_3 \\ +i(E_2 + iH_2) & +iH_3) \\ \hline -G + E_3 + iH_3 & -E_1 + iH_1 \\ & +i(E_2 + iH_2) \end{array}\right)$$

$$= \begin{pmatrix} -f_2 + if_1 & if_3 \\ if_3 & -f_2 + if_1 \end{pmatrix} \quad \text{or} \quad \begin{aligned} f_2 &= \tfrac{1}{2}(C_{11} + C_{22}) \\ f_1 &= +\tfrac{i}{2}(C_{11} - C_{22}) \\ f_3 &= +C_{12} = iC_{21} \end{aligned}$$

Thus we have $G = \frac{\partial A_0}{\partial x_0} + \sum_j \frac{\partial A_j}{\partial x_j} = 0$

$$f_j = E_j + iH_j, \quad j = 1, 2, 3$$

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Problem: Show that $\partial_{\dot r t} C^{t}_{s} = 0$ is equivalent to Maxwell's field equations. (65)

On the other hand, another set of relations

$$\partial_{\dot r t} C^{t}_{s} = 0$$

becomes

$$\begin{pmatrix} -\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} if_3 & -f_2 + if_1 \\ f_2 + if_1 & -if_3 \end{pmatrix} = 0$$

$$-i\frac{\partial f_3}{\partial x_0} - i\frac{\partial f_3}{\partial x_3} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_0} - i\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)$$
$$-i\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0$$

(11) $$-i\frac{\partial f_3}{\partial x_0} - i\frac{\partial f_3}{\partial x_3} - i\frac{\partial f_1}{\partial x_1} - i\frac{\partial f_2}{\partial x_2} - \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0$$

etc.

or

$$\frac{\partial H_3}{\partial x_0} + \frac{\partial H_3}{\partial x_3} + \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} = 0$$
$$\frac{\partial H_3}{\partial x_0} + \frac{\partial E_3}{\partial x_3} + \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} - \left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right) = 0$$

(22) $$+i\frac{\partial f_3}{\partial x_0} - i\frac{\partial f_3}{\partial x_3} - i\frac{\partial f_1}{\partial x_1} - i\frac{\partial f_2}{\partial x_2} + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0$$

$$\mathrm{div}\, \mathbf{F} = 0 \rightarrow \mathrm{div}\, \mathbf{E} = 0 \quad \mathrm{div}\, \mathbf{H} = 0$$

$$i\frac{\partial \mathbf{F}}{\partial x_0} + \mathrm{curl}\, \mathbf{F} = 0$$
$$\rightarrow \mathrm{curl}\, \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0$$
$$\mathrm{curl}\, \mathbf{H} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0$$

Thus, $\partial^{rt} a_{is} = C^{r}_{s}$ (or $\partial_{r}{}^{t} a_{ts} = C_{rs}$) $\Big\}$ $\boxed{\partial_{\dot r t} C^{t}_{s} = 0}$

$\boxed{\partial_{\dot r t} C^{t}_{s} = 0}$ with C_{ts}: Symmetric

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is nothing but the Maxwell's field equations for the electromagnetic field.

$$f_j = H_j + iE_j$$

are the components of a self-dual antisymmetric tensor, because

$$\begin{pmatrix} 0 & -H_3 & H_2 & -E_1 \\ H_3 & 0 & -H_1 & -E_2 \\ -H_2 & H_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix} = F_{\mu\nu}$$

$$\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{21}}{\partial x_2} + \frac{\partial F_{31}}{\partial x_3} - \frac{\partial F_{01}}{\partial x_0}$$

$$= (\text{curl } H)_1 - \frac{1}{c}\frac{\partial E_1}{\partial t}$$

$$\begin{pmatrix} 0 & -E_3 & E_2 & H_1 \\ E_3 & 0 & -E_1 & H_2 \\ -E_2 & E_1 & 0 & H_3 \\ -H_1 & -H_2 & -H_3 & 0 \end{pmatrix} = F^D_{\mu\nu}$$

dual tensor

~~also~~ satisfies another set of maxwell equation

$$F_{\mu\nu} + iF^D_{\mu\nu} = \begin{pmatrix} 0 & -f_3 & f_2 & if_1 \\ f_3 & 0 & -f_1 & if_2 \\ -f_2 & f_1 & 0 & if_3 \\ -if_1 & -if_2 & -if_3 & 0 \end{pmatrix}$$

$$(F_{\mu\nu} + i F^D_{\mu\nu})^D = -i(F_{\mu\nu} + i F^D_{\mu\nu})$$

$$= \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ -if_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & -f_3 \end{pmatrix}$$

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See Kapoorte-Uhlenbeck, P.R. 32 (1931), 1380.

More generally, one can set up a pair of spinor equations

$$\left.\begin{aligned} \partial^{rt} a_{ts} &= i\varkappa' c^r{}_s \\ \partial_{it} c^t{}_s &= i\varkappa'' a_{is} \end{aligned}\right\}$$

then we obtain

$$\partial_{it}\partial^{tia} a_{ius} = -\varkappa'\varkappa'' a_{is}$$

or K.G. equation for a vector field with the mass $\varkappa = \sqrt{\varkappa'\varkappa''}$.
The case of electromagnetic field is the particular one with

$$i\varkappa' = 1 \quad , \quad \varkappa'' = 0 .$$

If we put the contrary

$$\varkappa' = 0 \quad , \quad \varkappa'' \neq 0 ,$$

one obtains a strange case, which was proposed by Dirac recently
(Dirac, P.R.S. 209 (1951), 291)
(Schrödinger, Nature 169 (1952), 538.))

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\varkappa (A_\mu A^\mu - k^2)$$

$$\frac{\partial A_\mu}{\partial x_\mu} \neq 0 ; \quad \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \varkappa A_\mu \quad \lambda : \text{unknown function}$$

$$j_\mu = -\varkappa A_\mu \left(\frac{\partial j_\mu}{\partial x_\mu} = 0\right)$$

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Generalized wave equations

One can define similarly spinors of any order. One can define

$$\varphi^{\dot s\dot t\cdots}_{\lambda\mu\nu\cdots}$$

with $2k$ undotted indices and $2l-1$ dotted indices, k, l being integer or a half-integer. This is the essentially most general spinor, because one can freely raise or lower the indices so as to get more general spinors without introducing nothing new.

Thus, the most general spinor equation which for a particle with a definite mass $\kappa = \sqrt{\pi'\pi}$ which is linear in $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_j}$ takes the form

$$\partial^{\dot r\lambda}\varphi^{\dot s\dot t\cdots}_{\lambda\mu\nu\cdots} = i\kappa'\chi^{\dot r\dot s\dot t\cdots}_{\mu\nu\cdots}$$

$$\partial_{\dot r\lambda}\chi^{\dot r\dot s\dot t\cdots}_{\mu\nu\cdots} = i\kappa''\varphi^{\dot s\dot t\cdots}_{\lambda\mu\nu\cdots}$$

where χ has $2k-1$ undotted and $2l$ dotted indices. We regard $\psi = (\varphi, \chi)$ as the wave function with

$$2^{2k+1+2l} + 2^{2k-1+2l} = 2^{2k+2l}$$

components. But one can decompose them further into irreducible parts.

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Namely, ~~However~~, since any $\varphi^{\dot{s}\dot{t}\cdots}_{\kappa\lambda\mu\cdots}$

can be decomposed into sym. and anti. parts in the following way: First, ~~take~~ separate the sym. and anti. parts with respect to $\kappa\lambda\mu$, then the antisym transforms like a spinor with $2k-2$ undotted indices. The sym. part satisfies

$$\varphi^{\dot{s}\dot{t}\cdots}_{\kappa\lambda\mu\nu\cdots} + \varphi^{\dot{s}\dot{t}\cdots}_{\lambda\kappa\mu\nu\cdots} + \varphi^{\dot{s}\dot{t}\cdots}_{\lambda\mu\kappa\nu\cdots} = 0$$

Then, one can symmetrize it with respect to κ and μ (anti. part being spinor of low rank)

$$\varphi^{\dot{s}\dot{t}\cdots}_{\lambda\kappa\mu\nu\cdots} + \varphi^{\dot{s}\dot{t}\cdots}_{\lambda\mu\kappa\nu\cdots} = \varphi^{\dot{s}\dot{t}\cdots}_{\lambda\kappa\mu\nu\cdots}$$

Then it is sym. with respect to λ, κ, μ, because

$$\varphi_{\lambda\mu\nu} = \varphi_{\mu\nu\lambda} = \varphi_{\nu\mu\lambda} = \varphi_{\nu\lambda\mu}$$
$$= \varphi_{\lambda\nu\mu} = \varphi_{\mu\lambda\nu} = \varphi_{\mu\nu\lambda}$$

and one can proceed further and we obtain spinors symmetric with respect to $\lambda\mu\kappa\cdots$. Similarly, one can construct spinors which are sym. with respect to $\dot{s}\dot{t}\cdots$.

Number of independent components is now reduced to $(2k+1)\cdot 2l$

The total number of independent comp.

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of ψ is now

$$2l(2k+1)+2k(2l+1).$$

ψ satisfies the ~~K~~ second order wave eq.

$$\left(-\frac{\partial^2}{\partial x_0^2}+\sum_j \frac{\partial^2}{\partial x_j^2}-\varkappa^2\right)\psi=0$$

where

$$x^2=x'x''.$$

In particular, for $k=\frac{1}{2}$, $l=\frac{1}{2}$, ~~$x'=x''=\frac{mc}{\hbar}$~~, we have Dirac's wave equations for the electron.

$k=$ ~~$\frac{1}{2}$~~ 1, $l=\frac{1}{2}$:

$$\left.\begin{aligned}\partial^{\dot{r}}_{\dot{s}}\psi_{\dot{r}\dot{s}}&=ix'\chi^{\dot{r}}_{\dot{s}\lambda}\\ \partial_{\dot{r}x}\chi^{\dot{r}}_{\lambda}&=ix''\varphi_{x\lambda}\end{aligned}\right\}$$

wave equations for the vector mesons in general.

In particular $x'=0$, $x''=1$ gives Maxwell's eq. for el. mg. field.

The above procedure of symmetrization leaves us finally with the anti-spinor $\varphi_{x\lambda}$ (or $\chi^{\dot{r}\dot{s}}$) apart from the symmetric spinors.

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Problem:

If we write

$$\varphi_{11} = \varphi_{22} = 0 \qquad -\varphi_{12} = \varphi_{21} = \psi,$$

we obtain

$$\left.\begin{aligned} \partial^{\dot{a}\dot{r}}\psi &= ix'\chi^{\dot{r}}{}_1 \\ -\partial^{\dot{1}\dot{r}}\psi &= ix'\chi^{\dot{r}}{}_2 \end{aligned}\right\} (1)$$

$$\partial_{\dot{r}_1}\chi^{\dot{r}}{}_1 = \partial_{\dot{r}_2}\chi^{\dot{r}}{}_2 = 0 \quad (2)$$

$$\partial_{\dot{r}_2}\chi^{\dot{r}}{}_1 = -\partial_{\dot{r}_1}\chi^{\dot{r}}{}_2 = ix''\psi$$

or

$$\partial_{x^{\dot{r}}}\psi = ix'\chi_{x^{\dot{r}}\lambda} \qquad (1)'$$

~~the other four equations express $\chi_{\cdot\lambda}^{\dot{r}}$ in terms of ψ~~

and (2) becomes.

$$\left.\begin{aligned} \partial_{\dot{r}_1}\partial^{\dot{r}}{}_1\psi &= 0 \\ \partial_{\dot{r}_2}\partial^{\dot{r}}{}_2\psi &= 0 \\ \partial_{\dot{r}_2}\partial^{\dot{r}}{}_1\psi &= -x'x''\psi \\ \partial_{\dot{r}_1}\partial^{\dot{r}}{}_2\psi &= x'x''\psi \end{aligned}\right\}$$

these reduce to a single eq.

$$\partial_{\dot{r}_1}\partial^{\dot{r}_1}\psi = x'x''\psi$$

or

$$\left(-\frac{\partial^2}{\partial x_0^2} + \sum_j \frac{\partial^2}{\partial x_j^2} - x^2\right)\psi = 0 .$$

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In the more general cases,

$$\partial^{\dot r\lambda}\varphi^{\dot s\dot t\cdots}_{\lambda\mu\nu\cdots} = i\varkappa'\chi^{\dot r\dot s\dot t\cdots}_{\mu\nu\cdots}$$

$$\partial_{\dot r\lambda}\chi^{\dot r\dot s\dot t\cdots}_{\mu\nu\cdots} = i\varkappa''\varphi^{\dot s\dot t\cdots}_{\mu\nu\cdots}$$

we obtain the relations, if we restrict to symm. spinors:

$$\partial^{\dot r\lambda}\varphi^{\dot s\dot t\cdots}_{\lambda\mu\nu\cdots} = \partial^{\dot s\lambda}\varphi^{\dot r\dot t\cdots}_{\lambda\mu\nu\cdots}$$

Problem. Show that, in the case of φ_λ el.mg. field, this is the sup. cond.

$$\frac{\partial A_0}{\partial x_0} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = 0.$$

and $$\partial_{\dot r\lambda}\chi^{\dot r\dot s\dot t}_{\mu\nu\cdots} = \partial_{\dot r\mu}\chi^{\dot r\dot s\dot t\cdots}_{\lambda\nu\cdots}$$

From these conditions, the number of independent components reduces. (one can show that in the rest system, φ and χ are symm. with respect to $(\dot s,\lambda)$ and $(\dot r,\mu)$, so that they are symm. for all indices) The number is

$$2j+1 = 2k+2l+1+1 = 2k+2l.$$

This is a $2j+1$-dimensional repres. of the rotation group.

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In this sense,

$$j = k + l - \frac{1}{2}$$

can be regarded as the spin of the elementary particle. Now, since

$$k, l = \frac{1}{2}, 1, \frac{3}{2}, \cdots,$$

one can have a particle with arbitrary spin, either integer or half integer.

$k = \frac{1}{2}$, $l = \frac{1}{2}$: $j = \frac{1}{2}$

$k = 1$, $l = \frac{1}{2}$: $j = 1$

etc.

(The case $j = 0$ is not included here, but one can find it in the antisym. case $\varphi_{\lambda\mu} = -\varphi_{\mu\lambda}$)

The case $m = 0$ is exceptional. In this case, there are only two indep. components, in general, corresponding to the absence of the longitudinal photons.

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2^N components Ψ behave like a spinor of N-th rank,

$\Psi(\sigma) \to A\Psi(\sigma)$

$\sigma_j = 1 \times 1 \times \cdots \times \sigma_j \times \cdots \times 1$

$\sigma_j' A = A \sigma_j$ $\quad \sigma_j' = A \sigma_j A^{-1}$

13. Rotation group applies to Many Electron systems in N.R.Q.M. (69)

N-electrons

$x^{(1)}, \sigma^{(1)}, \ldots \ldots x^{(N)}, \sigma^{(N)}$

$p^{(1)}, p^{(2)} \ldots p^{(N)}$

H: invariant with respect to rotation

$x_j'^{(\nu)} = a_{jk} x_k^{(\nu)}$ $\quad (\nu = 1, 2, \cdots N)$

$p_j'^{(\nu)} = a_{jk} p_k^{(\nu)}$ $\quad (j, k = 1, 2, 3)$

$\sigma_j'^{(\nu)} = a_{jk} \sigma_j^{(\nu)}$ for proper rotation

$(= -a_{jk} \sigma_j^{(\nu)}$ for improper rotation)

$(A \sigma_j A^{-1} = a_{jk} \sigma_j^{(\nu)})$

because σ_j behave like $\varepsilon_{jkl} x_k p_l - x_l p_k$, where (j, k, l) is an even permutation of $(1, 2, 3)$. The matrix representation of σ. Thus ~~Ψ~~ changes under rotation, ~~just~~ like for the σ's. ~~Accordingly,~~ Instead, one can fix the matrix form

~~If the Hamiltonian is of the form~~

~~$H = \frac{1}{2m} \sum_{j,\nu} (p_j^{(\nu)})^2 + V(x^{(1)} \cdots x^{(N)} \sigma^{(1)} \cdots \sigma^{(N)})$~~,

for σ_j once for all, ~~σ_j~~ but ~~take~~ assume that $\Psi_{s_1 \cdots s_N}$ ~~which~~ changes ~~then~~ linearly under rotation as shown below.

Schrödinger equation (which transforms like a spinor of N-th rank)

$H \Psi = E \Psi$

$\Psi(x^{(1)} \cdots x^{(N)}, \sigma^{(1)} \cdots \sigma^{(N)}) \equiv \Psi_{s_1^{(1)} \cdots s_N^{(N)}}(x^{(1)} \cdots x^{(N)})$

$s^{(1)}, \ldots s^{(N)}$: eigenvalues of $\sigma_3^{(1)} \cdots \sigma_3^{(N)}$ or ~~s_i~~ can be ~~taken~~ 1 and 2.

discrete levels $E_1, E_2, \cdots$

$H \Psi = E_n \Psi$

If we give the specific form for the Hamiltonian familiar in the N.R.Q.M.

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(K.E. + Central potential due to the nucleus + C.R. between electrons + spin orbit coupling), then one can verify that the total angular momentum

$$\mathbf{M} = \sum_{\nu=1}^{N} \mathbf{m}^{(\nu)}$$

$$\mathbf{m}^{(\nu)} = [\mathbf{r}^{(\nu)} \times \mathbf{p}^{(\nu)}] + \frac{\hbar}{2}\boldsymbol{\sigma}^{(\nu)}$$

is commutative with H, so that eigenfunctions belonging to an energy level can be so chosen that they are eigenstates of

$$\mathbf{M}^2 = \mathbf{M}_1^2 + \mathbf{M}_2^2 + \mathbf{M}_3^2$$

and $\mathbf{M}_3$. They have the eigenvalues

$$\hbar^2 J(J+1)$$

and

$$M = -J, -J+1, \cdots J-1, J$$

where J is either an integer (including 0) or a half-integer according as the number of electrons is even or odd.

Thus, $2J+1$-eigenfunctions

$$\psi_{n,J,M}(r^{(1)} \cdots r^{(N)}, s^{(1)} \cdots s^{(N)})$$

belonging to the same J and the same energy E_n transform linearly among themselves under the rotation of the coordinate system.

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~~Now, if we~~ However, the drawbacks of the elementary q. m. is that :

(i) one has to assume a specific form for the Hamiltonian in order to verify the above statements

(ii) $F_{n,l,m}$ is, in general, a complicated functions of coordinates and $s^{(1)}, .. s^{(N)}$, so that it is ~~,~~ increasingly difficult to draw useful conclusions for ~~arbitrary~~ many electron systems, ~~unless~~ as N_f ~~is small, 2 or 3 say.~~ increases.

~~For instance,~~ Dirac has developed an algebraic method rather than the analytical method which required detailed information about the eigenfunctions. Dirac's method is, however, ~~a part of the~~ actually a particular and (implicit) application of q. t., which is more satisfactory and reveals the connection between algebraic method for operators and analytical method for eigenfunctions.

In order to develope this method, first of all, we have to show that the angular momentum is nothing ~~total~~ components of but the operator for the infinitesimal rotations around x- y- z- axes,

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Infinitesimal Rotations

$$x_j' = \sum_k a_{jk} x_k$$

$$\sum_j a_{jk} a_{jl} = \delta_{kl}$$

$$a_{jk} = \delta_{jk} + \varepsilon_{jk}$$

$$\Sigma(\delta_{jk}\delta_{jl} + \varepsilon_{jk}\delta_{jl} + \delta_{jk}\varepsilon_{jl}) = \delta_{kl}$$

$$\varepsilon_{lk} + \varepsilon_{kl} = 0$$

$$x_j' = x_j + \delta x_j \qquad \delta x_j = \sum_k \varepsilon_{jk} x_k$$

$$(\varepsilon_{jk}) = \begin{pmatrix} 0 & +\varepsilon_{12} & -\varepsilon_{31} \\ -\varepsilon_{12} & 0 & +\varepsilon_{23} \\ +\varepsilon_{31} & -\varepsilon_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_3 & +\alpha_2 \\ +\alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & +\alpha_1 & 0 \end{pmatrix}$$

$(\alpha_1 = -\varepsilon_{23} = \varepsilon_{32}$ etc.$)$

Rotation around z-axis

$\varepsilon_{13} = \varepsilon_{23} = 0$

$$\left.\begin{aligned} \delta x_1 &= +\varepsilon_{12} x_2 \\ \delta x_2 &= -\varepsilon_{12} x_1 \end{aligned}\right\} \qquad \left.\begin{aligned} \delta x &= +\varepsilon_{12} y \\ \delta y &= -\varepsilon_{12} x \end{aligned}\right\}$$

$$\psi(x', y', z') = \psi(x + \varepsilon_{12} y,\ y + \varepsilon_{12} x,\ z)$$

$$= \psi(x, y, z) + \varepsilon_{12}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x, y, z)$$

$$(a_{jk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta\varphi \\ 0 & \delta\varphi & 1 \end{pmatrix}$$

$$\alpha_3 = -\varepsilon_{12} = \delta\varphi \ ; \quad \psi(r, \theta, \varphi') = \psi(r, \theta, \varphi) + \delta\varphi \cdot \frac{\partial\psi}{\partial\varphi}$$

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$x_1' = x_1 - \alpha_3 x_2 + \alpha_2 x_3$

$\alpha_3 = \varepsilon_{12} = +\delta\varphi$

$\psi(x', y', z') \cong \psi(x, y, z) + \frac{i}{\hbar} m_z \delta\varphi \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x, y, z)$

$\theta' = \theta,\ \varphi' = \varphi + \delta\varphi$

$\psi(r, \theta, \varphi') = \psi(r, \theta, \varphi) + \delta\varphi \frac{\partial \psi(r, \theta, \varphi)}{\partial \varphi}$

Similarly, relations hold between infinitesimal rotations around x- y- axes and the operators

$I_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $I_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $I_z = \cdots$

If two successive infinitesimal rotations a and b are carried out successively, the product c is

$a_{jk} = \delta_{jk} + \varepsilon_{jk}$
$b_{jk} = \delta_{jk} + \omega_{jk}$

$c_{jl} = a_{jk} b_{kl} = (\delta_{jk} + \varepsilon_{jk})(\delta_{kl} + \omega_{kl})$
$= \delta_{jl} + \varepsilon_{jl} + \omega_{jl}$
$c_{jl} = \delta_{jl} + \theta_{jl}$
$\theta_{jl} = \varepsilon_{jl} + \omega_{jl}$ $(\delta_{jl} + \varepsilon_{jl})$

Conversely, any infinitesimal rotation can be given as the result of rotations around x-, y-, z- with the coefficients ε_{12}, ε_{31}, ε_{23} respectively.

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Or by a linear combination of the operators

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \text{etc.}$$

Problem, Show that

The direction of rotation axis is given by the direction of the vector $(\varepsilon_{12}, \varepsilon_{31}, \varepsilon_{23})$,

because

$$\begin{pmatrix} 0 & \varepsilon_{12} & -\varepsilon_{31} \\ -\varepsilon_{12} & 0 & \varepsilon_{23} \\ +\varepsilon_{31} & -\varepsilon_{23} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

for any vector in this direction ($x \neq y \neq z = \varepsilon_{12} \neq \varepsilon_{31} \neq \varepsilon_{23}$) does not change under the rotation.

and that the angle of rotation ω is given by

$$\omega = \pm\sqrt{\alpha_{23}^2 + \alpha_{31}^2 + \alpha_{12}^2}$$

where

~~because for a vector with, for instance,~~

~~$x = \varepsilon_{31}, \quad y = -\varepsilon_{23}, \quad z = 0$~~

which is $\perp$ar to the axis of rotation

~~$x' = x - \varepsilon_{12}\varepsilon_{23} = x + \varepsilon_{12} y$~~

~~$y' = y + \varepsilon_{12}$~~

because, if we take the new coord. system

~~$e_1'' = \varepsilon_{31} e_1 - \varepsilon_{23} e_2$~~

~~$e_2'' = \varepsilon_{12} e_2$~~ ~~$\varepsilon_{31} e_3$~~

$e_3' = \frac{1}{\varepsilon}(\varepsilon_{23} e_1 + \varepsilon_{31} e_2 + \varepsilon_{12} e_3)$

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(72)

$$(\alpha_{jk}) = \begin{pmatrix} \frac{\alpha_1}{\alpha} & \frac{\alpha_2}{\alpha} & \frac{\alpha_3}{\alpha} \\ \frac{\alpha_2}{\alpha'} & -\frac{\alpha_1}{\alpha'} & 0 \\ \frac{\alpha_1\alpha_3}{\alpha\alpha'} & \frac{\alpha_2\alpha_3}{\alpha\alpha'} & -\frac{\alpha'^2}{\alpha\alpha'} \end{pmatrix}$$

$\alpha' = \sqrt{\alpha_1^2 + \alpha_2^2}$

$\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$

If we go over to a new coord. system

α_{jl} : $\xi_j = \alpha_{jk} x_k$

$(\alpha^{-1})_{jl}$ $\alpha_{jl}\xi_j = x_l = (\alpha^{-1})_{lj}\xi_j$

α_{mj} : $x_j' = x_j + \varepsilon_{jk} x_k$

$= x_j + \varepsilon_{jk}(\alpha^{-1})_{kl}\xi_l$

$\xi_m' = \xi_m + \alpha_{mj}\varepsilon_{jk}(\alpha^{-1})_{kl}\xi_l$

$= \xi_m + \varepsilon'_{ml}\xi_l$

$\varepsilon'_{ml} = \alpha_{mj}\varepsilon_{jk}(\alpha^{-1})_{kl}$

$$(\alpha_{jk}) = \begin{pmatrix} \frac{1}{\alpha}\varepsilon_{23} & \frac{1}{\alpha}\varepsilon_{31} & \frac{1}{\alpha}\varepsilon_{12} \\ \frac{1}{\alpha'}\alpha_1 & \frac{1}{\alpha'}\alpha_2 & \frac{1}{\alpha'}\alpha_3 \\ \frac{\varepsilon_{12}}{\varepsilon'}\frac{\alpha_2}{\alpha'} & -\frac{\varepsilon_{23}}{\varepsilon'}\frac{\alpha_1}{\alpha'} & 0 \\ \alpha_1\alpha_3/\alpha\alpha' & \alpha_2\alpha_3/\alpha\alpha' & -(\alpha_1^2+\alpha_2^2)/\alpha\alpha' \\ \frac{\varepsilon_{12}\varepsilon_{23}}{\varepsilon\varepsilon'} & \frac{\varepsilon_{12}\varepsilon_{31}}{\varepsilon\varepsilon'} & -\frac{\varepsilon'^2}{\varepsilon\varepsilon'} \end{pmatrix} = \frac{-\alpha'^2}{\alpha\alpha'}$$

$\|\alpha_{jk}\|^2 = \frac{1}{\varepsilon^2\varepsilon'^2}(\varepsilon_{23}^2(\varepsilon_{31}^2+\varepsilon_{23}^2) + \varepsilon_{31}^2(\varepsilon_{31}^2+\varepsilon_{23}^2)$

$+ \varepsilon_{12}^2\varepsilon_{31}^2 + \varepsilon_{12}^2\varepsilon_{23}^2) = 1.$

$\varepsilon = \sqrt{\varepsilon_{12}^2+\varepsilon_{31}^2+\varepsilon_{23}^2}$, $\varepsilon' = \sqrt{\varepsilon_{31}^2+\varepsilon_{23}^2}$

$\varepsilon'_{12} = \varepsilon'_{13} = 0$

$\varepsilon'_{23} = \alpha_{2j}\varepsilon_{jk}\alpha_{3k}\varepsilon$

$= \frac{\varepsilon_{31}\varepsilon_{12}}{\varepsilon'}\varepsilon$

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$$\Psi(x',y',z') = \left(1 + i\frac{\alpha m}{\hbar}\right)\Psi(x,y,z)$$

$$(\varepsilon_{jk}) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\varepsilon_{12}\varepsilon_{23}}{\varepsilon'} & \frac{\varepsilon_{12}\varepsilon_{31}}{\varepsilon'} & -\frac{\varepsilon'^2}{\varepsilon'} \to -\frac{(\varepsilon_{31}^2+\varepsilon_{23}^2)}{\varepsilon'} \\ -\frac{\varepsilon_{31}\varepsilon}{\varepsilon'} & \frac{\varepsilon_{23}\varepsilon}{\varepsilon'} & -\frac{(\varepsilon_{12}\varepsilon_{23})^2}{\varepsilon'^2} \to 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\varepsilon}\varepsilon_{23} & \frac{1}{\varepsilon}\varepsilon_{31} & \frac{1}{\varepsilon\varepsilon'}\varepsilon_{12}\varepsilon_{23} \\ \frac{1}{\varepsilon}\varepsilon_{31} & -\frac{1}{\varepsilon}\varepsilon_{23} & \frac{1}{\varepsilon\varepsilon'}\varepsilon_{12}\varepsilon_{31} \\ \frac{1}{\varepsilon}\varepsilon_{12} & 0 & -\frac{1}{\varepsilon\varepsilon'}(\varepsilon_{31}^2+\varepsilon_{23}^2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & +\varepsilon & 0 \end{pmatrix}$$

This indicates a transformation

Thus, the components of the angular momentum vector

$$\mathbf{m} = (\mathbf{r} \times \mathbf{p}) = -i\hbar\,(\mathbf{r} \times \mathrm{grad})$$

are just $(-i\hbar)$ times the operators for the infinitesimal rotations. From the commutation relations

$$[m_x, m_y] = i\hbar\, m_z \quad \text{etc}$$

we obtain

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Now, if we take the Schrödinger fn

$$\Psi(x^{(1)}, \cdots x^{(N)}; s^{(1)}, \cdots s^{(N)})$$

$$= \Psi_{s_1, s_2, \cdots s_N}(x^{(1)}, \cdots x^{(N)})$$

$s_1, s_2, \cdots s_N = 1$ or 2,

it transforms like a covariant spinor of the N-th rank. Under rotation, (ξ_1, ξ_2) transform ~~etc~~ according to

$$\left.\begin{aligned} \xi_1' &= \alpha_{11}\xi_1 + \alpha_{12}\xi_2 \\ \xi_2' &= \alpha_{21}\xi_2 + \alpha_{22}\xi_2 \end{aligned}\right\}$$

For an infinitesimal rotation

$$\left.\begin{aligned} \xi_1' &= \xi_1 + \delta\alpha_{11}\xi_1 + \omega_{12}\xi_2 \\ \xi_2' &= \xi_2 + \omega_{21}\xi_1 + \omega_{22}\xi_2 \end{aligned}\right\}$$

For instance, for the rotation around z-axis of angle $\delta\varphi$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \cos\frac{\delta\varphi}{2} + i\sin\frac{\delta\varphi}{2}\cdot\sigma_3$$

$$(\omega_{rs}) = 1 + i\frac{\delta\varphi}{2}\sigma_3$$

$$(\omega_{rs}) = +i\frac{\delta\varphi}{2}\sigma_3$$

~~Thus~~, The product of a ~~num~~ infinitesimal operators is

$$(\alpha_{rs}) = 1 + \frac{i}{2}\alpha\sigma$$

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in contrast to

$(A_{jk}) = 1 - i\alpha m/\hbar$ (~~$\hbar \to 1$~~)

the simultaneous rotation of all coord. and spins will induce the change in Ψ which is characterized by the sum of operators,

$$\delta\Psi = -\frac{i}{\hbar}\alpha\{M\Psi = -\frac{i}{\hbar}\alpha\{\sum_{\nu=1}^{N} m^{(\nu)} + \frac{1}{2}\sum_{\nu=1}^{N}\sigma^{(\nu)}\}\Psi$$

Thus, the operator for the simultaneous infinitesimal rotation ~~$1 - i\alpha$~~ is given by

$$\frac{i}{\hbar}(\alpha M),$$

where M_1, M_2, M_3 obviously satisfy the C.R.

$$[M_j, M_k] = i\hbar M_\ell$$

where (j, k, ℓ) is an even permutation of $(1, 2, 3)$

More generally, one can deduce the C.R. without any reference to the ~~form~~ nature of variables the Schrödinger fcn.

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$$[I_x, I_y] = -I_z$$
$$[I_z, I_x] = -I_y$$
$$[I_y, I_z] = -I_x$$

(van der Waerden, p. 62.)

As was shown above, any infinitesimal rotation can be characterized by $(\varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}) = (\alpha_1, \alpha_2, \alpha_3)$, which three parameters defines a vector in the direction of axis of rotation and with the absolute value proportional to the angle of rotation. Thus, if $d_\gamma \equiv d(\gamma_1, \gamma_2, \gamma_3)$ is the result of rotations $d_\gamma \equiv d_\alpha(\alpha_1 \alpha_2 \alpha_3)$ and $d_\beta \equiv d(\beta_1, \beta_2, \beta_3)$, or

$$d_\beta d_\alpha = d_\gamma,$$

$\gamma_1, \gamma_2, \gamma_3$ are given as a definite function of α's and β's.

$$\gamma_j = \gamma_j(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)$$

The functional matrix

$$S^{(\alpha)}_{jk} = \left(\frac{\partial \gamma_j}{\partial \beta_k}\right)_{\beta=0}$$

does not zero determinant

or the Jacobian

$$\| S^{(\alpha)}_{jk} \| = \left\| \left(\frac{\partial \gamma_j}{\partial \beta_k}\right)_{\beta=0} \right\| \neq 0,$$

because β_j can be given as a function of γ's for a fixed α's, because there must

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be exist d_β which satisfies $d_\gamma = d_\beta d_\alpha$
or $d_\beta = d_\gamma (d_\alpha)^{-1}$, for any d_α.
Thus

$$T_{kl}(\alpha) = \left(\frac{\partial \beta_k^{(\gamma,k)}}{\partial \gamma_l}\right)_{\gamma=\alpha \text{ (for } \beta=0)}$$

exists and satisfies

$$S_{jk}^{(\alpha)} T_{kl}(\alpha) = \delta_{jl}$$

or $$S\,T = E$$

where E is the unit matrix.

We are now interested in the representation of the rotation group d_β, in which the rotations d_α in the neighborhood of the identity is given by a linear transformation D_α, the matrix of D_α is continuous and differentiable with respect to the parameters $\alpha_1, \alpha_2, \alpha_3$.
Of course, D_α must satisfy

$$D_\beta D_\alpha = D_\gamma$$

where $\gamma = \gamma(\alpha, \beta)$ is a given function of α and β. characteristic to the rotation group. Let the repres. be $v = (v_1, v_2, \cdots v_n)$ n-dimensional;

$$v = D_\beta u \qquad v_r = \sum_s (D_\beta)_{rs} u_s$$

$$u \equiv (u_1, u_2, \cdots u_n) \qquad (r, s = 1, 2, \cdots n)$$
$$v \equiv (v_1, v_2, \cdots v_n)$$

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$$v = D_\beta u = u + \left(\frac{\partial v}{\partial \beta_1}\right)_0 \beta_1 + \left(\frac{\partial v}{\partial \beta_2}\right)_0 \beta_2 + \left(\frac{\partial v}{\partial \beta_3}\right)_0 \beta_3$$

for infinitesimal β_j
or more precisely

$$v_r = \sum_s (D)_{\beta rs} u_s$$

$$= u_r + \frac{\partial}{\partial \beta_1}\left(\sum_s (D_\beta)_{rs} u_s\right) \beta_1 + \cdots + \cdots$$

Thus, $\left(\frac{\partial v_r}{\partial \beta_j}\right)_0 = \frac{\partial}{\partial \beta_j}\left(\sum_s (P_\beta)_{rs} u_s\right)$ are linear in $u_1, u_2, \cdots u_n$, so that one may put

$$\left(\frac{\partial v_r}{\partial \beta_j}\right)_0 = \sum_s (I_j)_{rs} u_s$$

or simply

$$\left(\frac{\partial v}{\partial \beta_j}\right)_0 = I_j u$$

In particular, if we take the three-dim. space itself, $u \equiv (x_1, x_2, x_3)$
$v \equiv (x_1', x_2', x_3')$, and $(\alpha_1, \alpha_2, \alpha_3)$ etc. are the angle of rotations around x_1- x_2- x_3-axes, then I_j are nothing but the I_x, I_y, I_z which were defined above.

Now take a fixed vector (or point)
$u_0^{(0)} \equiv (u_1^{(0)}, u_2^{(0)}, \cdots u_n^{(0)})$
and put (D_α is not necessary infinitesimal)

$$u = D_\alpha u_0^{(0)}.$$

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when

$$v = D_\beta u = D_\beta D_\alpha u^{(0)} = D_\gamma u^{(0)}$$

where $\gamma_j = \gamma_j(\alpha, \beta)$.

$$J_j u = \left(\frac{\partial v}{\partial \beta_j}\right)_{\beta=0} = \sum_k \left(\frac{\partial v}{\partial \gamma_k}\right)_{\gamma=\alpha} \left(\frac{\partial \gamma_k}{\partial \beta_j}\right)_{\beta=0}$$

$$= \sum_k \frac{\partial u}{\partial \alpha_k} S_{jkj}(\alpha) \qquad T_{jl}(\alpha)$$

$$\left(\because \quad \frac{\partial}{\partial \beta_j}(D_\beta u) = \sum_k \frac{\partial}{\partial \gamma_k}(D_\gamma u^{(0)}) \left(\frac{\partial \gamma_k}{\partial \beta_j}\right)\right.$$

For $\beta_j = 0$, we have

$$\left(\frac{\partial v}{\partial \beta_j}\right)_{\beta=0} = \sum_k \left(\frac{\partial v}{\partial \gamma_k}\right)_{\gamma=\alpha} \left(\frac{\partial \gamma_k}{\partial \beta_j}\right)$$

From $v = D_\beta D_\alpha u^{(0)} = D_\gamma u^{(0)}$, we have

~~for $\beta = 0$:~~ $D_\beta u =$

$$D_\beta \left(\frac{\partial u}{\partial \alpha_k}\right) = \left(\frac{\partial v}{\partial \gamma_k}\right)$$

For $\beta = 0$, this becomes

$$\left(\frac{\partial u}{\partial \alpha_k}\right) = \left(\frac{\partial v}{\partial \gamma_k}\right)_{\gamma=\alpha}$$

$$\boxed{\sum_j J_j u \cdot T_{jl}(\alpha) = \frac{\partial u}{\partial \alpha_l}}$$

This is called ~~the~~ Lie's characteristic differential equations of the representation. It should be notice that the matrix $T_{jl}(\alpha)$ is intrinsic to the rotation group and independent of the particular representation.

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Now, $u = D_\alpha u_0$ is completely determined by the Lie's diff. eq. and the initial conditions

$$u_r = u_r^{(0)} \qquad \text{for } \alpha_j = 0$$

(∵ By $\dfrac{\partial u_r^{(\alpha)}}{\partial \alpha_l} = \sum_{j,s} T_{jl}(\alpha)(J_j)_{rs} u_s(\alpha)$,

the values of $u_s(\alpha)$ in the neighborhood of $u_r = u_r^{(0)}$ is determined. and, more generally

$$u_r(\alpha) = u_r^{(0)} + \sum_l \left(\frac{\partial u_r}{\partial \alpha_l}\right)_{\alpha=0} \alpha_l + \frac{1}{2} \sum_{l,k} \frac{\partial^2 u_r}{\partial \alpha_l \partial \alpha_k} \alpha_l \alpha_k + \cdots$$

is completely determined by the char. diff. equations)

In other words, any representation of of the rotation group is completely determined by the infinitesimal operators J_1, J_2, J_3.

The infinitesimal operators must satisfy the integrability conditions:

$$\frac{\partial^2 u}{\partial \alpha_k \partial \alpha_l} = \sum_j J_j u \frac{\partial T_{jl}}{\partial \alpha_k} + J_j \frac{\partial u}{\partial \alpha_k} \cdot T_{jl}$$

$$= \sum J_j u \frac{\partial T_{jl}}{\partial \alpha_k} + J_j J_m u \cdot T_{mk} T_{jl}$$

$$= \sum J_k u \frac{\partial T_{kl}}{\partial \alpha_l} + J_k J_m u \, T_{ml} T_{jk}$$

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For $\alpha_j = 0$, we have

$$J_k J_\ell - J_\ell J_k = \sum \left(T_{ij} \frac{\partial T_{k\ell}}{\partial \alpha_k}\Big|_{\alpha=0} - J_j \frac{\partial T_{j\,k}}{\partial \alpha_\ell}\Big|_{\alpha=0} \right)$$

or

$$J_k J_\ell - J_\ell J_k = \sum_j C_{j,k\ell} J_j$$

$$C_{j,k\ell} = \left(\frac{\partial T_{j\ell}}{\partial \alpha_k} - \frac{\partial T_{j\,k}}{\partial \alpha_\ell} \right)_{\alpha=0}$$

The coefficients $C_{j,k\ell}$ depend only on the structure of the rotation group itself and are real. Hence, ~~if~~ we know that in the case of three dimensional representation

$C_{j,k\ell} = -1$ if $(j\,k\,\ell)$ is an even perm. of $(1,2,3)$

$= 0$ if any two of j, k, ℓ are equal to each other.

Thus, we have for any representation

$$J_k J_\ell - J_\ell J_k = -J_j$$

(j,k,ℓ): even perm. of $(1,2,3)$

If we introduce the operators (hermitian)

$$M_j = -i\hbar J_j,$$

we have the C.R.

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$[M_{jk}, M_{lk}] = i\delta_{kl} M_j \quad (\delta_{kl})$

which is the same as the commutation relations for the components of angular momentum operators.

Further, we introduce

$$M_1 + i M_2 = \pm M_+$$
$$M_1 - i M_2 = \pm M_-$$

then, we have

$$\left.\begin{aligned} M_3 M_+ - M_+ M_3 &= \pm M_+ \\ M_3 M_- - M_- M_3 &= -\pm M_- \\ M_+ M_- - M_- M_+ &= 2\pm M_3 \end{aligned}\right\}$$

Irreducible Representations of Rotation Group

Now, let us ~~take~~ consider any representation in finite dimensional space R. Then the rotations around the z-axis constitute Abelian group and its representations are reduced completely to one dimensional representations. A rotation of angle ω is represented by

$$u_M = D_\omega u_M = e^{iM\omega} u_M \quad (\ldots 0\ u_M\ 0 \ldots)$$

where v_M is a basic vector (or ~~a component of~~ one of the coordinates in the coordinate system, in which the representation matrices for the rotations around the z-axis is completely reduced)

On the other hand

$$\left(\frac{\partial v_M}{\partial \beta_3}\right)_0 = iM u_M = J_3 u_M$$

$\beta_3 = \omega$

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or

$$M_z v_M = M v_M$$

From this, it follows that

$$M_z(M_+ v_M) = M_+(M_z+1) v_M$$
$$= (M+1)(M_+ v_M)$$

and $M_z(M_- v_M) = (M-1)(M_- v_M)$

Thus, if v_M is an eigenvector belonging to the eigenvalue M of M_z, then $(M_+ v_M)$ and $(M_- v_M)$ are eigenvectors of M_z with the eigenvalues $M+1$ and $M-1$ respectively.

Further arguments about possible eigen-values of M_z is the same as in Q.M.

$$M = -J, -J+1, \cdots\cdots J-1, J$$

$$J = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\cdots$$

$M^2 = (M_1^2 + M_2^2 + M_3^2) v = J(J+1) v$ for all vectors in R_{2J+1}.

~~The corresponding~~ Thus, we have a $2J+1$-dimensional repres. One can normalize v_M in such a way that, for $M = -J+1, -J+2, \cdots J, J-1, \cdots -J+1$

$$M_- v_M = \sqrt{J(J+1) - M(M-1)}\, v_{M-1},$$

~~then we have~~ or

$$M_- v_M = \sqrt{(J+M)(J-M+1)}\, v_{M-1}$$

then we have, from

$$M_+ M_- = (M_1 + iM_2)(M_1 - iM_2)$$
$$= M_1^2 + M_2^2 - i[M_1, M_2]$$
$$= M_1^2 + M_2^2 + M_z = M^2 - M_z^2 + M_z$$

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$$M_+ M_- v_M = \{J(J+1) - M(M-1)\} v_M$$

~~$M_- M_+ v_M = \{J(J+1) - M(M+1)\} v_M$~~

~~$M_- (M_+ v_M) =$~~

$$= M_+ \sqrt{\{J(J+1) - M(M-1)\}}\, v_{M-1}$$

or $$M_+ v_{M-1} = \sqrt{J(J+1) - M(M-1)} \cdot v_M$$

or $$M_+ v_M = \sqrt{J(J+1) - M(M+1)} \cdot v_{M+1}$$

$$M_3 v_M = M v_M$$

The representation in R_{2J+1} is irreducible, because, ~~if otherwise, there is a sub-space with R_{2J+1} is further reduced, it must be a sub-space~~ spanned by only ~~(a part of)~~ a part of $v_{-J}, \ldots v_J$ ~~(not all of them because~~ (if it is a linear combination of two or more of ~~them~~ any vector in R_{2J+1} can be represented by a superposition of eigenvectors of M_z. But, any vector which is a superposition of only a part of $v_{-J} \ldots v_J$ ~~is not~~ will be transformed into another vector which is outside the original subspace. ~~Now~~ Thus, only a linear combination of all of $v_{-J} \ldots v_J$ can be invariant. Now, ~~this subspace~~ the representation in the subspace of the rotations around z-axis can again be completely

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reduced and the representation thus obtained is, obviously, $2J+1$-dimensional, so that the subspace is nothing but the original space.

It is obvious that the following holds:
Now we can prove the important theorem: The representation with v_M $(M=-J, \dots, +J)$ is equivalent to the representation D_J.

Proof: D_J was the repr. by means of monomials

$$\zeta_{J,M} = \frac{\xi^j \eta^k}{\sqrt{j!\,k!}}, \qquad \begin{aligned} M &= \frac{k-j}{2} \\ J &= \frac{k+j}{2} \end{aligned}$$

By a rotation around z-axis of angle ω,

$$\zeta'_{J,M} = e^{iM\omega} \zeta_{J,M},$$

which must coincide with v_M except for a numerical factor. So the whole vector space must be the same as that of v spanned by v_M.

Incidentally, it is even true that the numerical factor can be taken to be one, because the operator

$$M_z \zeta_{J,M} = -i(J_x - iJ_y)$$

corresponds to the infinitesimal rotation around y with the matrix:

$$\delta\theta \cdot J_x: \quad \begin{pmatrix} \cos\frac{\delta\theta}{2} & -i\sin\frac{\delta\theta}{2} \\ -i\sin\frac{\delta\theta}{2} & \cos\frac{\delta\theta}{2} \end{pmatrix} = \begin{pmatrix} 1 & -i\frac{\delta\theta}{2} \\ -i\frac{\delta\theta}{2} & 1 \end{pmatrix}.$$

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Further, if J is an integer, I say, then the above representation is equivalent to the representation with $Y_{l,m}(\theta,\varphi)$. ~~as basis vectors~~ The help of The reason is the same as for D_l.

Finally, one can prove the theorem: Every irreducible representation of the order $2J+1$ is equivalent to D_J. This is because the vector space must be the same $2J+1$-dimensional complex space (which evidently cannot be less than $2J+1$). ~~For~~ any two representations have the same structure

---

Remark

$$D(\alpha) = \begin{pmatrix} 1 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 1 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 1 \end{pmatrix}^T = 1 + \alpha_1 I_1^T + \alpha_2 I_2^T + \alpha_3 I_3^T$$

$$I_j^T \to \frac{i M_j}{\hbar} = J_j \qquad M_j = -i\hbar J_j$$

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T, \quad I_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^T$$

$$I_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^T$$

$$x'_i = (D_\alpha)_{ik} x_k$$

$$J_3 = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$I_1 I_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2 I_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_1 I_2 - I_2 I_1 = -I_3 \to [M_1, M_2] = -i\hbar M_3 \qquad [J_1, J_2] = -J_3$$

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~~X~~ Correction:

$$\begin{pmatrix} z' \\ \eta' \end{pmatrix} = \left(\cos\frac{\delta\varphi}{2} + i\sin\frac{\delta\varphi}{2}\cdot\sigma_3\right)\begin{pmatrix} z \\ \eta \end{pmatrix}$$

$$= \begin{pmatrix} e^{+i\frac{\delta\varphi}{2}} z \\ e^{+i\frac{\delta\varphi}{2}} \eta \end{pmatrix}$$

Inf. Rotations

$$\begin{pmatrix} z' \\ \eta' \end{pmatrix} = \left(1 - \frac{i\delta\varphi}{2}\sigma_j\right)\begin{pmatrix} z \\ \eta \end{pmatrix}.$$

$$+\frac{i}{2}\alpha\sigma$$

$$(a_j k) = (1 + i\alpha m/\hbar)$$

$$\text{z-axis}\begin{cases} \begin{pmatrix} z' \\ \eta' \end{pmatrix} = \left(1 + \frac{i\delta\varphi}{2}\sigma_3\right)\begin{pmatrix} z \\ \eta \end{pmatrix} \\ (a_j k) \simeq \left(1 + i\,\delta\varphi\, m_3/\hbar\right) = \left(1 + \delta\varphi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right. \end{cases}$$

$$1 - \frac{i\,\delta\varphi}{\hbar}\left(\underbrace{m_3}_{\text{orb}} + \underbrace{\frac{1}{2}\sigma_3}_{\text{spin}}\right)$$

$$\psi(x', y', z') = \left(1 + i\,\alpha\, m/\hbar\right)\psi(x, y, z)$$

$$c_n'\psi_n = \sum x_n\psi_n + \frac{i\alpha}{\hbar}\sum_{n'}\psi_{n'}\, m_{n'n}\, c_n$$

$$c_n' = c_n + \frac{i\alpha}{\hbar}\sum_{n'} m_{n'n}\, c_{n'}$$

$$m_j\psi_n = \sum_{n'}\psi_{n'}(m_j)_{n'n}$$

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$$x_j' = a_{jk} x_k \qquad \psi_r' = \alpha_{rs} \psi_s$$

$$\sigma_j' \psi_r' = (\sigma_j')_{rs} \alpha_{st} \psi_t$$

$$= \alpha_{rs} (\sigma_j)_{sp} \alpha^{-1}_{pq} \alpha_{qt} \psi_t$$

$$= \alpha_{rs} (\sigma_j)_{st} \psi_t$$

$$(\sigma_j')_{rq} = \alpha_{rs} (\sigma_j)_{sp} \alpha^{-1}_{pq} = a_{jk} \sigma_k$$

$$\psi^{*\prime}_r (\sigma_j')_{rs} \psi_s' = \psi^*_r (\sigma_j)_{rs} \psi_s$$

~~$(\sigma_j')_{rs} \psi_s' =$~~

Rotation around x, y, z-axes:

(x)
$$(a_{jk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \qquad \alpha_{rs} = \begin{pmatrix} \cos\frac{\varphi_1}{2} & i\sin\frac{\varphi_1}{2} \\ i\sin\frac{\varphi_1}{2} & \cos\frac{\varphi_1}{2} \end{pmatrix}$$

$$(a_{jk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta\varphi_1 \\ 0 & \delta\varphi_1 & 1 \end{pmatrix} + \delta\varphi_1 \, y$$

$$\Psi_r(x, y - \delta\varphi_1 z, z + \delta\varphi_1 y) = \Psi_r(x,y,z) + \delta\varphi_1 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Psi_r$$

$$= \left(1 + \frac{i\delta\varphi_1}{\hbar} m_1\right)\Psi_r \qquad \frac{i}{\hbar} m_1$$

(y)
$$\alpha_{rs}\Psi_s = \left(1 + \frac{i\delta\varphi_1}{2}\sigma_1\right)_{rs}\Psi_s$$

$$\begin{cases} \Psi_r' = \left(1 + \frac{i 2\delta\varphi_2}{\hbar} m_2\right)\Psi_r \\ \alpha_{rs}\Psi_s = \left(1 + \frac{i\delta\varphi_2}{2}\sigma_2\right)_{rs}\Psi_s \end{cases}$$

(z) $\left(1 + \frac{i\delta\varphi_3}{2}\sigma_3\right)$:
$$\frac{\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{(J+M)!(J-M)!}} = \frac{\left(1 + \frac{i\delta\varphi_3}{2}\right)^{J+M}\left(1 - \frac{i\delta\varphi_3}{2}\right)^{J-M}\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{(J+M)!(J-M)!}}$$

$$= (1 + i\delta\varphi_3 \cdot M)\frac{\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{(J+M)!(J-M)!}}$$

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(x) $\dfrac{\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{(J+M)!\,(J-M)!}} = \dfrac{\left(\xi_1+\frac{i\delta\varphi_1}{2}\xi_2\right)^{J+M}\left(\xi_2+\frac{i\delta\varphi_1}{2}\xi_1\right)^{J-M}}{\sqrt{(J+M)!\,(J-M)!}}$

$= \dfrac{\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{\quad}} + \dfrac{\xi_1^{J+M-1}\,\xi_2^{J-M+1}}{\sqrt{(J+M)!\,(J-M)!}}\,\dfrac{J+M}{2}\,i\delta\varphi_1$

$+ \dfrac{\xi_1^{J+M+1}\,\xi_2^{J-M-1}}{\sqrt{(J+M)!\,(J-M)!}}\,\dfrac{J-M}{2}\,i\delta\varphi_1$

$M_1\,u_{J,M} = \frac{\hbar}{2}\sqrt{(J+M)(J-M+1)}\,u_{J,M-1} + \frac{\hbar}{2}\sqrt{(J+M)(J-M)}\,u_{J,M+1}$

(y) $\dfrac{\left(\xi_1+\frac{i\delta\varphi_1}{2}(-i\xi_2)\right)^{J+M}\left(\xi_2+\frac{i\delta\varphi_1}{2}(+i\xi_1)\right)^{J-M}}{\sqrt{(J+M)!\,(J-M)!}}$

$= \dfrac{\xi_1^{J+M}\,\xi_2^{J-M}}{\sqrt{\quad}} + \dfrac{\xi_1^{J+M-1}\,\xi_2^{J-M+1}\,(-i)(J+M)}{\sqrt{(J+M)!\,(J-M)!}}\,\dfrac{i\delta\varphi_1}{2}$

$+ \dfrac{\xi_1^{J+M+1}\,\xi_2^{J-M-1}\,(+i)(J-M)}{\sqrt{(J+M)!\,(J-M)!}}\,\dfrac{i\delta\varphi_1}{2}$

$M_2\,u_{J,M} = \frac{\hbar}{2}\sqrt{(J+M)(J-M+1)}\,(-i)\,u_{J,M-1} + \frac{\hbar}{2}\sqrt{(J+M+1)(J-M)}\,(i)\,u_{J,M+1}$

$u'_{J,M} = (M_1+iM_2)u_{J,M} = \hbar\sqrt{(J+M)(J-M+1)}\cdot u_{J,M-1}$
$(M_1-iM_2)u_{J,M} = \hbar\sqrt{(J+M+1)(J-M)}\,u_{J,M+1}$

$(M_1+iM_2)_{M,M-1} = \hbar\sqrt{(J+M)(J-M+1)}$

$(M_1-iM_2)_{M,M+1} = \hbar\sqrt{(J+M+1)(J-M)} = (M_1+iM_2)_{M+1,M}$

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Vector space with $2J+1$-components $C_{J,M}$, where

$$\Psi = \sum_M C_{J,M}\,\Psi_{JM}$$

Rotation T: $\Psi' = T\Psi = \sum_M C'_{J,M}\,\Psi_{J,M}$

$T = m_j$: $\Psi' = m_j\cdot\Psi = m_j\cdot\sum_M C_{J,M}\Psi_{JM}$

$$= \sum C_{J,M}\,(m_j)_{M',JM}\,\Psi_{J,M'}$$

$$C'_{JM} = (m_j)_{M'M}\,C_{JM}$$

Thus, the matrices $(m_j)_{M'M}$ give the representation of infinitesimal rotations with $\Psi_{J,M}$ as basic vectors.

Problem: Prove that the relations

$$(m_1 + i m_2)_{\cancel{M,M-1}\ M+1,M} = (m_1 - i m_2)_{M,M+1}$$

$$= \hbar\sqrt{(J+M+1)(J-M)},$$

hold, if we take suitable phase factors for $\Psi_{J,M}$. They above derived from the C.R. for m_j:

$$[m_1, m_2] = -i\hbar m_3$$

$$[m_2, m_3] = i\hbar m_1$$

$$[m_1 \pm i m_2,\ m_3] = \mp\hbar\,(m_1 \pm i m_2)$$

$$(m_1 \pm i m_2)_{MM'}\cdot M' = (M \mp 1)(m_1 \pm i m_2)_{MM'}$$

$(m_1 \pm i m_2)_{MM'} \neq 0$ only for $M' = M \mp 1$

$$(m_1 - i m_2)(m_1 + i m_2) = m_1^2 + m_2^2 - \hbar m_3$$

$$= \hbar^2 J(J+1) - m_3^2 - \hbar m_3$$

$$(m_1 - i m_2)_{M,M+1}\,(m_1 + i m_2)_{M+1,M} = \hbar^2\big(J(J+1) - M(M+1)\big)$$

$$|(m_1 + i m_2)_{M+1,M}| = |(m_1 - i m_2)_{M,M+1}|$$

$$= \hbar\sqrt{J(J+1) - M(M+1)} = \hbar\sqrt{(J+M+1)(J-M)}$$

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Irreducibility of D_J

Schur's lemma:
(Waerden, p47
Weyl, p.

~~If all of the representation matrices $A, B, C \ldots$ of a group $(a, b, c, \ldots)$ commute with a certain matrix E, E must be a multiple of unit matrix 1, if the repr. is irreducible.~~

~~Conversely~~, If there is no matrix other than the numeral multiple of 1 which commute with all matrices of group repr., then this repres. is irreducible.

Proof: If the repres. is reducible, by a suitable transf. S

$$A' = SAS^{-1}, \quad \ldots$$

will have the form

$$A' = \left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right) \qquad B' = \left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right) \quad \ldots \quad ,$$

then

$$E' = \left(\begin{array}{c|c} \lambda_1 \ldots \lambda & \\ \hline & \mu \ldots n \end{array}\right) \qquad \lambda \neq \mu$$

commute with all $A', B', C' \ldots$
Thus, $E = S^{-1} E' S$ commute with all $A, B, \ldots$, because

$$AE = S^{-1}A'S \cdot S^{-1}E'S = S^{-1}A'E'S = S^{-1}E'A'S$$
$$= SEA .$$

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E' is not a multiple of $\mathbf{1}$, because, if $E' = \lambda \mathbf{1}$

$$\lambda \mathbf{1} = S^{-1} E' S$$

or $$E' = \lambda S \mathbf{1} S^{-1} = \lambda \mathbf{1},$$

Therefore, if there is no matrix other than $\lambda \mathbf{1}$ which commutes with all matrices then the repr. must be irred.

~~Conversely, if the representation is irreducible, E can be used a certain E exists which is not $\lambda \mathbf{1}$, then E will can be brought to the a diagonal form~~ like E'

~~Then~~,

Now, D_J; If A is commutative with M_3 in the repr. in which M_3 is diagonal,

$$A M_3 = M_3 A$$

or $$A_{M,M'} M' = M A_{M,M'}$$

since different M (from $-J$ to J) appear only once in D_J, $A_{M,M'}$ must be diagonal.

$$A_{MM''} (M_1 + i M_2)_{M''M'} = (M_1 + i M_2)_{MM''} A_{M''M'}$$

$$A_{M,M-1} (M_1 + i M_2)_{M-1,M-1} = (M_1 + i M_2)_{M,M-1} A_{M-1,}$$

~~$A_M \sqrt{(J+M)(J-M+1)} = \sqrt{J-}$~~

which holds, only if $A_M = A_{M-1}$ for $M = J, J-1, \ldots -J+1$, so that A become a numerical multiple of $\mathbf{1}$. Q. E. D.

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Correction

$$x_j' = a_{jk} x_k$$

Infinit. rot.

$$x_j' = (1 + \sum_j a_j I_j) x$$

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad I_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [I_1, I_2] = I_3$$

$$\Psi(xyz) \to \Psi(x'y'z') = (1 + \sum_j \alpha_j J_j) \Psi(xyz)$$

$$J_1 = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad J_2 = \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$J_3 = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad [J_1, J_2] = -J_3.$$

$$\Psi(xyz) \to \Psi(x''y''z'') \text{ where } x'' = (1 - \sum_j \alpha_j I_j) x$$

$$\Psi(xyz) \to \Psi(x''y''z'') = (1 + \sum_j \alpha_j J_j') \Psi(xyz)$$

$$J_j' = -J_j \qquad [J_1', J_2'] = J_3'$$

$$\Psi \to \left(1 - \frac{i}{\hbar}\alpha_j \Lambda_j\right) \Psi(r^{(1)}, \cdots r^{(n)})$$

$$\left(1 - \frac{i}{\hbar}\alpha_j M_j\right) \Psi(r^{(1)} \cdots r^{(n)}; s_1 \cdots s_N)$$

$$M_j = \sum_\nu m_j^{(\nu)} + \frac{\hbar}{2} \sum_\nu \sigma_j^{(\nu)}$$

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Relation between group theoretical Representation and Quantum Mechanical Expressions.

There is some confusion due to a slight difference between q.t. treatment and q.m. treatment of the same object. Namely, in q.t., one can start from the representation

$$x_j' = a_{jk} x_k \qquad (1)$$

of rotation group d_3. Here, x_1, x_2, x_3 are the coordinates of a point in the three dimensional real space. In q.m., however, each of x_j is regarded as an operator which satisfies a number of commutation relations together with p_j, m_j, etc. Just like σ_j, x_j are to operate on the Schrödinger function. ~~This is why the relation (1) should be replaced by~~

$$x_j' = x_k a_{kj} \qquad (1)'$$

Alternatively, x_j can be regarded as a part of ~~eigenfunctions~~, into which arbitrary function is expanded. ~~a certain set of~~

Then, the transformation (1)

~~$x_j' = x_k a_{kj}$~~

induces the representation of d_3:

$$c_j' = a_{jk} c_k a_{kj}$$

where c_j, c_k are the coeff. of expansion of

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an arbitrary function into power series of x_j before and after the rotation as characterized by (1).
For an infinitesimal rotation charact. by parameters $(\alpha_1, \alpha_2, \alpha_3)$, we have

$$(a_{jk}) = \begin{pmatrix} 1 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 1 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 1 \end{pmatrix}$$

$$= \delta_{jk} + \sum_l \alpha_l (I_l)_{jk}$$

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [I_1, I_2] = -I_3 \quad \text{etc.}$$

or $$c'_j = \left(1 + \sum_l \alpha_l I_l\right)_{jk} c_k .$$

Thus, the original ~~rota~~ formula for the rotation

$$x'_j = \left(1 - \sum \alpha_l (I_l)\right)_{jk} x_k$$

n-dimensional
induces the representation, in general,

$$u'_r = \sum_{s=1}^{n} \left(1 + \sum_{l=1}^{3} \alpha_l J_l\right)_{rs} u_s$$

with $$[J_1, J_2] = -J_3 \quad \text{etc.}$$

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Example 1.

$$[J_j, J_k] = -J_l$$

is satisfied by

$$J_1 = i\sigma_1/2 \qquad J_2 = i\sigma_2/2$$
$$J_3 = i\sigma_3/2$$

Thus we have a two-dimensional representation

$$u_r' = \sum_{s=1}^{2}\left(1 + \sum_j \alpha_j J_j\right)_{rs} u_s$$
$$= \sum_{s=1}^{2}\left(1 + \frac{i}{2}\alpha_j \sigma_j\right)_{rs} u_s$$

2.

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}; \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_r' = \sum_{s=1}^{3}\left(1 + \sum_j \alpha_j J_j\right)_{rs} x_s$$

This is slightly different from the transformation of x_j or x_r

$$x_r' = \sum_{s=1}^{3}\left(1 - \sum_j \alpha_j J_j\right)_{rs} x_s$$

This is only because of the gap between q.m. and q.t.

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Expansion into series of ortho-normal functions

In the above exposition, there are two ways of looking at the Schrödinger fn for many-electron system with spins. Namely, one may write it as

$$\Psi_{s_1 s_2 \cdots s_N}(x^{(1)} \cdots x^{(N)})$$

which has 2^N-components, or one can regard it as a function

$$\Psi(x^{(1)} \cdots x^{(N)}; \sigma^{(1)}, \sigma^{(2)} \cdots \sigma^{(N)})$$
$$\equiv \Psi_{s_1 s_2 \cdots s_N}(x^{(1)} \cdots x^{(N)})$$

where $\sigma^{(\nu)}$'s and s_ν's are related by a relation

$$\sigma^{(\nu)} = 3 - 2 s_\nu$$

or $\sigma^{(\nu)} = 1$ for $s_\nu = 1$
$= -1$ for $s_\nu = 2$.

Then $\sigma^{(\nu)}$ denotes the eigenvalue of $\sigma_3^{(\nu)}$, because, if we know that the system is in the state $\sigma_3^{(\nu)} = +1$, then $\Psi(\cdots \overset{\nu}{+1} \cdots) \neq 0$ and $\Psi(\cdots \overset{\nu}{-1} \cdots) = 0$ or

$$\Psi_{s_1 s_2 \cdots \overset{\nu}{1} \cdots s_N} \neq 0, \quad \Psi_{s_1 s_2 \cdots \overset{\nu}{2} \cdots s_N} = 0,$$

~~in conformity with~~ the ~~interpretation that~~ ~~that~~ red and satisfies the requirement:

$$(\sigma_3^{(\nu)})_{s_1 s_2 \cdots s_N; t_1 \cdots t_N} \Psi_{t_1 \cdots t_N}$$
$$= (-1)^{s_\nu - 1} \Psi_{s_1 s_2 \cdots \cdots s_N} = \sigma^{(\nu)} \cdot \Psi_{s_1 s_2 \cdots s_N}$$

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One electron system:

$\Psi(x,1) \equiv \Psi_1(x)$, $\Psi(x,2) \equiv \Psi_2(x)$

σ_3 ~~$\Psi \to \Psi' \equiv$~~

$$\Psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \to \Psi' = \begin{pmatrix} \psi_1(x) \\ -\psi_2(x) \end{pmatrix}$$

$\Psi'(x,s) = s\,\Psi(x,s)$

σ_1 :

$$\Psi' = \begin{pmatrix} \psi_2(x) \\ \psi_1(x) \end{pmatrix}$$

$\Psi'(x,s) = \Psi(x,-s)$

σ_2 :

$$\Psi' = \begin{pmatrix} -i\psi_2(x) \\ i\,\psi_1(x) \end{pmatrix}$$

$\Psi'(x,s) = (-i)^s \Psi(x,-s)$

N-electron system:

$\Psi(x^{(1)} \cdots x^{(N)}; s^{(1)} \cdots s^{(N)}) \equiv \Psi_{s_1 \cdots s_N}(x^{(1)} \cdots x^{(N)})$

$s_\nu = 1 \to s^{(\nu)} = 1$ }

$s_\nu = 2 \to s^{(\nu)} = -1$

σ_3 : $\Psi'(x^{(1)} \cdots x^{(N)}; s^{(1)} \cdots s^{(N)})$

$\equiv s^{(\nu)} \Psi(x^{(1)} \cdots, s^{(1)} \cdots)$

σ_1 : $\Psi'(\quad , \quad)$

$\equiv \Psi(x^{(1)} \cdots; s^{(1)} \cdots -s^{(\nu)} \cdots)$

σ_2 : $\Psi' \equiv (-i)^{s^{(\nu)}} \Psi(\cdots \quad -s^{(\nu)} \cdots)$

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(83)

$$\Psi(r_1^{(1)} \cdots r^{(N)}; s^{(1)} \cdots s^{(N)}) = \sum_{n,J,M} C_{n,J,M} \Psi_{n,J,M}(r^{(1)} \cdots r^{(N)}; s^{(1)} \cdots s^{(N)})$$

$$\sum \int \tilde{\Psi}_{n',J',M'}(r^{(1)} \cdots r^{(N)}; s^{(1)} \cdots s^{(N)}) \Psi_{n,J,M}(r^{(1)} \cdots s^{(N)} \cdots) = \delta_{nn'} \delta_{JJ'} \delta_{MM'}$$

$$F(r^{(1)} \cdots r^{(N)}, p^{(1)} \cdots p^{(N)}, \sigma^{(1)} \cdots \sigma^{(N)})$$

$p^{(\nu)} \to -i\hbar\,\mathrm{grad}_\nu$

$\sigma_1^{(\nu)} \to \Psi'(\cdots s^{(\nu)} \cdots) = s^{(\nu)} \Psi(\cdots s^{(\nu)} \cdots)$

$\sigma_2^{(\nu)} \to \Psi'(\cdots s^{(\nu)} \cdots) = \cdots$

$\sigma_3^{(\nu)} \to \Psi'(\cdots s^{(\nu)} \cdots) = (-i)^{\delta^{(\nu)}} \Psi(\cdots s^{(\nu)} \cdots)$

$$\Psi' = F\Psi_{n,J,M} = \sum_{n'J'M'} \Psi_{n'J'M'} (n'J'M'|F|n,J,M)$$

$$(n'J'M'|F|n,J,M) = \sum_{s^{(1)} \cdots s^{(\nu)}} \int \tilde{\Psi}_{n'J'M'} F \Psi_{n,J,M} \, dr^{(1)} \cdots dr^{(N)}$$

$$\Psi' = F\Psi = \sum_{n,J,M} C_{n,J,M} (F\Psi_{n,J,M})$$

$$= \sum_{n,J,M,n'J'M'} \Psi_{n'J'M'} (n'J'M'|F|n,J,M) C_{n,J,M}$$

On the other hand, one can expand:

$$\Psi' = \sum_{n'J'M'} \Psi_{n'J'M'} C_{n'J'M'}$$

or the operator F induces a linear transf.

$$C_{n'J'M'} = (n'J'M'|F|n,J,M) C_{n,J,M} \quad (1)$$

~~If we go back to the original notation, the effect of $\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \sigma_3^{(\nu)}$ can be best seen, if~~

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Problem: Verify that for $J=\frac{1}{2}$, according to (1), $(c_{n,\frac{1}{2},\frac{1}{2}}, c_{n,\frac{1}{2},-\frac{1}{2}})$ transform like a spinor (ξ_1, ξ_2).

We take the case, $J=\frac{1}{2}$, $M=1/2$ and $-1/2$.

$$\psi_{n,\frac{1}{2},\frac{1}{2}}(\cdots s^{(\nu)}\cdots) = 1 \quad \text{for } s^{(\nu)}=1$$
$$= 0 \quad \text{for } s^{(\nu)}=-1$$

$$\psi_{n,\frac{1}{2},-\frac{1}{2}}(\cdots s^{(\nu)}\cdots) = 0 \quad \text{for } s^{(\nu)}=1$$
$$= 1 \quad \text{for } s^{(\nu)}=-1$$

σ_3: $\sigma_3 \psi_{n,\frac{1}{2},\frac{1}{2}} = \psi_{n,\frac{1}{2},\frac{1}{2}}$

$\sigma_3 \psi_{n,\frac{1}{2},-\frac{1}{2}} = -\psi_{n,\frac{1}{2},\frac{1}{2}}$

σ_1: $\sigma_1 \psi_{n,\frac{1}{2},\frac{1}{2}}(\cdots s^{(\nu)}\cdots) = \psi_{n,\frac{1}{2},\frac{1}{2}}(\cdots s^{(\nu)}\cdots)$

$\sigma_1 \psi_{n,\frac{1}{2},-\frac{1}{2}}(\cdots s^{(\nu)}\cdots) = \psi_{n,\frac{1}{2},\frac{1}{2}}(\cdots s^{(\nu)}\cdots)$

~~σ_1: $\sigma_1 \psi_{n,\frac{1}{2},\frac{1}{2}} \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$~~

$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ ~~$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$~~

σ_2: $\sigma_2 \psi_{n,\frac{1}{2},\frac{1}{2}}(\cdots s_n^{(\nu)}=-1 \cdots) = (+i)\psi_{n,\frac{1}{2},-\frac{1}{2}}$

$\sigma_2 \psi_{n,\frac{1}{2},-\frac{1}{2}}(\cdots s_n^{(\nu)}=+1 \cdots) = (+i)\psi_{n,\frac{1}{2},\frac{1}{2}}$

$\cdot\left(-\frac{1}{2}\,|\,\sigma_2\,|\,\frac{1}{2}\right) = +i$

$\left(\frac{1}{2}\,|\,\sigma_2\,|\,-\frac{1}{2}\right) = -i$

$\sigma_2 = \begin{pmatrix}0&-i\\ i&0\end{pmatrix}$

$\begin{pmatrix}c_{n,\frac{1}{2},\frac{1}{2}}\\ c_{n\,\frac{1}{2},-\frac{1}{2}}\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$

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Selection Rules (84)

~~Hence~~, we have confirmed that the coeffs of expansion ~~induce a linear Fraunt~~. transform linearly, if we ~~multiply~~ operate on an arbitrary wave function Ψ by a linear operator F. a number of

In particular, if the operators $F_1, F_2, \ldots F_{2j+1}$ themselves are basic vectors of a $2j+1$-dim. representation of the rotation groups, the totality of $F_m \Psi_{u,J,M}$ can be regarded as the basis vectors for the direct product

$$\mathbb{D}_j \times \mathbb{D}_J$$

of two representations $\mathbb{D}_j$, $\mathbb{D}_J$. These $(2j+1)\times(2J+1)$ vectors can be ~~regard~~ written as a linear combination of basis vectors for the representations

$$\mathbb{D}_{J+j} + \mathbb{D}_{J+j-1} + \cdots + \mathbb{D}_{|J-j|}$$

into which $\mathbb{D}_j \times \mathbb{D}_J$ can be decomposed.

~~Corres~~ Parallel to this, one can introduce $(2j+1)\times(2J+1)$ dimensional ~~basic~~ vector with components

$$\begin{pmatrix} \frac{\xi^2}{\sqrt{2}} \\ \eta\zeta \\ \frac{\eta^2}{\sqrt{2}} \end{pmatrix} \times \left(\frac{X^{J+M} Y^{J-M}}{\sqrt{(J+M)!(J-M)!}} \right)$$

which can be decomposed into a linear combination of

$$W_{J',M'} \qquad J' = J-1, J, J+1$$

The coefficient of decomposition must be

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the same as the decomposition of the basic vectors

$$\begin{pmatrix} \frac{x+iy}{\sqrt{2}} \\ z \\ \frac{x-iy}{\sqrt{2}} \end{pmatrix} \times \psi_{n,J,M}$$

into series of basic vectors $\psi_{n',J',M'}$, except for the indeterminacy of common factor $\phi(J')$ for a given J' and different M', which is related not only to the behavior of x, y, z and $\psi_{n,J,M}$ with respect to the rotation, but also to the variables that do not change under rotation. (Normalization with respect to all the variables of a function $(x+iy)\psi_{n,J,M}$, such as for instance, is necessary to know exactly the expansion coef., while the corresponding situation is missing in $\frac{z^2}{\sqrt{2}} \cdot \frac{x^{J+M} y^{J-M}}{\sqrt{\quad}}$ etc.

$$u_{jm} U_{JM} = \sum C^{j,m}_{J,M;J'M'} W_{J',M'}$$

$$\psi_{jm} \cdot \psi_{JM} = \sum C^{jm}_{J'M';JM} \psi_{J'M'}$$

$$\mathbf{D}_j \times \mathbf{D}_J = \sum_{J'} \mathbf{D}_{J'}$$

$$\sum_{mM} u_{jm} U_{JM} \mathbf{e}_{jm} \mathbf{e}_{JM} = \sum C^{jm}_{J,M;J'M'} W_{J',M'} \mathbf{e}_{J'M'}$$

$$u_{jm} U_{JM} = \sum C^{jm}_{J,M;J'M'} W_{J'M'} (\mathbf{e}_{mM} \mathbf{e}_{J'M'})$$

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$\Psi(x^{(1)} \cdots x^{(N)}; s_1 \cdots s_N) = \sum_{m,M} u^{n,J}_{m,M} F_m \psi_{n,J,M}$

$(2j+1)(2J+1)$ fcts

$u^{n,J}_{m,M}$ (n, J: fixed): components of $(2j+1)(2J+1)$-dimensional vector.

Expansion:

$$F_m \psi_{n,J,M} = \sum_{n'J'M'} C^{m}_{n'J'M';\,n,J,M}\, \psi_{n'J'M'}$$

$$C_{n'J'M';\,nJM} = (n'J'M' | F_m | nJM)$$

vector in $2J'+1$-dimensional space with components (with fixed m, n, J):

$$v^{n'J',nJ}_{M'} = \sum_{m,M} C_{n'J'M';\,n,JM}\, u^{n,J}_{mM}$$

Rotation A:

$$u'^{n,J}_{mM} = A_{mM;m'M'}\, u^{n,J}_{m'M'}$$

$(2j+1)(2J+1)$ dimensional repr.

$$(v^{n'J',nJ}_{M'})' = \sum_{M''} a^{J'}_{M'M''} (v^{n'J',nJ}_{M''})$$

$(2J'+1)$-dimensional repr.

(∵ $\Psi = u^{n,J}_{mM} F_m \psi_{n,J,M} = \sum_{n'J'} v^{n'J',nJ}_{M'} \psi_{n'J'M'}$

Rotation changes Ψ into

$$\Psi' = \sum A_{mM;m'M'} u^{n,J}_{m'M'} F_m \psi_{nJM} = \sum (u'^{nJ}_{mM}) F_m \psi_{nJM}$$

$$= \sum v^{n'J',nJ}_{M''} \psi_{n'J'M'} a^{J'}_{M'M''}$$

$$= \sum (v'^{n'J',nJ}_{M'}) \psi_{n'J'M'}$$

$$= \sum u'^{nJ}_{mM} C^{m}_{n'J'M';\,nJM} \psi_{n'J'M'} \quad)$$

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one can give the interpretation:
Thus, by a change of the coordinate system as given by the coord. transf.

$$(C^{m}_{n'J'M';nJM}) \qquad m, n' : \text{fixed}$$

from

$$U^{nJ}_{mM} \quad (n, J : \text{fixed})$$

to

$$U^{n'J';nJ}_{M'} \quad (n, J : \text{fixed}), \quad {}^{m}_{n'} : \text{fixed}$$

the representation matrix

$$(A_{mM;m'M'}).$$

with $(2j+1)(2J+1)$ rows and columns is completely reduced to

$$\begin{pmatrix} A^{J}_{M'M''} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

with different J's. Of course, this is true only if we take one n'.
According to the formula

$$\mathbb{D}_j \times \mathbb{D}_J = \mathbb{D}_{J+j} + \mathbb{D}_{J+j-1} \cdots + \mathbb{D}_{|J-j|}, \quad (1)$$

we can conclude that J' must be restricted to

$$J' = J+j, J+j-1, \cdots |J-j|. \quad (2)$$

In other words, the coefficients

$$C_{n'J'M';nJM}$$

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are to be zero unless J' satisfy the above selection rule.
(If $C_{n'J'M';nJM} \neq 0$ identically, by the coordinate transformation, ~~Λ~~ $2J'+1$-dim subspace vector with comp. $V_M^{n'J';nJ}$ must give the reprs. ~~$2J'+1$~~ $D_{2J'+1}$, which is impossible unless (2) is satisfied)

This give the selection rule for $j=1$
$$J' = J+1, J, J-1.$$

In q.m., the same rule can be derived from the C.R. (Condon, Shortley, T.A.S. p.60)
$$[m_j, x_k] = i\hbar x_l$$
$$[m^2, x_k] = m_j[m_j, x_k] + [m_j, x_k]m_j$$
$k=1$: $= -i\hbar(m_2 x_3 - m_3 x_2 + x_3 m_2 - x_3 m_2)$
~~$= (-i\hbar)\, m_l\, (i\hbar)$~~
$$[m^2,[m^2, x_1]]$$
$$= -i\hbar\{m_2[m^2, x_3] - m_3[m^2, x_2] + [m^2, x_3]m_2 - [m^2, x_3]m_2\}$$
$$= (-i\hbar)^2\{m_2(m_1 x_2 - m_2 x_1 + x_2 m_1 - x_1 m_2) - m_3(m_3 x_1 - m_1 x_3 + x_1 m_3 - x_1 m_3) + (m_1 x_2 - m_2 x_1 + x_2 m_1 - x_1 m_2)m_2 - (m_3 x_1 - m_1 x_3 + x_1 m_3 - x_3 m_3)m_3\}$$
$$\pm -\hbar^2\{-(m_2^2 + m_3^2)x_1 + x_1(m_2^2 + m_3^2) - (m_2^2 + m_3^2)x_1 - i\hbar m_1 x_3 + i\hbar x_2 - x_1(m_2^2 + m_3^2) + i\hbar x_3 - i\hbar x_2\} + 4m_1(m_2 x_2 + m_3 x_3)\}$$

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or $(M^2)^2 x_j - 2M^2 x_j M^2 + x_j (M^2)^2$

$= 2t^2 (M^2 x_j + x_j M^2) - 4t^2 M_j (M_k x_k)$

$[M_j, (M_k x_k)] = 0$

$\because [M_1, (M_1 x_1 + M_2 x_2 + M_3 x_3)]$

$= [M_1, M_2] x_2 + M_2 [M_1, x_2]$

$+ [M_1, M_3] x_3 + M_3 [M_1, x_3]$

$= it M_3 x_2 + M_2 (it) x_3$

$= it M_2 x_3 - M_3 (it) x_2$

$= 0$

$[M^2, (M_k x_k)] = 0$

~~$(J(J+1))^2 \cdots 2$~~

$J = J'$:

$(J(J+1))^2 (\cdots J | x_j | J \cdots) - 2(J(J+1))^2 (\cdots J | x_j | \cdots J \cdots)$

$+ \cdots$

$= 2 J(J+1) (\cdots J \cdots | x_j | \cdots J \cdots) - 4 t^2$

$(\cdots J \cdots | M_j | \cdots J \cdots)(\cdots J \cdots | M_k x_k | \cdots J \cdots)$

$J \neq J'$:

$\{J(J+1)\}^2 - \{J'(J'+1)\}^2$

$= 2\{J(J+1) + J'(J'+1)\}$

$(J-J')^2 (J+J'+1)^2 = (J+J'+1)^2 - (J-J')^2 - 1$

$\{(J-J')^2 - 1\}\{(J+J'+1)^2 - 1\} = 0$

$J - J' = \pm 1$

$J + J' + 1 = \pm 1,$

or $J' = J \pm 1,$

(and also $J' = J$.)

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Selection rule for M:

Let us take the rotation around z-axis of an angle φ:

$$A_{mM;m'M'} = \begin{pmatrix} e^{i(J+1)\varphi} & & & 0 \\ & \ddots & & \\ & & e^{i(m+M)\varphi} & \\ & & & \ddots \\ 0 & & & \end{pmatrix}$$

$m = 1, 0, -1$

$$a^{J'}_{M'M''} = \begin{pmatrix} \ddots & & & 0 \\ & \ddots & & \\ & & e^{iM'\varphi} & \\ & & & \ddots \\ 0 & & & \end{pmatrix}$$

$M' = m + M \qquad m = 1, 0, -1$

or $M' = M \pm 1, M.$

From the C.R.

$$[M_3, x_1] = i\hbar x_2$$
$$[M_3, x_2] = -i\hbar x_1$$
$$[M_3, x_3] = 0$$

or $[M_3, x_1 + i x_2] = \hbar(x_1 + i x_2)$

$[M_3, x_1 - i x_2] = \hbar(x_1 - i x_2)$

$(n' J', M' | x_3 | n J M) = 0$ unless $M' = M$

$(n', J', M' | x_1 \pm i x_2 | n J M) = 0$ unless $M' = M \pm 1$

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Intensity Rule

$C^{m}_{n'J'M', nJM}$

depends on $n'J'M'$ for a fixed m. The dependence on $n'J'$ can determined, only if we know more about the nature of F_m, $\psi_{n,J,M}$. However, once $n'J'$ are fixed, (i.e. ($J' = J-1$ or J or $J+1$), relative values of C's for different m, M, M' (and fixed J, J') can be determined group theoretically.

(Weyl, p. 199; Waerden, p. 77)

However, in this case it is much easier to deduce the same result from elem. q. m. method.

Condon, Shortley, p. 61.

$$[M_1 - iM_2,\ x_1 - ix_2] = [M_1, x_1] - i[M_2, x_1] - i[M_1, x_2] - [M_2, x_2] = 0$$

$$(n, J, M-1 | M_1 - iM_2 | n, J, M)(n, J, M | x_1 - ix_2 | n', J', M+1)$$
$$= (n, J, M-1 | x_1 - ix_2 | n', J', M)(n', J', M | M_1 - iM_2 | n', J', M+1)$$

$$(n, J, M-1 | x_1 - ix_2 | n'J'M) : (n, J, M | x_1 - ix_2 | n', J', M+1)$$
$$= (n, J, M-1 | M_1 - iM_2 | n, J, M) : (n', J', M | M_1 - iM_2 | n', J', M+1)$$
$$= \sqrt{(J+M)(J-M+1)} : \sqrt{(J'+M+1)(J'-M)}$$

etc.

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$[M_1 - iM_2, x_3] = +k(x_1 - ix_2)$

$(n,J,M|M_1-iM_2|n,J,M+1)(n,J,M+1|x_3|n'J'M+1)$
$-(n,J,M|x_3|n'J'M)(n'J'M|M_1-iM_2|n'J'M+1)$
$= +k(n,J,M|x_1-ix_2|n'J'M+1)$

$[M_1+iM_2, x_3] = -k(x_1+ix_2)$

$(n,J,M+1|M_1+iM_2|n,J,M)(n,J,M|x_3|n'J'M)$
$-(n,J,M+1|x_3|n'J'M+1)(n'J'M+1|M_1+iM_2|n'J'M)$
$= -k(n,J,M+1|x_1+ix_2|n'J'M)$

From these two equations, one can determine the ratio of $(n,J,M+1|x_3|n'J'M+1)$ and $(n,J,M|x_3|n'J'M)$ in terms of the ratio of $(n,J,M|x_1-ix_2|n'J'M+1)$ and $(n,J,M+1|x_1+ix_2|n'J'M)$, ~~which can be made 1 if we suitably normalize the eigenfunctions for a pair of $n,J;n',J'$.~~

$[M_1+iM_2, x_1+ix_2] = 0$

$(n,J,M+1|M_1+iM_2|n''J''M)(nJM|x_1+ix_2|n'J'M-1)$
$= (n,J,M+1|x_1+ix_2|n'J'M)(n'J'M|M_1+iM_2|n'J'M-1)$

$(nJM|x_1+ix_2|n'J'M+1):(nJM+1|x_1+ix_2|n'J'M)$
$= (n'J'M|M_1+iM_2|n'J'M-1):(nJM+1|M_1+iM_2|nJM)$
$= \sqrt{(J'+M)(J'-M+1)} : \sqrt{(J+M+1)(J-M)}$

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Russell Saunders Coupling or LS-Coupling.

$H = H_0 + H'$

H_0: spin indep. $\quad \Psi = \varphi\chi$

$H_0\varphi = E\varphi$

$\varphi \rightarrow \varphi_{L,\lambda}$

$\chi \rightarrow \chi_{S,\sigma}$

$m \simeq \Lambda + \Sigma$

$\Lambda = \sum m^{(\nu)} \qquad \Sigma = \frac{\hbar}{2}\sum_\nu \sigma^{(\nu)}$

$\Lambda^2 : \hbar^2 L(L+1)$

$\Lambda_3 : \hbar\lambda \; (= -L, \ldots +L)$

$\Sigma^2 : \hbar^2 S(S+1)$

$\Lambda_3 : \hbar\sigma \quad (-S \ldots +S)$

$D_L \times D_S = \sum_J D_J$

$J = L+S, \ \ldots |L-S|$

Multiplet structure with the ~~the reas~~ separation as determined by H'.

The reason why different S give ~~diffo~~ widely different levels is connected with the symmetry of H with respect to permutation.

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Problem:
Deduce Landé g-formula for the Zeeman effect in LS-coupling.

$$H' = \frac{e\mathcal{H}}{2mc}(\Lambda_3 + 2\Sigma_3)$$

$$= \frac{e\mathcal{H}}{2mc}(M_3 + \Sigma_3)$$

$$(M,L,S,JM|\,H'\,|nLSJM)$$

$$= \frac{e\hbar\mathcal{H}M}{2mc}\,g(LSJ)$$

$$g(L,S,J) = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

Proof

$$[M^2[M^2,\Sigma_3]]$$

$$= (M^2)^2\Sigma_3 - 2M^2\Sigma_3 M^2 + \Sigma_3(M^2)^2$$

$$= 2\hbar^2(M^2\Sigma_3 + \Sigma_3 M^2) - 4\hbar^2 M(M\Sigma)$$

$$[M_3, M\Sigma] = 0. \qquad (JM|M\Sigma|J'M') = 0 \text{ for } J \neq J',\ M \neq M'$$

$$\Lambda^2 = (M - \Sigma)^2$$

$$\hbar^2 L(L+1) = \hbar^2 J(J+1) + \hbar^2 S(S+1) - 2(JM|M\Sigma|JM)$$

$$(JM|M\Sigma|JM) = \hbar^2\frac{J(J+1) - L(L+1) + S(S+1)}{2}$$

~~$(J(J+1))^2(JM|\Sigma_3|JM)$~~

$$J(J+1)(JM|\Sigma_3|JM)$$

$$= M\hbar\cdot(JM|M\Sigma|JM)$$

$$(JM|\Sigma_3|JM) = \hbar M\cdot\frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}.$$

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Selection Rules in L-S-coupling

~~$\tilde{\varphi}_{L',\lambda'} \chi_{S',s'} \quad \varphi_{L,\lambda} \chi_{S,s}$~~

$$\Psi_{n,L,S;J,M} = \sum_{\lambda,s} C_{\lambda,s}\, \varphi_{n,L,\lambda}\, \chi_{S,s}$$

$$\sum \int \tilde{\Psi}_{n',L',S';J',M'}\, x_k^{(\omega)}\, \Psi_{n,L,S;J,M} \qquad k=1,2,3$$

$$= \sum \tilde{C}^{J'M'}_{\lambda's'}\, C^{JM}_{\lambda s} \int \tilde{\varphi}_{n'L'\lambda'}\, x_k^{(\omega)}\, \varphi_{nL\lambda} \sum \tilde{\chi}_{S's'}\, \chi_{Ss}$$

- $S' = S$. $\mathbb{D}_1 \times \mathbb{D}_L = \mathbb{D}_{L'}$

$L' = L, L \pm 1$. ($L=0 \to L'=0$ forbidden)

$L = 0, 1, 2, \ldots\ldots$

25+1 | S | $S, P, D, \ldots\ldots$

1 | 0 | ^1S ^1P .

2 | 1/2 | ^2S ^2P .

3 | 1 | ^3S ^3P .

4 | 3/2 | ^4S ^4P .

~~$J = 0, 1/2, \ldots$~~

~~$^1S_{1/2}$~~

$L=0$, $S=1/2$ $J=1/2$: $^2S_{1/2}$

$L=1$, $S=1/2$ $J=1/2, 3/2$: $^2P_{1/2}$ $^2P_{3/2}$

$J' = J, J \pm 1$ ($J=0 \to J'=0$ forbidden)

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One single particle Approximation

$H = H_0 + H'$

H' → electronic repulsion

or Hartree approximation

common central potential

$\varphi_{n_1 l_1 m_1}(\vec{r}^{(1)})\, \varphi_{n_2 l_2 m_2}(\vec{r}^{(2)}) \cdots \varphi_{n_N l_N m_N}(\vec{r}^{(N)})$

$D_{l_1} \times D_{l_2} \times \cdots \times D_{l_N}$

$= (D_{(l_1+l_2)} + \cdots + D_{|l_1-l_2|}) \times \cdots \times D_{l_N}$

$= D_{l_1+l_2+\cdots l_N} + \cdots$

$\psi_{n, l_1, l_2, \cdots l_N, L, \lambda}$

$L = l_1+l_2+\cdots l_N,\ l_1+l_2+\cdots l_N - 1, \cdots$

Coulomb repulsion

$\int \tilde{\psi}_{n, l_i, L, \lambda} \sum_{\substack{\mu<\nu \\ =1}}^{N} \frac{e^2}{r_{\mu\nu}} \psi_{n, l_i, L, \lambda}$

For a given $l_1, l_2, \cdots l_N$, the energy levels splits widely due to the difference in Coulomb energy for different L.

among the levels of a multiplet for with definite L, S, J, the separation of levels is determined by additional term in the Hamiltonian

$H' = \frac{1}{2m^2c^2} \sum_{\nu=1}^{N} \left(\frac{1}{r^{(\nu)}} \frac{dV(r^{(\nu)})}{dr^{(\nu)}} \right) \sum (\vec{m}^{(\nu)} \vec{s}^{(\nu)})$

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where $V(r)$ is the effective central potential common to all electrons and

$$S^{(\nu)} = \frac{\hbar}{2}\sigma^{(\nu)}, \quad *$$

~~Example. 2-electron atom~~

On this approximation

$$(m^{(\nu)})^2 = \hbar^2 l_\nu(l_\nu+1)$$
$$\nu = 1, 2, \cdots N$$

are all commutative with H_0 and also with $\mathfrak{M}$, $\mathfrak{M}_3$, Σ, Σ_3; Λ, Λ_3, so that the eigenfunctions can be characterized by

$$J, S, L, l_1, l_2, \cdots l_N$$

and some other q.n. $n_1, n_2, \cdots n_N$ suitably chosen.

If we include the Coulomb repulsion between electrons, this is no longer true, and one must look for appropriate linear comb. for one-particle eigenfunctions.

---

* H' commutes with $\mathfrak{M}$ because

$$[(m^{(\nu)} + S^{(\nu)}),\ m^{(\nu)}S^{(\nu)}] = 0$$

or

$$[m_j^{(\nu)}, m^{(\nu)}S^{(\nu)}]S^{(\nu)} + m^{(\nu)}[S_j^{(\nu)}, S^{(\nu)}]$$
$$= -i\hbar(m_3^{(\nu)}S_2^{(\nu)} - m_2^{(\nu)}S_3^{(\nu)}) - i\hbar(m_2^{(\nu)}S_3^{(\nu)}$$
$$- m_3^{(\nu)}S_2^{(\nu)}) = 0$$

Thus, J, $\mathfrak{M}$ are good q.n. Λ^2 and Σ^2 ~~also~~ do not commute with H', because so that H' is not diagonal with respect to L and S, but diagonal elements give the change of energy levels approximately, if H' small enough.

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Parity

As for the ~~coordinate~~ orbital eigenfunction $\varphi_{n,l,\lambda}$, the reflection

$$x_j \to x_j' = -x_j$$

induces the transformation

$$\varphi_{n,l,\lambda} \to \pm \varphi_{n,l,\lambda}$$

according as l is even or odd, because ~~one~~ we can always chose the eigenfunctions such that they are simultaneously the eigenfunction for the operators of S, R ($S^2=1$). The eigenfunction belonging to the eigenvalue $+1$ should consist of only even powers of $x_j^{(\nu)}$: for instance const., $x_j^{(\nu)} x_k^{(\mu)}$, $x_j^{(\nu)} x_k^{(\mu)} x_l^{(\nu)} x_m^{(\nu)}$, ...

Whereas the eigenfunctions belonging to -1 should consist of only odd powers of $x_j^{(\nu)}$: $x_j^{(\nu)}$, $x_j^{(\nu)} x_k^{(\mu)} x_l^{(\nu)}$, etc. . . . .

Alternatively, one can expand any function $\varphi(x^{(1)} \cdots x^{(N)})$ into series of product of

$$Y_{l_1 m_1}(\theta_1, \varphi_1) \cdots Y_{l_N m_N}(\theta_N, \varphi_N).$$

Then, if we take up a portion consisting of

$$(2l_1+1) \cdots (2l_N+1)$$

spherical harmonics, it gives the repres.

$$D_{l_1} \times D_{l_2} \times \cdots \times D_{l_N} \quad (-1)^{l_1+l_2+\cdots+l_N}$$

If we decompose it into irreducible parts

$$\sum_l D_l, \quad l = l_1+l_2+\cdots l_N,\ l_1+l_2+\cdots+l_N-1, \cdots$$

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then it is obvious that the parity is even or odd according as Σl_ν is even or odd.*

$D_L^{(+)}$ for ~~$l_1+l_2+\cdots+l_N$: even~~ $l_1+l_2+\cdots+l_N$ even

$D_L^{(-)}$ for ~~$l_1+l_2+\cdots+l_N$: odd~~ $l_1+l_2+\cdots+l_N$ odd

The spin function, the parity is always even, because ~~σ~~ is an axial vector which does not change under reflection.

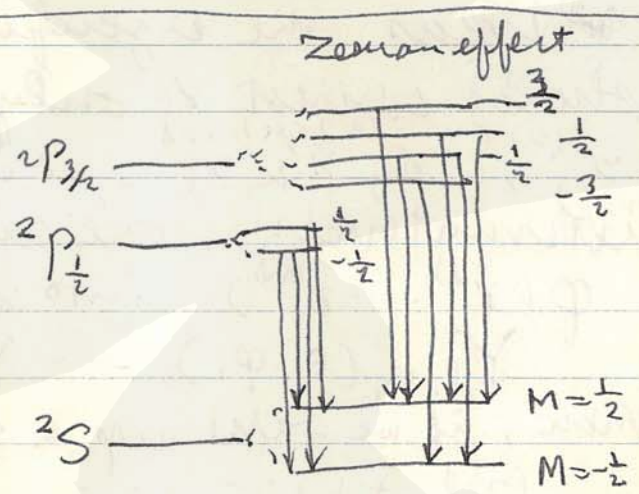
$D_L^{(+)} \times D_S^{(+)} = \sum_J D_J^{(+)}$ $\sum_\nu l_\nu$: ~~even odd~~

$D_L^{(-)} \times D_S^{(+)} = \sum D_J^{(-)}$ $\sum_\nu l_\nu$: odd

---

* In general, $D_L^{(+)}$ is a mixture of $l_1, \cdots l_N$ with even Σl_ν or odd Σl_ν alone.

Zeeman effect



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Permutation Group S_N
(Symmetric Group)

a ~~my~~ q.m. system consisting of N identical particles such as electrons has Hamiltonian H which is invariant with respect to the permutation of between coordinates, momenta and spin ~~variables~~ operators of ~~all~~ any number of particles. Thus, if

$$\Psi(x^{(1)}, s_1; x^{(2)}, s_2; \cdots x^{(N)}, s_N)$$

is a solution of Schrödinger eq.

$$H\Psi = E\Psi$$

for a discrete eigenvalue E, then

$$\Psi(x^{(2)}, s_2; x^{(1)}, s_1; \cdots x^{(N)}, s_N),$$

for instance, is also the eigenfn for the same eigenvalue E. In general, there are N! ~~different~~ eigenfns which can be obtained from an eigenfn Ψ by N! - permutations of coordinates ~~of~~ and spins of N-particles. ~~If~~ Some of them differ from each other, in general, so that we have an extra-degeneracy in addition to the degeneracy, for instance, due to, the invariance ~~of H~~ with respect to rotations.

Since the totality of N! - permutations constitutes the symmetric group, we expect that ~~the application of~~ group theory would ~~be~~ ~~b~~ find another application

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to q.m. in this connection. We expect that the application of theory of symm. groups would be applied even much more widely to q.m. systems than the theory of rotation group, because the symmetry ~~with respect to~~ due to the identity of particles is something more fundamental than the rotational symmetry.

~~In q.m. we~~ Now the symmetry properties of wave fn as ~~wi~~ with respect to permutation ~~can be~~ there are various possibilities as will be discussed in the following. However, in q.m. so far we know only two cases.

(i) Symmetric wave fn

(ii) antisymmetric wave fn.

The latter corresponds to Pauli's exclusion principle and, if we confine our attention to many electron system, we ~~may~~ may forget all other possibilities other than the case (ii).

In order to simplify the problem, let us take the case approximately ~~Hamilt.~~ Hamilt. Now as before ~~and regarding by regarding~~ and we correspond. approx. eigenfn

$$\varphi(x^{(1)} \cdots x^{(N)}) \chi(s_1 \cdots s_N)$$

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Two Electron System

In order to see the effect of Pauli principle, let us consider the well-known example of ~~Two~~ Helium-like atom. In the approx.,

$$\varphi(x_1^{(1)}, x_2^{(2)})\, u(s_1, s_2),$$

$$\varphi(x_1^{(1)}, x_2^{(2)}) \cong C_1 \varphi^{sym}(x^{(1)}, x^{(2)}) + C_2 \varphi^{anti.}(x^{(1)}, x^{(2)})$$

$$u(s_1, s_2) = \begin{cases} \alpha(s_1)\alpha(s_2) \\ \frac{1}{\sqrt{2}}\left(\alpha(s_1)\beta(s_2) + \beta(s_1)\alpha(s_2)\right) \\ \beta(s_1)\beta(s_2) \end{cases}$$

$S=1$ triplet $u_{1,1}$, $u_{1,0}$, $u_{1,-1}$

$S=0$ singlet $u_{0,0} = \frac{1}{\sqrt{2}}\left(\alpha(s_1)\beta(s_2) - \beta(s_1)\alpha(s_2)\right)$

$$\Psi = \varphi^{sym} u_{0,0}$$

or $= \varphi^{anti} u_{1,M_s}$ $\quad M = 1, 0, -1.$

In individual orbit approximation,

$$H' = \frac{e^2}{r_{12}}$$

$$\varphi^{sym}_{anti} = \frac{1}{\sqrt{2}}\left(\varphi_{n_1 l_1 m_1}(r_1)\varphi_{n_2 l_2 m_2}(r_2) \pm \varphi_{n_2 l_2 m_2}(r_2)\varphi_{n_2 l_2 m_2}(r_1)\right)$$

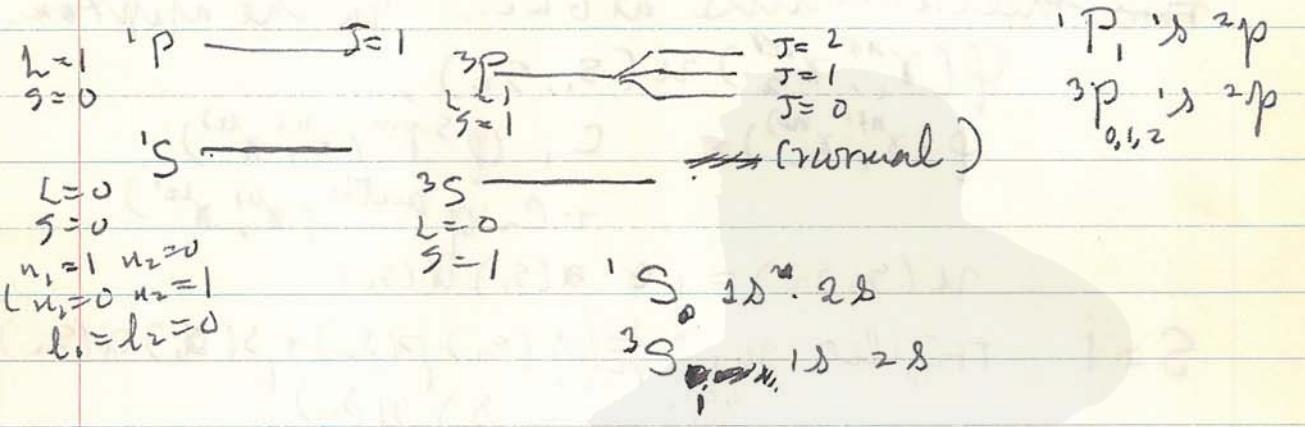
$$\Delta E = \iint \tilde{\varphi}_{n_1 l_1 m_1}(r_1)\tilde{\varphi}_{n_2 l_2 m_2}(r_2)\frac{e^2}{r_{12}}\varphi_{n_1 l_1 m_1}(r_1)\varphi_{n_2 l_2 m_2}(r_2)$$

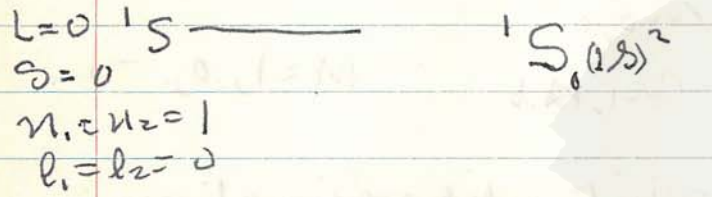
$$\pm \iint \tilde{\varphi}_{n_1 l_1 m_1}(r_2)\tilde{\varphi}_{n_2 l_2 m_2}(r_1)\frac{e^2}{r_{12}}\varphi_{n_1 l_1 m_1}(r_1)\varphi_{n_2 l_2 m_2}(r_2)$$

The exchange integral give rise to a wide separation of sym. and anti. levels with same L and different S. In this case,

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the integral is positive and so that the sym. (parhelium) levels are considerably higher than anti. (orthohelium) level!





~~As clear~~ Now, if we go over to a more general case like three electron atoms, the classification of electron states according to symmetry ~~properties~~ with respect to permutation is not as simple as above. In addition to the symmetric state (with respect to spins), there are states with more complicated symmetry characteristics. In order to deal with the general ~~case~~ N-electron system effective, one has to know ~~more about~~ something about permutation groups.

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Now any permutation can be written as a product of cycles. For instance,

$$\begin{matrix}1&2&3&4&5\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\ 3&1&2&5&4\end{matrix} \rightarrow (132)(45)$$

$N=5$

No two of the cycles written the same number. The procedure is unique. Whichever number one may start, but the same cycle has the same cycle, only the notation being different;

$$(132) \equiv (321) \equiv (213)$$

If we decompose the permutation into cycles;

$$P_r = (r_1\, r_2 \cdots r_{n_1})(r_{n_1+1} \cdots r_{n_2})(r_{n_2+1} \cdots) \cdots (\cdots r_N)$$

P_t = ~~(t_1 t_2 ...)~~ ; permutation

then the net result of permutation

$$P_t^{-1} P_r P_t$$

is

$$P_S = P_t^{-1} P_r P_t = (s_1 s_2 \cdots s_{n_1})(s_{n_1+1} \cdots) \cdots (\cdots s_N)$$

which has the same structure as P_r. Thus all the permutations of the same class have the same structure,

Proof: P_t : $s_1 \rightarrow r_1$ $s_2 \rightarrow r_2$ $s_3 \rightarrow r_3 \cdots$

P_r : $r_1 \rightarrow r_2$ $r_2 \rightarrow r_3$ $\cdots$ $r_{n_1} \rightarrow r_1$

P_t^{-1} : $r_1 \rightarrow s_1$ $r_2 \rightarrow s_2$ $r_3 \rightarrow s_3 \cdots$

$P_S = P_t^{-1} P_r P_t$: $s_1 \rightarrow s_2$ $s_2 \rightarrow s_3 \cdots s_{n_1} \rightarrow s_1$

Thus, the class is determined by the number

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of cycles and the number of integers they contain.

Of course P_r and P_r^{-1} belong to the same class:

$$P_r = (r_1 r_2 \cdots r_{n_1})(\cdots)\cdots(\cdots r_N)$$
$$P_r^{-1} = (r_{n_1} \cdots r_2 r_1)(\cdots)\cdots$$

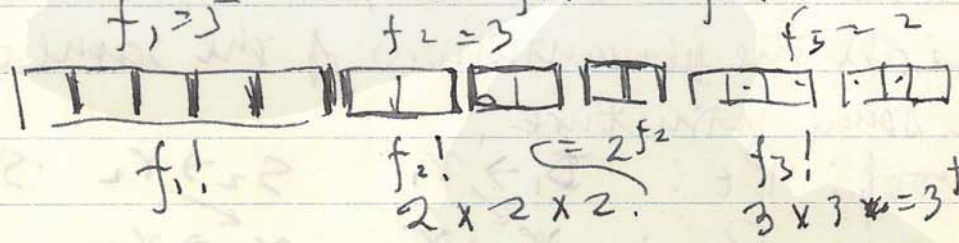
In other words, to each class one can assign a definite way of decomposition (or partition) of N:

$$N = 1\cdot f_1 + 2 n_2 + 3 n_3$$
$$1\cdot f_1 + 2 f_2 + 3 f_3 + \cdots$$
$$+ N \cdot f_N$$

$(f_1, f_2, f_3 \cdots f_N)$: each 0 or positive integer. *

The number of element of the class $(f_1, f_2 \cdots f_N)$ is

$$n_{f_1 f_2 \cdots f_N} = \frac{N!}{1^{f_1} f_1!\ 2^{f_2} f_2! \cdots N^{f_N} f_N!}$$



---

* Example: $N = 5$

$f_1 = 2$, $f_2 = 1$, $f_3 = 1$.

$(132)(45)$

Weyl, 328.

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Group Character

In (any finite) dimensional representation of a group. Of (a-)

$$a, b, c$$
$$A, B, C \cdots$$

the trace of the matrix A

$$\chi(a) = \sum_{r=1}^{n} A_{rr}$$

is called the (group) character.

Equivalent representations have the same character. $a \to A' = SAS^{-1}$ etc.

$$\chi'(a) = \sum_{r=1}^{n} (SAS^{-1})_{rr} = \sum_{r} S_{rs} A_{st} S_{tr}$$
$$= \sum_{s,t} A_{st} \delta_{ts} = \sum_{s} A_{ss} = \chi(a)$$

In any representation, the elements a, b of G belonging to the same class have the same character for the similar reason. Thus, character is a class function, satisfying the relation

$$\chi(ab) = \chi(ba) \quad \text{or} \quad \chi(aba^{-1}) = \chi(b)$$

(i.e. $\sum_{rs} A_{rs} B_{sr} = \sum_{rs} B_{rs} A_{sr}$)

If, in particular, the representation is unitary

$$a \to A \qquad a^{-1} \to A^{-1} = A^{*}$$
$$\tilde{\chi}(a) = \chi(a^{-1}).$$

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The character of the unit element 1 is always the same as the dimension n

$$\chi(1) = n$$

of the ~~repres~~. one valued repres.

If the representation is irreducible, we call the character primitive.

If a ~~gro~~ representation $\mathfrak{R}$ of a group G is reduced completely

$$\mathfrak{R} = m_1 \{\mathfrak{R}_1\} + m_2 \{\mathfrak{R}_2\} + \cdots$$

where equivalent representations are just together, then

$$\chi(a) = m_1 \chi_1(a) + m_2 \chi_2(a) + \cdots$$

For a representation

$$\mathfrak{R}_1 \times \mathfrak{R}_2 ,$$

the each with the character χ_1, χ_2, we have

$$\chi(a) = \chi_1(a)\,\chi_2(a)$$

because

$$a \to A^{(1)}_{rs} \cdot A^{(2)}_{\mu\nu} = A_{r\mu;s\nu}$$

$$\chi(a) = \sum_{r\mu} A_{r\mu;r\mu} = \sum_{r,\mu} A^{(1)}_{rr} A^{(2)}_{\mu\mu}$$

$$= \chi_1(a)\,\chi_2[a]$$

For one dimensional representation,

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there is no distinction between the character and the representation itself.

Now let us go back to the operators $P_1, P_2 \cdots P_{N!}$ and construct a quantity

$$Q_r = \frac{1}{N!} \sum_t (P_t) P_r (P_t^{-1}),$$

then one can immediately see that ~~two~~ two permutations P_r, P_s which belong to the same class

$$Q_r = Q_s,$$

so that there are ~~only~~ different Q's, the number of which is equal to the number of ~~π~~ classes in π_N.

$Q_1, Q_2, \cdots Q_f$ f: number of classes are all commutative with all P_r and hence commutative with each other. Let us take a representation, in which $Q_1, Q_2, \cdots Q_f$ are all diagonal:

$$\sum_k (Q_\rho)_{kk} = \frac{1}{N!} \sum_{k,l,m} (P_t)_{kl} (P_r)_{lm} (P_t^{-1})_{mk}$$

$$= \sum_k (P_r)_{kk}$$

or, of course, the character of Q_r is the same as that of P_r. If we take a set of simultaneous eigenfunctions of $Q_1, Q_2, \cdots Q_f$ belonging to a set of eigenvalues ~~$\chi_1, \chi_2, \cdots \chi$~~ $q_1, q_2 \cdots q_f$.

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There are degeneracy, in general, and so that the number of linear independent eigenfunctions be m,

$u_1, u_2 \cdots u_m$.

Then we have a m-dim. repres. of T_N, and the corresponding matrix element $(Q_r)_{kk} = q_r$ $k=1, 2, \cdots m$ is independent of k. Thus

$$\chi_r = \sum_k (Q_r)_{kk} = m q_r$$
$$r = 1, 2, \cdots f$$

give the character of the representation.

If we take orthogonal and normalized eigenfunctions, then the representation is unitary.

$$(Q_r) u_k = u_\ell \cdot (Q_r)_{\ell k},$$

$$u'_k = (P_r) \cdot u_k = \sum_\ell u_\ell (P_r)_{\ell k}$$

$$(P_s)(P_r) u_k = \sum_\ell P_s u_\ell (P_r)_{\ell k}$$
$$= \sum u_{\nu j} (P_s)_{\nu j, l} (P_r)_{\ell k}$$

$$(P_r)^{-1}_{\nu j, l} \cdot u_k = -$$

$$(P_r)_{\ell k} = \sum_{s_1 \cdots s_N} \tilde{u}_\ell (P_r) u_k$$

$$(P_r^{-1})_{\ell k} = \sum \tilde{u}_\ell \cdot (P_r^{-1}) u_k$$
$$= \sum (P_r \tilde{u}_\ell) \cdot u_k$$
$$= \sum \left(\sum_{\nu j} \tilde{u}_{\nu j} (\tilde{P}_r)_{\nu j, \ell} \cdot u_k \right)$$
$$= (\tilde{P}_r)_{k\ell}$$

$$P_r^{-1} = P_r^*$$

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(Weyl, p. 15 Σ)

Fundamental Theorems of the theory of Group Representations

Schur's lemma: Assumption

Σ: irreducible system of linear correspondences of m-dimensional vector space r on itself.

Ω: similar system in n-dim vector space s.

Define A such that

$$\Sigma A = A \Omega,$$

i.e. for each U of Σ there exists V of Ω such that

$$UA = AV$$

and conversely for each V of Ω there exists U of Σ such that the same relation is fulfilled.

Conclusion statement: Either

(i) $A = 0$, or

(ii) $m = n$ and $\det A \neq 0$, Σ and Ω being equivalent.

$$A : \begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2k} & \cdots & A_{2n} \\ \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{mn} \end{pmatrix}$$

$$\sum_{j=1}^{m} U_{ij} A_{jk} = \sum_{j=1}^{n} A_{ij} V_{jk}$$

$$a \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{pmatrix}$$

If we regard $(A_{1k}, \cdots A_{mk})$ the components of a vector in r, the subspace of r spanned

$A^{(k)} a^{(k)}$

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by n vectors $a^{(1)} \cdots a^{(n)}$ in r transforms into itself, under any U of Σ, because

~~$U a^{(k)} = v_{jk}$~~

~~$\sum_{s,k} v_{js} a_{sk} e_k = \sum_{j,k} a_{rj} v_{jk} e_k$~~

and the right hand ($\sum_{j=1}^{m} U_{rs} A_{sk} = \sum_{j=1}^{n} A_{rj} V_{jk}$

the above relation means that

$A^{(k)} \to A^{(k)'}$ $(a^{(k)})' = U a^{(k)}$

$= v_{1k} a^{(1)} + v_{2k} a^{(2)} \cdots + v_{nk} a^{(n)}$

$(a_r^{(k)})' = \sum_j v_{jk} a_r^{(j)}$

with components

Now, since Σ is irreducible, there is no invariant subspace of r which transform into itself under Σ, this subspace must be either 0 (or i.e. $a^{(k)} = 0$, $A = 0$) or r itself. In the latter case, m of $a^{(k)}$ $(k = 1, 2, \cdots n)$ are linearly independent. This is only possible for $n \geq m$.

In the latter case, one can construct the transposed matrices U^*, V^*, Λ^* which satisfy the relation

$$V^* A^* = A^* U^*.$$

Then, the same reasoning as above leads us to $m \geq n$.

Thus, the conclusion is either

(i) $A = 0$, or

(ii) $m = n$ and $a^{(k)}$ $k = 1, 2 \cdots m (= n)$

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are linearly independent, so that

$$\det A \neq 0.$$

(∵ If $\det A = 0$, the simultaneous equations

$$\sum_k a_{jk} w_k = 0$$

has a non-zero solution w_k, or

$$\sum a^{kl} w_k = 0$$

contradictory to the assumption)

Hence we found that

I, If two irreducible systems Σ, Ω (of matrices) are inequivalent,

$$UA = AV$$

is satisfied only by $A = 0$.

II, If Σ is an irreducible system, and a correspondence A commutes with all correspondences U of Σ:

$$UA = AU,$$

if and only if A is a multiple of unit matrix 1

(Proof of II. From such a A, one can construct

$$B = A - \alpha 1$$

so that $\det B = 0$

(∵ If $\det A \neq 0$, the simultaneous eq.

$$\sum_k a_{jk} w_k = \alpha w_j$$

has a solution for a certain (eigenvalue) of α, so that $\det B = 0$.)

$$UB' = BU$$

~~$\sum_{k,k'} a_{jk} a_{kl} = a_{jk} a_{kl}$~~

which is only true for $B = 0$ (because

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, as before, n- (or n-) vectors ~~$f^{(k)}$~~ must span the whole n-dim. vector space. (In the latter case they must be linearly independent with $\det B \neq 0$.)

We can apply the above results to the representations of a group.

Fundamental theorem:

1. If $s \to U(s)$, $s \to V(s)$ are two inequivalent irreducible representations of a group G, $U(s) A = A V(s)$ is satisfied (indep of s) only by $A = 0$.
2. $U(s) A = A U(s)$ is satisfied for all s only by $A = \lambda 1$

Remark:

Σ: multiplicative system; if U, U': correspondences in Σ, then UU' is also in Σ. U may not necessarily have posers U^{-1} — its determinant may be 0.

Burnside's Theorem:

Σ: irreducible and $U = (u_{jk})$ are correspondences of an n-dim. space on to itself. then u_{jk} are linearly independent. Namely, the relation

$$\mathrm{Tr}(LU) = \sum_{jk} l_{jk} u_{jk} = 0$$

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Orthogonality of Group Characters

G: a finite group

Σ: a representation of G. One can always ~~take~~ construct an equivalent unitary representation as follows

$$u'_s(b) = \sum A^{(b)}_{rs} u_s$$

$$\tilde{u}'_s(b) = \sum \tilde{A}^{(b)}_{rs} \tilde{u}_s$$

$$\sum_b \sum_s \tilde{u}'_s(b) u'_s(b) = \sum \tilde{u}_t A^*_{tr}(b) A_{rs}(b) u_s$$

$$= \sum \tilde{u}_t H_{ts} u_s$$

this is invariant, because

$$\sum \tilde{u}_t A^*_{tq}(c) A^*_{qp}(b) A_{pr}(b) A_{rs}(c) u_s$$

$$= \sum \tilde{u}_t H_{ts} u_s$$

Since H is Hermitian, i.e.

$$H_{ts} = \sum A^*_{tr}(b) A_{rs}(b)$$

$$\tilde{H}_{st} = \sum \tilde{A}^*_{sr}(b) \tilde{A}_{rt}(b) = H_{st},$$

one can change the coordinate system, so that $H'_{ts} = \delta_{ts}$. ($= (S H_{tb} S^{-1})_{ts}$) Then, the new representation is unitary.

In the case of continuous group, the summation over the elements of group is to be replaced by integration with respect parameters. This is possible for a rotation group. In the case of Lorentz group, the situation is different.

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Now let $b \to U(b) \to A(b)$ and $b \to A'(b)$ be two inequivalent irreducible repres. of a finite group G.

$A = (A_{jk}(b))$ n-dim.

$A' = (A'_{\lambda\kappa}(b))$ n'-dim.

$(A'(b))^{-1}_{\kappa\lambda} = \tilde{A}^{(b)}_{\lambda\kappa}$

B: arbitrary matrix with n rows and n'-columns.

$$\sum_b A(b) \cdot B\, A'^{-1}(b) = C$$

is invariant in the sense

$$A(b') B A'^{-1}(b') = B C$$

because

$$\sum_b A(b')A(b) = B \cdot A'^{-1}(b) A'^{-1}(b')$$
$$= \sum_{b''} A(b'') B\, A'^{-1}(b'') = C$$

or $A(b')\, C = C\, A'(b')$

Thus, C must be zero.

$$\sum_b \sum_k \sum_\kappa A_{jk}(b) B_{k\kappa} (A'(b))^{-1}_{\kappa\lambda} = 0$$

or, since $B_{k\kappa}$ are arbitrary, we have $n^2 \cdot n'^2$ relations

$$\sum_b A_{jk}(b) A'_{\lambda\kappa}(b) = 0 \qquad (1)$$

Thus, the matrix elements of two inequivalent repres. must be orthogonal.

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If we take a single irred. repres. of sn-dim.
$b \to A(b)$:

$$\sum_b \sum_{k,l} A_{jk}(b)\, B_{kl}\, A^{-1}_{lm}(b) = C_{jm}$$

~~is~~ satisfies

$$AC = CA$$

for arbitrary B, so that C must be an integral multiple of 1,
or

$$\sum_b \sum_{k,l} A_{jk}(b)\, B_{kl}\, A^{-1}_{lm}(b) = \alpha \cdot \delta_{jm}$$

~~or~~ If we take B_{kl} ~~differentiated~~ from 0 only for one pair of values of k,l:

$$\sum_b A_{jk}(b)\, A^{-1}_{lm}(b) = \alpha_{kl}\, \delta_{jm}$$

Hence, take $j = m$ and sum over j:

$$\sum_b A_{jk}(b)\, A^{-1}_{lj}(b) = \alpha_{kl} \cdot n$$

or

$$g\,\delta_{kl} = \alpha_{kl} \cdot n$$

where g is the order of the group.

$$\sum_b A_{jk}(b)\, A^{-1}_{lm}(b) = \frac{g}{n}\,\delta_{kl}\,\delta_{jm}$$

or

$$\sum_b \tilde{A}_{ml}(b)\, A_{jk}(b) = \frac{g}{n}\,\delta_{kl}\,\delta_{jm}$$

Take the average

$$\frac{1}{g}\sum_b \tilde{A}_{ml}(b)\, A_{jk}(b) = \frac{1}{n}\,\delta_{kl}\,\delta_{jm}$$

in particular,

$$\frac{1}{g}\sum_b \tilde{A}_{jk}(b)\, A_{jk}(b) = \frac{1}{n}.$$

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Thus, the components of one or more inequivalent irreducible reprs. constitute a (mutually) orthogonal set of functions on the group manifold. (~~they are linearly ind.~~)
Hence, the components $A_{jk}(b)$, $A_{\kappa\lambda}(b)$ ... etc. are linearly independent. (~~Each of them~~ ~~They~~ can be regarded as the components of a vector in ~~an~~ a g-dimension space. They are linearly independent, because they are mutually orthogonal.)
They are at most g such vectors:

$$n_1^2 + n_2^2 + \cdots \leq g$$

where $n_1, n_2 \ldots$ are dimensions of inequiv. irr. repr.
Orthogonality relations for the group characters can be obtained by taking in (1)

$$j = k, \quad \lambda = \kappa$$

and summing over j and λ

$$\sum_b \chi(b)\,\tilde{\chi}(b) = 0$$

and $j = k$ in (1) and summing over j

$$\sum_b \tilde{\chi}(b)\chi(b) = g$$

Thus, the primitive characters of inequivalent representations constitutes a set of orthogonal (and ~~normalized~~ if we take $\chi(b)/\sqrt{g}$) functions on the group manifold.

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$\chi(b)$: charact. of regular repres.: $b = 1 \quad g = m_1^2 + m_2^2 + \cdots$

other elements have no diagonal elements.

They are linearly independent.

A representation of G_f can be unambiguously described by its character, no distinction being made between equivalent representations. The numbers of times $m_1, m_2 \ldots$ an irreducible representations occurs $\chi_1, \chi_2, \ldots$ occurs are m from $\chi(b) = m_1 \chi_1(b) + m_2 \chi_2(b) + \cdots$

$$\frac{1}{g} \sum_b \tilde{\chi}_r(b) \chi(b) = m_r$$

and

$$\frac{1}{g} \sum_b \tilde{\chi}(b) \chi(b) = m_1^2 + m_2^2 + \cdots$$

Thus the criterion for the irreducibility of a given representation is:

> It is necessary and sufficient that the $\frac{1}{g} \sum_b \tilde{\chi}(b) \chi(b) = 1$!!!

Since the characters are class functions, there can be no more than f linearly independent class functions, where f is the number of classes. Hence a finite group has no more inequivalent irreducible representations than classes.

(The primitive characters constitute a complete set of orthogonal class functions (there exist f irreducible representations) and the totality of matrix elements

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Constitute orthogonal vectors in the g-dimensional space:

$g = n^2 + n'^2 + \cdots\cdots$)

Problem 1. Verify that all the irred. repr. are there in

Example 1. π_3. $g = 3! = 6$

1 (12), (13), (23)
(123), (132)

$\chi_1^{(1)} = \chi_{(12)}^{(1)} = \chi_{(123)}^{(1)} = 1$ $\quad n^{(1)} = 1$

$\chi_1^{(2)} = \chi_{(123)}^{(2)} = 1$ $\quad \chi_{(12)}^{(2)} = -1$ $\quad n^{(2)} = 1$

$\chi_1^{(3)} = 2$ $\quad \chi_{(12)}^{(3)} = 0$ $\quad \chi_{(123)}^{(3)} = -1$, $n^{(3)} = 2$

$(12) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ $\quad (13) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

$(23) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad (1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(123) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad (132) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\chi_1^{(1)}\chi_1^{(3)} + 3\chi_{(12)}^{(1)}\chi_{(12)}^{(3)} + 2\chi_{(123)}^{(1)}\chi_{(123)}^{(3)}$

$= 2 - 0 - 2 = 0$

$\chi_1^{(2)}\chi_1^{(3)} + 3\chi_{(12)}^{(1)}\chi_{(12)}^{(3)} + 2\chi_{(123)}^{(1)}\chi_{(123)}^{(3)} = 0$

$6 = 1^2 + 1^2 + 2^2$.

Example 2. π_4: $g = 4! = 24$

$f = 5$: $\underline{1} \times 1$ $\quad (12) \times \underline{6}$

$(123) \times \underline{8}$ $\quad (12)(34) \times \underline{3}$

$(1234) \times \underline{6}$ $\quad (1243)$ (1324)

(1342) $(14\cdots)$

$24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$

$(\chi_1^{(3)})^2 + 2(\chi_{(123)}^{(3)})^2 = 4 + 2 = 6$, irreducible repres. !!!

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Three Electron-Atom

1. $u_1 = \alpha(1)\alpha(2)\alpha(3)$

$u_2 = \frac{1}{\sqrt{3}}\{\alpha(1)\alpha(2)\beta(3) + \alpha(1)\beta(2)\alpha(3) + \beta(1)\alpha(2)\alpha(3)\}$

$u_3 = \frac{1}{\sqrt{3}}\{\alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3) + \beta(1)\beta(2)\alpha(3)\}$

$u_4 = \beta(1)\beta(2)\beta(3)$

Symmetric triplet state, for $L=0$,
for example ${}^3S_1$

$\chi_1^{(1)} = \chi_{(123)}^{(1)} = \chi_{(123)}^{(1)} = 1$,

the corresponding orbital eigenfunction must be anti-symmetric.

$$\varphi(r_1, r_2, r_3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \varphi_{n_1 l_1 m_1}(r_1) & \varphi_{n_1 l_1 m_1}(r_2) & \varphi_{n_1 l_1 m_1}(r_3) \\ \varphi_{n_2 l_2 m_2}(r_1) & \varphi_{n_2 l_2 m_2}(r_2) & \varphi_{n_2 l_2 m_2}(r_3) \\ \varphi_{n_3 l_3 m_3}(r_1) & \varphi_{n_3 l_3 m_3}(r_2) & \varphi_{n_3 l_3 m_3}(r_3) \end{vmatrix}$$

This is different from zero, only if $(n_1 l_1 m_1)$, $(n_2 l_2 m_2)$, $(n_3 l_3 n_3)$ are different from each other.

2. Anti-symmetric state with respect to the spins can be occur, because comb. of
$\alpha(1)\alpha(2)\beta(3)$ etc.
cannot be antisym. unless it is identically zero.

3. $v_1 = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\}\alpha(3)$ | $w_1 = \frac{1}{\sqrt{2}}\{\alpha_1\beta_3 - \beta_1\alpha_3\}\alpha_2$

$v_2 = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\}\beta(3)$ | $w_2 = \frac{1}{\sqrt{2}}\{\alpha_1\beta_3 - \beta_1\alpha_3\}\beta_2$

$P_{12} v_1 = -v_1$, ~~$P_{12} v_2 = -1$~~ $P_{12} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$

$P_{12} v_2 = \frac{1}{\sqrt{2}}\{\beta_1\alpha_3\alpha_2 - \alpha_1\alpha_3\beta_2\}$

$w_1 = v_{21} - v_1$

~~$P_{12} v_4 = \frac{1}{\sqrt{2}}\{\beta_1\beta_3\alpha_2 - \beta_1\alpha_3\beta_2\} = v_4 = v_2$~~

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$(1|P_{12}|1) = u_1 \cdot P_{12} v_1 = -1$ $\quad u_2 P_{12} v_1 = 0 = (2|P_{12}|1)$

$(1|P_{12}|2) = v_1 P_{12} u_1 = -1$ $\quad u_2 P_{12} u_2 = 1 = (2|P_{12}|2)$

$$P_{12} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \qquad P_{23} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_{13} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad \text{etc,}$$

a complete set of orthogonal rows.

Thus, the ~~whole spin function is~~ decomposed into eigenfunctions are obtained: First four ~~six~~ are ~~basis~~ basis vectors of 4-one dimensional symmetric representation of ~~the~~ π_3. ~~Fifth~~ and Sixth together constitute ~~a~~ basic vectors of two dimensional repres. (irred.) Seventh and Eighth ~~belo~~ constitute basic vectors of the same ~~dimen~~ two dimensional representation.

They are at the same time the basic vectors for the 4-, 2- 2- dimensional representations of the rotation group (which are all irreducible)

(u_1, u_2, u_3, u_4) $\quad S = 3/2$

(v_1, v_2) $\quad S = \frac{1}{2}$

(w_1, w_2) $\quad S = \frac{1}{2}$

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Four Electron Atom

$N = 4$. π_4 : $g = 4! = 24$

Classes	1	(12)	(123)	(12)(34)	(1234)
$f = 5$	1	6	8	3	6
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	-1	1	1	-1
$\chi^{(3)}$	2	0		-2	
$\chi^{(4)}$					
$\chi^{(5)}$					

$m_1^2 + m_2^2 + m_3^2 + \cdots \leq 24$

Subgroups: 1, (12), (34), (12)(34)

(abelian)

1	1	1	1
1	-1	-1	1
1	x	y	z
	+1	-1	+1
	-1	+1	-1

$1 + x^2 + y^2 + z^2 = 4$

$1 + x + y + z = 0$) $1 + z = 0$

$1 - x - y + z = 0$ $x + y = 0$

$z = -1$. $x^2 + y^2 = 2$

$x = +1$, $y = -1$

$x = -1$, $y = +1$

Thus, we take a two dimensional representation of π_4 in which the elements of Abelian subgroups are diagonalized.

$(12) \to \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\chi_{(12)} = 0$ $(34) \to \begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$ $\chi_{(34)} = 0$

$(12)(34) \to \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ $\chi_{(12)(34)} = -2$?

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1	6	8	3	6
1	−1	1	1	−1
2	0	x	−2	y

$4 + 8x^2 + 12 + 6y^2 = 24$

$2 + 8x - 6 + 6y = 0$

$2 + 8x - 6 - 6y = 0$

$4x + 3y = 2$

$4x - 3y = 2$

$x = \frac{1}{2}$

$y = 0$

$4 + 2 + 12 = 20 \neq 24.$

1	(12)	(34)	(12)(34)
$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\+1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$
3	1	1	−1

3 (1) 1 (6) x (8) −1 (3) y (6)

$3 + 6 + 8x^2 - 3 + 6y^2 = 24 \times h$

$8x^2 + 6y^2 = 24 \times h \to 6$

$h = 5$ $= 114$

$4x^2 + 3y^2 = 57$

$x =$

1 (+)	6 (−)	8 (+)	3 (+)	6 (−)
2	0	x	2	y

$4 + 8x^2 + 12 + 6y^2 = 24$

$2 + 8x + 6 + 6y = 0$

$2 + 8x + 6 - 6y = 0$

$x = -1, \quad y = 0$

(3) 2 0 −1 2 0

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$1^2+1^2+2^2+?+?\leq 24$

$?+?\leq 18$

3-dimensional repres.

$1=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ $(12)=\begin{pmatrix}\cdot&+1&\\&&-1\end{pmatrix}$ $(34)=\begin{pmatrix}\cdot&-1&\\&&+1\end{pmatrix}$

$(12)(34)=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix}$

	(1)(+)	(6)(−)	(8)(+)	(3)(+)	(6)(−)
	3	±1	x	−1	y

$9+6+x^2+3+6y^2=24$

$x=0 \quad y=\mp 1.$

$3\pm 6-3\mp 6y=0$

$3\mp 6-3\pm 6=0.$

$\chi^{(4)}$:	3	+1	0	−1	−1
$\chi^{(5)}$:	3	−1	0	−1	+1

	(1)	(12)	(123)	(12)(34)	(1234)
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	−1	1	1	−1
$\chi^{(3)}$	2	0	−1	2	0
$\chi^{(4)}$	3	+1	0	−1	−1
$\chi^{(5)}$	3	−1	0	−1	+1

~~Invariant Subgroup : Alternate group A_4 consisting of even permutations of $q=12$. this is invariant subgroup of π_4 and the factor group $\pi_4/A_4=\pi_3$~~

~~$(1\cdot(123)\ (12)(34))\ ((12)\ (12)(34))$~~

~~$((123),\ (12)(34))\ (\qquad)$~~

~~$(\qquad ??$~~

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spin function　$2^N = 2^4 = 16$.

D_2
- $\alpha_1\alpha_2\alpha_3\alpha_4$
- $\beta_1\alpha_2\alpha_3\alpha_4 + \alpha_1\beta_2\alpha_3\alpha_4 + \cdots$ (5)
- $\beta_1\beta_2\alpha_3\alpha_4 + \cdots$　$S = 2$
- $\beta_1\beta_2\beta_3\alpha_4 + \cdots$　$2S+1 = 4$
- $\beta_1\beta_2\beta_3\beta_4$

π_4　$S = 1$,　$2S+1 = 3$

$\frac{1}{\sqrt{2}}(\beta_1\alpha_2 - \alpha_1\beta_2)\alpha_3\alpha_4$

$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\beta_1\alpha_2 - \alpha_1\beta_2)(\alpha_3\beta_4 + \beta_3\alpha_4)$

$\frac{1}{\sqrt{2}}(\beta_1\alpha_2 - \alpha_1\beta_2)\alpha_3\beta_4$

(3)

$\frac{1}{\sqrt{2}}(\beta_1\alpha_3 - \alpha_1\beta_3)\alpha_2\alpha_4$

$\cdots$

(3)

$\frac{1}{\sqrt{2}}(\beta_1\alpha_4 - \alpha_1\beta_4)\alpha_2\alpha_3$

$\cdots$

$S = 0$

$\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\beta_2)(\alpha_3\beta_4 - \beta_3\alpha_4)$

$S = 0$

$\frac{1}{2}(\beta_1\alpha_3 - \alpha_1\beta_3)(\alpha_2\beta_4 - \beta_2\alpha_4)$

π_4

D_1

—	—	—
—	—	—
—	—	—

$\chi^{(5)}$?　$\chi^{(4)}$?

(12)	(34)	
-1	-1	two dimensional
	(12)(34)	
	+1	

$\chi^{(3)}$

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$\Psi(x^{(\nu)}, s_\nu) \equiv \Psi(1, 2, \cdots N)$

Dirac's method

(Dirac, Chapter IX) Let us introduce the self-adjoint operator P_r defined by

$$P_r \Psi(x^{(\nu)}, s_\nu) = \Psi(P_r(x^{(\nu)}, s_\nu))$$

$$\sum \int \tilde{\Psi}'(x^{(\nu)}, s_\nu) \cdot P_r \Psi(x^{(\nu)}, s_\nu) \qquad \int \tilde{\Psi}' P_r^{-1} P_r \Psi$$

$$= \sum \int \tilde{\Psi}'(P_r^{-1}(x^{(\nu)}, s_\nu)) \Psi(x^{(\nu)}, s_\nu) \qquad = \int \tilde{\Psi}' \Psi$$

$$= \sum \int \tilde{\Psi}' P_r^{-1} \cdot \Psi$$

$\Psi \to P_r \Psi$ induces $\tilde{\Psi}' \to \tilde{\Psi}' P_r^{-1}$

On the other hand

$P_r \Psi$ induces $\tilde{P}_r \tilde{\Psi}' = \tilde{\Psi}' P_r^*$,

for arbitrary Ψ and Ψ', so that

$$P_r^{-1} = P_r^*$$

$$\int \tilde{\Psi}' P_r^* \cdot P_r \Psi = \int \tilde{\Psi}' \Psi$$

i.e. P_r is a unitary operator.

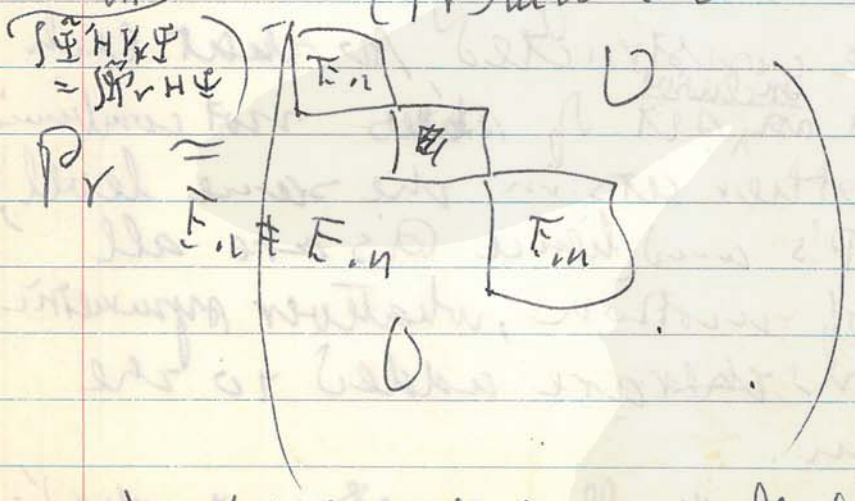
P_r $(r = 1, \cdots N!)$ are not necessarily commutative with each other. The Hamiltonian H is commutative with all of them and hence, in each repr,

$$(P_r)_{mn} H_{nn'} = E_m (P_r)_{mn}$$

$$(P_r)_{mn} \neq 0 \quad \text{only for } E_n = E_m$$

(cf. $\int \tilde{\Psi}' H P_r \Psi = \int \tilde{\Psi}' H \Psi$ or $\int \tilde{\Psi}' H P_r \Psi = \int \tilde{\Psi}' P_r H \Psi$)

$P_r \cong$

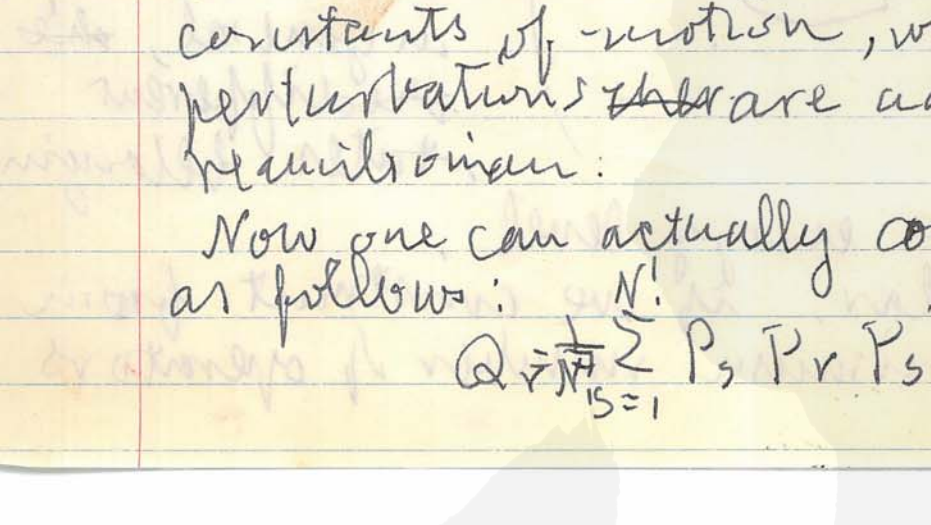


If P_r is diagonal, its eigenvalues (diagonal elements) designate, in general, the different states belonging to the same energy level.

In particular, if we construct from P_r, maximum number of operators

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$Q_1, Q_2, \ldots Q_f$ which commute with all P_r, then they can be diagonalized simultaneously, and a set of ~~diag~~ eigenvalues of Q's ~~will be the~~ characteristic for one value of $(Q_s^{(m)})$, ~~the~~ P's must have a definite ~~set of~~ value, because $(P_r)_{mn} \neq 0$ only for $(Q_s^{(m)})_{mm} = (Q_s^{(n)})_{nn}$ for $s = 1, 2, \ldots f$. This is possible, only if a set of values $Q_s^{(m)}$ $(s = 1, 2 \ldots f)$ subdivides the submatrices with a definite ~~En~~ energy value.



~~This means that~~ In this way, ~~the~~ a number of sets of eigenfunctions belonging to the same energy are constructed so that each constitute an exclusive set of states not combining with ~~the~~ other sets in the same level, because P's and hence Q's are all constants of motion, whatever symmetric perturbations ~~that~~ are added to the Hamiltonian.

Now one can actually construct ~~the~~ Φ's as follows:

$$\Phi_r = \frac{1}{N!} \sum_{s=1}^{N!} P_s P_r P_s^{-1}$$

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Q_r, Q_t are different, only if they belong to different classes. Thus we have as many independent Q's as there are classes in π_N. There is no more, because ~~if there exist~~ any 1-1 of P_r's can be written as a linear combination

$$Q' = \sum_{r=1}^{f} c_r P_r$$

In order that Q' is commutative with P_t, say,

$$Q' P_t = P_t Q'$$

or

$$\sum_r c_r P_t P_r P_t^{-1} = \sum_r c_r P_r$$

By taking different P_t, $P_t P_r P_t^{-1}$ runs through different permutations of the class of P_r, so that all c_r for the same class must be equal. This means that Q' must be a linear combination of $Q_1, Q_2 \ldots Q_f$ (f: number of classes)

There is a simple relation between

Now $Q_1, Q_2 \ldots Q_f$ have ~~are~~ only one set of characters $(\chi_1 \ldots \chi_f)$. ~~If~~ We ~~consider~~ can have a representation of π_N, (by the help of eigenfunctions), in which $Q_1, Q_2 \ldots Q_f$ are represented by a diagonal matrices together with P_t, ~~Then~~ This representation is completely

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reducible and each of the irreducible part is characterized by a set of eigenvalues $(\alpha_1, \alpha_2 \cdots \alpha_f)$. On the other hand, since the diagonal sum (or the character) of the matrices of P_r belonging to the same class are the same, they must also be equal to the diagonal sum of Ω_r (obtained from P_r). If the dimension of the representation is m,

$$\chi_r = m\alpha_r' \qquad r = 1, 2, \cdots f.$$

since the diagonal elements of α_r are all equal to α_r'.

Now from the fact that any fn of P_r can be rewritten as a linear comb. of $\chi_p \chi_q \cdots$ then any fn of P_r which commutes with all P_r is a linear combination of α's, we have a definite relation for π_N:

$$\Omega_p \Omega_q = \sum_{j=1}^{N} a_{pqj} \Omega_j .$$

The number of possible solutions of this equation is equal to the number of classes of π_N.

Proof: It must be equal to the number of different eigenvalues of an arbitrary function of $\Omega_1 \cdots \Omega_f$, because one can construct B in such a way that the eigenvalues of B are all

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different for all different sets of values of Q's $(s=1,2,\cdots f)$. From the relations

$$P_1 = \sum_{p=1}^{f} b_p^{(1)} Q_p$$

$$P^2 = \sum b_p^{(2)} Q_p$$

$$\vdots$$

$$P^f = \sum b_p^{(f)} Q_p$$

and the relation

$$Q_1 = 1,$$

one can eliminate $Q_1 \cdots Q_f$ and obtains an relation of the type

$$B^f + c_1 B^{f-1} + \cdots + c_f = 0$$

There are no more than f-solutions of this equation for B. Correspondingly ~~set of~~ the number of possible ~~the~~ sets of values of $Q_1, \cdots Q_f$.

It cannot be less than f, because, if so, a linear relation

$$\sum_{p=1}^{f} c'_p Q'_p = 0$$

could be satisfied for all these sets of values of Q's $(s=1,2,\cdots f)$, so that

$$\sum_{p=1}^{f} c'_p Q_p = 0,$$

~~cont~~ which is a contradiction to the

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fact that they are linearly independent.

Thus, we conclude that there are just f - mutually exclusive sets of states, all physically important quantities are symmetric ~~with respect to~~ the and commutative with all Pr, so that they must be functions of Q's and other operators which are also Pr.
Thus, any ~~At~~ two states belonging to the same set of eigenvalues of Q's are physically indistinguishable. If there are m linearly independent eigenfunctions belonging to the same set of Q's ($s=1,2,\ldots f$) m is ~~at the same time~~ the degree of degeneracy and is equal to the dimension of irreducible representation.*

---

* The reason why each set of values $Q_1, Q_2 \ldots Q_f$ (or $\chi_1^* \ldots \chi_f$) ~~gives~~ characterizes an irreducible repres. is that ~~Q's~~ the only matrices which can commute with all the matrices of Pr ~~of~~ this repres. of π_N, is a linear comb. of ~~$Q_1, Q_2, \ldots$~~ those for $Q_1, Q_2 \ldots Q_f$ or a multiple of ~~I~~ unit matrix.

Such a degeneracy cannot be removed by any perturbation whatsoever!!!

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Remarks:

(i) $(Q_1', Q_2' \cdots Q_f')$ determines an irreducible representation, because the only ~~operators~~ matrices which are commutative with all P's are a linear combination of Q's and, therefore, they are multiples of the unit matrix. (This completes the proof of theorems)

(ii) The ~~set of~~ states belonging to ~~the~~ a ~~set~~ set of values $(Q_1' Q_2' \cdots Q_f')$ is exclusive set in that arbitrary perturbation V does not give rise to the transition to other ~~sets~~ of states belonging to other sets, because

$$[Q_r, V] = 0$$

or $\langle Q_r^{(m)}| V_{mn} = V_{mn} \langle Q_r^{(n)}|$

for $r = 1, 2, \cdots f$

or $V_{mn} \neq 0$ only if $Q_r^{(m)} = Q_r^{(n)}$

for $r = 1, 2, \cdots f$.

(iii) If we ignore the spin dependent terms in H, ~~$\Lambda^2, \Lambda_3, \Sigma^2, \Lambda\Sigma_3$~~, $Q_r^{(\sigma)} = \frac{1}{N!} \sum_{t=1}^{g} P_t^{(\sigma)} P_t^{(\sigma)} (P_t^{(\sigma)})^{-1}$ are all commutative with each other and with H. ~~Further M, M_3 are~~ (Or one can take the combination ~~Further~~ $M^2, M_3, \Lambda^2, \Sigma^2, Q_r^{(\sigma)}$.

A set of eigenvalues of these operators characterizes ~~the~~ a set of states which is exclusive in this approximation.

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One can further introduce the operators $Q_r^{(\sigma)}$ $(r=1,2,\cdots)$ which commute with all others. However, the exclusion principle may restrict the combination of eigenvalues of $Q_r^{(\sigma)}$ and $Q_r^{(\lambda)}$ as will be discussed below.

(iv) Evaluation of eigenvalues of $Q_r^{(\sigma)}$:

$$Q_{(12)}^{(\sigma)} = \frac{1}{2}\sum_{\mu<\nu}(1+\sigma^{(\mu)}\sigma^{(\nu)}) \Big/ \frac{N(N-1)}{2}$$

$$= \frac{1}{2}\left\{1 + \frac{2}{N(N-1)}\sum_{\mu<\nu}\sigma^{(\mu)}\sigma^{(\nu)}\right\}$$

$$2\sum_{\mu<\nu}\sigma^{(\mu)}\sigma^{(\nu)} = \left(\sum_{\mu}\sigma^{(\mu)}\cdot\sum_{\nu}\sigma^{(\nu)}\right) - \sum_{\mu}\sigma^{(\mu)}\sigma^{(\mu)}$$

$$= \left(\frac{2}{\hbar}\sum\right)^2 - 3N$$

$$Q_{12}^{(\sigma)\prime} = 4S(S+1) - 3N$$

$$Q^{(\sigma)\prime} = \frac{1}{2}\left\{1 + \frac{4S(S+1)-3N}{N(N-1)}\right\}$$

$$= -\frac{N(N-4)+4S(S+1)}{2N(N-1)}$$

where

$$S = \frac{1}{2}N,\ \frac{1}{2}N-1,\ \frac{1}{2}N-2,\ \cdots$$

Thus, $Q_{12}^{(\sigma)\prime}$ is uniquely determined by S and N. Similarly, all other characters are determined by N and S, because they are commutative with Σ and symmetric functions $\sigma^{(\mu)}$, so that they must be functions of $(\Sigma)^2$ and N.

Conversely, one can actually construct other characters in terms of S and N. (For instance, the components

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(the linear symmetric part of Θ's)

$\sum \sim \sum_\mu \sigma^{(\mu)}$ are ~~itself is~~ not commutative with each other, so that there is no linear combination of $\sigma^{(\mu)}$ which can appear in Q's. The bilinear combinations $\sum \sigma^{(\mu)}\sigma^{(\nu)}$, $\sum_\mu \sigma^{(\mu)} \sum_\nu \sigma^{(\nu)}$ (~ symm $\sum^2$) and $\sum_\mu \sigma^{(\mu)}\sigma^{(\mu)}$ are all functions of $\sum^2$ and N. Further, only symmetric functions of Θ's of even degrees can appear in Q's, because the functions of odd degrees can not be invariant with respect to rotations. Now the high degree terms such as

$$\sum_{\pi<\lambda} (\Theta^{(\pi)}\cdot\Theta^{(\lambda)})(\Theta^{(\mu)}\Theta^{(\nu)}) = \sum_{\pi<\lambda}(\Theta^{(\pi)}\Theta^{(\lambda)}) \cdot \sum_{\mu<\nu}(\Theta^{(\mu)}\Theta^{(\nu)})$$

$$\sum_{\substack{\pi<\lambda \\ \pi<\mu}} (\Theta^{(\pi)}\cdot\Theta^{(\lambda)})(\Theta^{(\pi)}\Theta^{(\mu)}) = \sum_{\pi<\lambda<\mu} \{(\Theta^{(\pi)}\Theta^{(\lambda)})(\Theta^{(\pi)}\Theta^{(\mu)}) + (\Theta^{(\pi)}\Theta^{(\mu)})(\Theta^{(\pi)}\Theta^{(\lambda)})\}$$

$$\therefore (\Theta^{(\pi)}\Theta^{(\lambda)})(\Theta^{(\pi)}\Theta^{(\mu)}) = (\Theta^{(\pi)}\Theta^{(\mu)})(\Theta^{(\pi)}\Theta^{(\lambda)}) + i(\Theta^{(\pi)}\cdot[\Theta^{(\lambda)}\times\Theta^{(\mu)}])$$

$$(\Theta^{(\pi)}\cdot\Theta^{(\lambda)})(\Theta^{(\pi)}\cdot\Theta^{(\mu)}) + (\Theta^{(\pi)}\cdot\Theta^{(\mu)})(\Theta^{(\pi)}\Theta^{(\lambda)}) = 2\,\Theta^{(\lambda)}\Theta^{(\mu)} \quad (\pi \neq \lambda, \mu)$$

$$\sum_{\pi<\lambda} (\Theta^{(\pi)}\Theta^{(\lambda)})^2 = \frac{N(N-1)}{2}(\Theta^{(\lambda)})^2 = 3\frac{N(N-1)}{2}$$

Problem: show that

$$\therefore \quad Q_{(123)} = \frac{3}{4N(N-1)(N-2)} \sum_{\pi=1}^{N-2} \sum_{\pi<\lambda} \sum_{\pi<\mu} (1+\Theta^{(\pi)}\Theta^{(\lambda)})(1+\Theta^{(\pi)}\Theta^{(\mu)})$$

$$= \frac{3}{4N(N-1)(N-2)} \left\{ \frac{N(N-1)(N-2)}{3} + (N-2)\sum_{\lambda<\mu}(\Theta^{(\lambda)}\Theta^{(\mu)}) + 2\frac{N(N-1)(N-2)}{3}\sum_{\lambda<\mu}(\;) + 3\frac{N(N-1)}{2} \right.$$

$\pi<\lambda<\mu$: $2(N-2)\sum_{\lambda<\mu}\Theta^{(\lambda)}\Theta^{(\mu)} + \frac{3}{2}N(N-1)$

$\pi<\lambda$: $\sum(\Theta^{(\pi)}\Theta^{(\lambda)})(\Theta^{(\pi)}\Theta^{(\lambda)}) = \frac{N(N-1)}{2}$

$\pi\lambda\mu$: $\lambda,\mu = 2(\pi\lambda\mu) \simeq (\pi\mu)(\pi\lambda)$

$\lambda\pi\mu$: $(\pi\mu\lambda) = (\pi\lambda)(\pi\mu)$

$\pi\lambda\mu$: $\mu\lambda\pi$

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$$Q_{(123)} = \frac{3}{4N(N-1)(N-2)} \sum_{x=1}^{N-2} \sum_{\lambda>x} \sum_{\mu>x} (1+\sigma^{(x)}\sigma^{(\lambda)})(1+\sigma^{(x)}\sigma^{(\mu)})$$

$$= \frac{3}{4N(N-1)(N-2)} \left\{ \frac{N(N-1)(N-2)}{3} + 2(N-2)M \right\}$$

$$+2\sum_{x=1}^{N-2}\sum_{\lambda>x}(\sigma^{(x)}\sigma^{(\lambda)}) \quad \text{(N-2)} \quad + \sum_{x=1}^{N-2}\sum_{\lambda>x}\sum_{\mu>x}(\sigma^{(x)}\sigma^{(\lambda)})(\sigma^{(x)}\sigma^{(\mu)})$$

$$2\sum_{x=1}^{N-2}\sum_{\lambda>x}(\sigma^{(x)}\sigma^{(\mu)}) + \quad 2(N-2)\sum_{\lambda>x}\sum_{\lambda<\mu}\sigma^{(\lambda)}\sigma^{(\mu)}$$

$$\lambda=1 \quad \mu=2 \qquad \lambda=2 \quad \mu=3$$

$$= 2(N-2)\sum_{x=1}^{N-1}\sum_{\lambda>x}(\sigma^{(x)}\sigma^{(\mu)}) \qquad \lambda=N-1 \quad \mu=N$$

$$M = \sum_{x=1}^{N-1}\sum_{\mu>x}\sigma^{(x)}\sigma^{(\mu)} = \frac{1}{2}\{4S(S+1)-3N\}$$

$$Q_{(123)} = \frac{3}{4N(N-1)(N-2)} \left\{ \frac{N(N-1)(N-2)}{3} \right.$$

$$\left. + 8(N-2)S(S+1) - 6N(N-2) \right\}$$

etc.

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Problem.

Verify the relation $\quad Q_r' = \frac{1}{n}\chi_r$

$\chi_1^{(3)} = 2, \quad \chi_{(12)}^{(3)} = 0, \quad \chi_{(123)}^{(3)} = -1$

in the case of π_3.

$$P_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad P_{(13)} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \quad P_{(23)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_{(12)} = \frac{1}{3}\{P_{(12)} + P_{(13)} + P_{(23)}\} = \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$P_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad P_{(132)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$P_{(123)} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\cdot(-1)$$

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Eigenvalue of Energy levels
of Exclusive sets

$$\Psi_\alpha = \psi_{\alpha_1}(x^{(1)}, s_1) \cdots \psi_{\alpha_N}(x^{(N)}, s_N)$$
$$\equiv \psi_{\alpha_1}(1) \cdots \psi_{\alpha_N}(N)$$
$$P_t \Psi_\alpha \equiv \psi_{\alpha_1}(t_1) \cdots \psi_{\alpha_N}(t_N)$$
$$\equiv \psi_{\beta_1}(1) \cdots \psi_{\beta_N}(N) \equiv \Psi_\beta$$

where $t_1 \cdots t_N$ is a certain rearrangement of $1, 2, \cdots N$ and $\beta_1 \cdots \beta_N$ is the corresponding same one rearrangement of $\alpha_1 \alpha_2 \cdots \alpha_N$.

$$P_t \Psi_\alpha = \psi_\alpha(t_1) \cdots \psi_{\alpha_N}(t_N)$$
$$= P_t^{-1} \Psi_\alpha = \psi_{\beta_1}(1) \cdots \psi_{\beta_N}(N) = \Psi_\beta \quad \text{or} \quad P_t'' \Psi_\beta \equiv \Psi_\alpha$$
$$R_t P_t'' \Psi = P_t R_t \Psi = \Psi$$

More generally,

$$P_s R_t \Psi = R_t P_s \Psi = P_s \psi_{\beta_1}(1) \cdots \psi_{\beta_N}(N)$$
$$\equiv R_t \psi_{\alpha_1}(s_1) \cdots \psi_{\alpha_N}(s_N)$$
$$= \psi_{\beta_1}(s_1) \cdots \psi_{\beta_N}(s_N)$$

R_t has meaning only when it is operated on the product function of N-different functions, or such functions as obtained by permutation of variables and also their linear combinations.

If $\psi_{\alpha_1} \cdots \psi_{\alpha_N}$ are orthogonal and normalized, we have

$$\int \tilde{\Psi}_\alpha P_s \Psi_\alpha = 0 \quad \text{unless } P_s = 1.$$
$$= 1 \quad \text{for } P_s = 1.$$

Let us define

$$V_t = \int \tilde{\Psi}_\alpha V P_t \Psi_\alpha,$$

which is independent of the permutation of α's,

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$$V_{r,s} = \int \tilde{\Psi}_\alpha P_r V P_s \Psi_\alpha$$
$$= \int \tilde{\Psi}_\alpha V P_r P_s \Psi_\alpha$$
$$= V_t \quad \text{if} \quad P_r P_s = P_t$$

or one can write the matrix equation

$$V = \sum_t V_t \cdot P_t ,$$

where V_t is a number and P_t is a matrix with elements

$$(P_t)_{rs} = \int \tilde{\Psi}_\gamma P_r P_t P_s \Psi_\alpha$$

Proof: $(P_t)_{rs} = \int \tilde{\Psi}_\alpha P_t P_r P_s \Psi_\alpha$

$= 1$ if $P_r P_s = P_t$

$= 0$ if $P_r P_s \neq P_t$

Hence, $\sum_t V_t \cdot (P_t)_{rs} = V_t$ for $P_r P_s = P_t$

The above matrix relation is only true for the submatrix with rows and columns characterized by $P_s \Psi_\alpha \quad s = 1, 2, \cdots N!$

Now the perturbation energy of the first order in the above individual orbit approximation, is given as the eigenvalues of the perturbation V_{rs}. Instead of finding the individual levels, let us content ourselves merely determining the average energy of states belonging to each of the exclusive set as characterized by a set of values of $\chi_1 \chi_2 \cdots \chi_f$.

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$P_t \to R_t$ $\quad P_s R_t \to R_s R_t$ (∵ $P_s R_t P_s P_t \Psi_\alpha = P_s R_t \Psi_\alpha$) 120(2)

Namely, if there are n-states in this set

$$\Delta E = \frac{1}{n}\sum_r V_{rr} = \frac{1}{n}\sum V_t (R_t)_{rr}$$

which is independent of the choice of the eigenfns as basic vectors.

Although the ~~group~~ R_t is ~~slightly~~ matrix corresponding to R_t is ~~slightly~~ different from that of P_t ~~and equal~~ (R_t is the representation of π_N and P_t are the same as those of the repres. P_t of π_N are ~~the same~~, so that the primitive characters of R_t, ~~and P_t~~ ,

$$\Delta E = \frac{1}{n}\sum_t V_t \chi_t = \frac{N!}{n}\sum_t V_t Q'_t$$

but it is convenient to apply Slath's method in this case

We can generalize the above considerations. in the case, in which some of $\psi_{\alpha_1} \cdots \psi_{\alpha_N}$ are identical, For instance, if $\alpha_1 = \alpha_2$, $R_{(12)}$ reduces to the identity

$$R_{(12)} \Psi_\alpha = \Psi_\alpha$$

Thus, all R_t, for which $(\alpha_1 \cdots \alpha_N) \to (\beta_1 \cdots \beta_N)$ is an identity, so that $R_t = \mathbf{1}$, runs through a certain subgroup of π_N which leaves

$$\Psi_\alpha = \psi_{\alpha_1}(1) \cdots \psi_{\alpha_N}(N)$$

invariant..

Correspondingly, in the relation

$$V_{r,s} = \sum \tilde{\Psi}_\alpha V P_r P_s \Psi_\alpha$$
$$= V_t \quad \text{for} \quad P_r P_s = P_t$$

does not change, but

$$(R_t)_{rs} = \sum \tilde{\Psi}_\alpha P_r R_t P_s \Psi_\alpha$$
$$= \sum \tilde{\Psi}_\alpha R_t P_r P_s \Psi_\alpha$$

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is 1 for all R_t which satisfy

$$R_t P_r P_s \Psi_\alpha = \Psi_\alpha .$$

There are h such R_t for each pair of P_r, P_s, where h is the order of the subgroup of π_N which leaves Ψ_α invariant, because if we take one R_t such that

$$R_t P_r P_s \Psi_\alpha = \Psi_\alpha ,$$

then all other $R_t' = R_s R_t R_s$, where $R_s \Psi_\alpha = \Psi_\alpha$ satisfy

$$R_t' P_r P_s \Psi_\alpha = \Psi_\alpha .$$

Thus, we have

$$\sum_{t=1}^{g} V_t (R_t)_{rs} = V_t \quad \text{for } P_r P_s = P_t$$

where R_t (or P_t) runs through $g' = g/h$ elements of π_N which give different $R_t \Psi_\alpha$.

Thus,

$$V = \sum_{t=1}^{g'} V_t R_t .$$

If one takes the diagonal sum for an exclusive set and takes the average

$$\frac{1}{n} \sum_r V_{rr} = \frac{1}{n} \sum_t V_t (R_t)_{rr}$$
$$= \frac{1}{n} \sum_t V_t \chi_t$$

where t runs through $1, 2 \ldots g'$.

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Three Electron Atom

$\Psi_1 = \psi_1(1)\psi_2(2)\psi_3(3)$

$= \varphi_1(1)\alpha(1)\cdot\varphi_2(2)u_2(2)\cdot\varphi_3(3)u_3(3)$

u's are either α or β. (φ's are orthogonal)

$$V = \frac{e^2}{r_{12}} + \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}}$$

~~$V_{(12)}$~~

$$V_0 = \int \tilde{\varphi}_1(1)\tilde{\varphi}_2(2)\tilde{\varphi}_3(3)\left(\frac{e^2}{r_{12}} + \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}}\right) \varphi_1(1)\varphi_2(2)\varphi_3(3)$$

$$= \int \tilde{\varphi}_1(1)\tilde{\varphi}_2(2)\frac{e^2}{r_{12}}\varphi_1(1)\varphi_2(2)$$
$$+ \int \tilde{\varphi}_1(1)\tilde{\varphi}_3(2)\frac{e^2}{r_{12}}\varphi_1(1)\varphi_3(2)$$
$$+ \int \tilde{\varphi}_2(1)\tilde{\varphi}_3(2)\frac{e^2}{r_{12}}\varphi_2(1)\varphi_3(2)$$

$$V_{(12)} = \int \tilde{\varphi}_1(1)\tilde{\varphi}_2(2)\tilde{\varphi}_3(3)\left(\frac{e^2}{r_{12}} + \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}}\right) \varphi_2(1)\varphi_1(2)\varphi_3(3)\cdot\sum u_1(1)u_2(2)u_3(3) \times u_2(1)u_1(2)u_3(3)$$

$V_{(13)}$ ~~$V_{(23)}$~~ etc; $V_{(123)} = 0$

$V_{(12)} \neq 0$ only if $u_1 = u_2$

$V_{(13)} \neq 0$ only if $u_1 = u_3$

$V_{(23)} \neq 0$ only if $u_2 = u_3$

They are, in general, positive.

Pauli exclusion principle: $n=1$

$\chi_0 = 1$ $\chi_{(12)} = -1$ ~~$\chi_{(123)}$~~ $= 1$.

$$\Delta E = -(V_{(12)} + V_{(13)} + V_{(23)})$$

Thus, we obtain, in general, lower energy ~~levels, on the average~~ if more electron spins are parallel to each other. Thus, one can guess that high S (multiplicity) gives

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lower level.
However, this is not the complete picture, because the corresponding eigenfcn of Slater type does not stand alone under rotations, but we have to take a number of such Slater functions and take suitable linear combinations corresponding to L, S, J.
For this purpose, again one can use either of two methods
(i) Dirac-Wigner: Let us first ignore the spin-dep. Term in H. and assume start from the product of coord. and spin eigenfcns.
(ii) Slater: First construct antisym wave fcns from individual eigenfunctions and then go over to the question whether a certain pair of L, S is contained in ~~that~~ the set of antisym. wave fcns.

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(See below: slater's method) 4.9

However, under the rotation, they are mixed up.)
If, on the contrary, one started from the irreducible representation of the rotation group like

$$\varphi_1(\alpha'')\varphi_2(\alpha^{(2)})\varphi_3(\alpha^{(3)}) \cdot \begin{pmatrix} \alpha(s_1)\alpha(s_2)\alpha(s_3) \\ \frac{1}{\sqrt{3}}\left(\alpha(s_1)\alpha(s_2)\beta(s_3)+\cdots+\cdots\right) \\ \frac{1}{\sqrt{3}}\left(\beta(s_1)\beta(s_2)\beta(s_3)\right. \end{pmatrix}$$

one obtains, first,

$$\Delta E = V_0 \text{ etc.},$$

and then

$$\Delta E = V_0 - V_{12}^{(exch)} - V_{13}^{(exch)} - V_{23}^{(exch)}.$$ comb.

by replace $\varphi_1(\alpha'')\varphi_2(\alpha^{(2)})\varphi_3(\alpha^{(13)})$ by the correct

$$\begin{vmatrix} \varphi_1(\alpha'') & \varphi_2(\alpha'') & \varphi_3(\alpha'') \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

For other states belong to (iii') with respect to permutations, we obtain

$$\Delta E = V_0.$$

The reason why these latter results are better approximation correct can be understood in following way by using Dirac method, in which explicit no assumption is made on the form of the wave functions, except that they are certain linear combinations of product functions.

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~~The energy levels above obtained are~~ average ~~of~~ ~~the~~ ~~system~~ ~~In order to find~~ $\chi_{(12)}$ ~~for N-electron~~ angle and one can have a ~~above considerations~~ diagonalization as follows: ~~can be readily extended to N-electron atoms. One assumes that~~

$$\Psi = \Phi(x^{(1)} \cdots x^{(N)}, \cdots S_N)$$

$$P_r \Psi = P_r^{(x)} \Phi(x^{(1)} \cdots x^{(N)} \cdots S_1 \cdots S_N)$$

$$P_r^{(\sigma)} = \pm \Psi$$

In particular, (see Remark (*) below)

$$P_{(12)}^{(\sigma)} = \frac{1}{2}(1 + \sigma^{(1)}\sigma^{(2)}),$$

and hence

$$P_{(12)}^{(x)} \cdot \frac{1}{2}(1 + \sigma^{(1)}\sigma^{(2)})\Psi = -\Psi,$$

or

$$P_{(12)}^{(x)} \Psi = -\frac{1}{2}(1 + \sigma^{(1)}\sigma^{(2)})\Psi$$

$$Q_{(12)}^{(x)} \Psi = -\frac{1}{2}\left\{1 + \frac{2}{N(N-1)} \sum_{\mu<\nu} \sigma^{(\mu)}\sigma^{(\nu)}\right\}\Psi$$

Since $\frac{\hbar^2}{4} \sum_{\mu<\nu} \sigma^{(\mu)}\sigma^{(\nu)} = \hbar^2 S(S+1)$, we have

$$2\sum_{\mu<\nu} \sigma^{(\mu)}\sigma^{(\nu)} = \left(\sum_\mu \sigma^{(\mu)} \cdot \sum_\nu \sigma^{(\nu)}\right) - \sum_\mu \sigma^{(\mu)}\sigma^{(\mu)}$$

$$= \cancel{4S(S+1)}$$

$$= 4S(S+1) - 3N$$

or

$$Q_{12}^{(x)} = -\frac{1}{2}\left\{1 + \frac{4S(S+1) - 3N}{N(N-1)}\right\}$$

$$= -\frac{N(N-4) + 4S(S+1)}{2N(N-1)}$$

where

$$S = \frac{1}{2}N,\ \frac{1}{2}N-1,\ \frac{1}{2}N-2,\ \cdots$$

We have already chosen a set of spin functions which constitute an exclusive set with respect to the permutation as well as with respect to the rotation.

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Now, let us again take the approximate wave function of the form

$$\Psi_{\mu\nu} = P_{\mu\nu}(x^{(1)} \cdots x^{(N)})\, u_\mu(s_1 \cdots s_N)$$

where P_μ are linear combinations of

$$P_\nu^{(x)} \varphi_1(x_1^{(1)}) \cdots \varphi_n(x^{(n)})$$

and u_μ are linear combinations of products $\eta_\alpha(1)$ etc.

Further, if $\varphi_1, \cdots \varphi_n$ are all different, the average energy of levels corresponding to an exclusive set is

$$\Delta E = V_0 + \sum_{\mu<\nu} V_{(\mu\nu)} Q_{(\mu\nu)}^{(x)}$$

where

$$V_{(\mu\nu)} = \int \tilde{\varphi}_\mu(x_2) \tilde{\varphi}_\nu(x_1) \frac{e^2}{r_{12}} \varphi_\mu(x_1) \varphi_\nu(x_2)$$

Since $Q_{12}^{(x)}$ is, in general, the smaller algebraically the large S, and $V_{(\mu\nu)}$ is in most cases positive, one can conclude that, among the individual orbitals functions $(\varphi_1 \cdots \varphi_n)$, the states with the higher multiplicity lies deeper.

Example. Three electron atom.
$N=3$. If three orbitals are all different, two have two exclusive sets of levels:

$S = \frac{3}{2}$ (quartet): $Q_{12}^{(x)} = -\frac{-3+15}{12} = -1$

$(\Delta E)_{\frac{3}{2}} = V_0 - \sum_{\mu<\nu} V_{(\mu\nu)}$

$S = \frac{1}{2}$ (doublet): $Q_{12}^{(x)} = 0$

$(\Delta E)_{\frac{1}{2}} = V_0$.

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Thus, the quartet lies ~~higher~~ lower than the doublet.*

If two of the orbitals are identical,

$\varphi_1 = \varphi_2$ $\quad \Phi = \varphi_1(x_1)\varphi_1(x_2)\varphi_2(x_3)$,

$P_1 = 1$, $P_2 = (1,2)$, $P_3 = (1,3)$ $P_4 = (23)$,

$R_1 = 1$ $\quad R_2 = 1$ $\quad R_3 = R_{(12)}$ $\quad R_4 = R_{(12)} = R_3$

$P_5 = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $P_6 = (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$R_5 = R_{(123)}$ $\quad R_6 = R_{(13)}$ $\quad \varphi_1(x_3)\varphi_1(x_1)\varphi_2(x_2)$

$\varphi_1(x_2)\varphi_1(x_3)\varphi_2(x_1)$

There are only two different R's, R_1 and $R_{(12)}$ and the corresponding V_0 and

$$V_{(12)} = \int \varphi_1(x_1)\varphi_2(x_2)\,\tilde{\varphi}_1(x_3)\left(\frac{e^2}{r_{12}} + \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}}\right)\varphi_1(x_1)\varphi_2(x_2)\varphi_2(x_3)$$

$$= \int \tilde{\varphi}_2(x_2)\varphi_1(x_3)\frac{e^2}{r_{23}}\,\varphi_1(x_2)\varphi_2(x_3)$$

$S = \frac{3}{2}$: ~~$\Delta E = V_0$~~ is impossible, because Ψ cannot be antisymmetric.

$S = \frac{1}{2}$

$$\Delta E = V_0$$

---

* What we obtain in the above was the further average over all possible L, S.

~~$\Delta E = \frac{1}{4}(\Delta E)_{\frac{3}{2}} + 2\cdot\frac{2}{4}(\Delta E)_{\frac{1}{2}}$~~

$=$

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(93)

spin permutation operators

Remarks (*) (2) $N=2$ case, Interval,

Let us take a complete set of orthogonal spin eigenfunctions

$$\alpha(s_1)\alpha(s_2)\cdots\alpha(s_N) \equiv \chi^{11\cdots1}(s_1\cdots s_N)$$
$$\beta(s_1)\alpha(s_2)\cdots\alpha(s_N) \equiv \chi^{21\cdots1}(s_1\cdots s_N)$$
$$\alpha(s_1)\beta(s_1)\cdots\alpha(s_N) \equiv \chi^{12\cdots1}(s_1\cdots s_N)$$
$$\vdots$$
$$\beta(s_1)\cdots\beta(s_N) \equiv \chi^{22\cdots2}(s_1\cdots s_N)$$

$P_r = P_r^{(\alpha)} P_r^{(\pi)}$

where

$$\left.\begin{aligned}\alpha(1)&=1\\ \alpha(2)&=0\end{aligned}\right\} \qquad \left.\begin{aligned}\beta(1)&=0\\ \beta(2)&=1\end{aligned}\right\},$$

so that arbitrary spin function can be written as

$$\chi(s_1\cdots s_N) = \sum_{r_1 r_2\cdots r_N} c_{r_1 r_2\cdots r_N}\chi^{r_1 r_2\cdots r_N}(s_1\cdots s_N)$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = \beta$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\left.\begin{aligned}\sum_s(\sigma_1)_{rs}\alpha(s) &= \beta(r) & \sum_s(\sigma_1)_{rs}\beta(s) &= \alpha(r)\\ \sum_s(\sigma_2)_{rs}\alpha(s) &= +i\beta(r) & \sum_s(\sigma_2)_{rs}\beta(s) &= -i\alpha(r)\\ \sum_s(\sigma_3)_{rs}\alpha(s) &= \alpha(r) & \sum_s(\sigma_3)_{rs}\beta(s) &= -\beta(r)\end{aligned}\right\}$$

$$\left.\begin{aligned}\sigma_1\alpha &= \beta & \sigma_1\beta &= \alpha\\ \sigma_2\alpha &= i\beta & \sigma_2\beta &= -i\alpha\\ \sigma_3\alpha &= \alpha & \sigma_3\beta &= -\beta\end{aligned}\right\}$$

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$\sigma_1^{(1)}\sigma_1^{(2)}\cdot\alpha^{(1)}\alpha^{(2)} = \beta^{(1)}\beta^{(2)}$

$\sigma_2^{(1)}\sigma_2^{(2)}\quad\alpha^{(1)}\quad\alpha^{(2)} = -\beta^{(1)}\beta^{(2)}$

$\sigma_3^{(1)}\sigma_3^{(2)}\quad\alpha^{(1)}\quad\alpha^{(2)} = \alpha^{(1)}\alpha^{(2)}$

$$\frac{1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2}\alpha^{(1)}\alpha^{(2)} = \alpha^{(1)}\alpha^{(2)}$$

$$\frac{1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2}\cdot\alpha^{(1)}\beta^{(2)}$$

$$= \frac{1}{2}\{\alpha^{(1)}\beta^{(2)} + \beta^{(1)}\alpha^{(2)} + \beta^{(1)}\alpha^{(2)} - \alpha^{(1)}\beta^{(2)}\}$$

$$= \beta^{(1)}\alpha^{(2)}$$

etc.

Thus, in general, we have the relation

$$\frac{1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2}\chi^{r_1 r_2\cdots r_N}(s_1 s_2\cdots s_N)$$

$$= P_{12}^{(\sigma)}\chi^{r_1 r_2\cdots r_N}(s_1 s_2\cdots s_N)$$

$$= \chi^{r_1 r_2\cdots r_N}(s_2 s_1\cdots s_N)$$

still

or, more generally,

$$\frac{1+\boldsymbol{\sigma}^{(\mu)}\boldsymbol{\sigma}^{(\nu)}}{2}\chi(s_1\cdots s_N)$$

$$= P_{\mu\nu}^{(\sigma)}\chi(s_1\cdots s_N)$$

In this way, the interchange $P_{\mu\nu}^{(\sigma)}$ is expressed in terms of the spin vector operators. Now we introduce

$$\frac{1}{2}\sum\boldsymbol{\sigma} = \frac{\hbar}{2}\sum_{\nu=1}^{N}\boldsymbol{\sigma}^{(\nu)}$$

$$\left[\sum_3, \frac{1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2}\right] = \frac{\hbar}{2}\left[\sigma_3^{(1)}+\sigma_3^{(2)},\ \sigma_1^{(1)}\sigma_1^{(2)} + \sigma_2^{(1)}\sigma_2^{(2)}\right]$$

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$$= \frac{\hbar}{4}\left\{ [\sigma_3^{(1)}, \sigma_1^{(1)}\sigma_1^{(2)} + \sigma_2^{(1)}\sigma_2^{(2)}] + [\sigma_3^{(2)}, \sigma_1^{(1)}\sigma_1^{(2)} + \sigma_2^{(1)}\sigma_2^{(2)}] \right\}$$

$$= \frac{\hbar}{4}\left\{ i(\sigma_2^{(1)}\sigma_1^{(2)} - \sigma_1^{(1)}\sigma_2^{(2)}) + i(\sigma_1^{(1)}\sigma_2^{(2)} - \sigma_2^{(1)}\sigma_1^{(2)}) \right\}$$

$$= 0$$

Thus, the total spin angular momentum is commutative with

$$P_{\mu\nu} = \frac{1 + \sigma^{(\mu)}\sigma^{(\nu)}}{2}.$$

This is quite obvious, because Σ is symmetric with respect to all particles, so that the permutation of any pairs of

$$P_{\mu\nu}\,\Sigma\,P_{\mu\nu} = \Sigma$$

(∵ $\Sigma\, P_{\mu\nu}\psi(s_1 s_2 \cdots s_N) = \Sigma\,\psi(s_2 s_1 \cdots s_N)$

$= P_{12}\Sigma\,\psi(s_1 s_2 \cdots s_N)$

$= \Sigma\,\psi(s_2 s_1 \cdots s_N)$)

Thus, if we construct a number of mutually commutative operators, among P_r ($r = 0, \cdots N!$), then $\psi(s_1 s_2 \cdots s_N)$ can be a simultaneous eigenfunction of Σ^2, Σ_z and these commutative operators.

In order to find these operators, Dirac developed the following method, which is actually intimately connected with the

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theory of repres. of sym. group.

(Dirac. Chap. IX.)

Let an operator χ which ~~is~~ can be constructed algebraically from P_r, ~~but~~ is commutative with all of them

$$P_r \chi = \chi P_r \qquad r = 1, 2, \ldots N!$$

the question is how to construct such operators and how many such operators there are which are independent of each other.

$$\chi = \sum_r P_r$$

is certainly one of them, because

$$P_s \sum_r P_r = \sum_r P_r = \sum_r P_r \cdot P_s$$

a

However, there are many more and ~~the number~~ as many as the number of the classes of ~~the~~ the group Π_N

Two permutations P_r, P_s belong to the same class, if there exists a certain P_t satisfying

$$P_t P_r P_t^{-1} = P_s .$$

Thus, if two permutations P_r, P_s belong to the same class they are similar, ~~in that~~ the difference being ~~the change in the names of particles.~~ the different choice of particles.

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Remark (**) (1) Proof that $n_1+n_2+\cdots = g$.

Regular representation

$\Psi(1\cdots N) \equiv \varphi_1(1)\varphi_2(2)\cdots\varphi_N(N)$

all in different individual states

$\Psi_1 = P_1\Psi(1\cdots N) = \Psi(1\cdots N)$

$\Psi_2 = P_{(12)}\Psi(1\cdots N) = \Psi(21\cdots N) = \psi_1(2)\psi_2(1)\cdots\psi_N(N)$

$\Psi_g = P_g\Psi(1\cdots N)$

$g = N!$ $\quad P_1 = 1$

(# $\Psi = \sum c_r\Psi_r$

$P_r\Psi_t = \sum_s \Psi_s (P_{s\leftarrow r})_{st}$)

We have $\tilde{\Psi}$ a g-dimensional repres. of Π_N:

$(P_r)_{st} = \sum \tilde{\Psi}_s P_s^{-1} P_r P_t \Psi = \sum \tilde{\Psi}_{g\,s} P_r \Psi_t$

$(P_1)_{st} = \delta_{st}$ $\quad s,t = 1,2,\cdots g$

$(P_r)_{ss} = 0$ $\quad$ for $r = 2,\cdots g$.

$\chi_1 = \sum_s (P_1)_{ss} = g$

$\chi_r = 0$ $\quad$ for $r = 2,\cdots g$

$\chi_r = m_1\chi_r^{(1)} + m_2\chi_r^{(2)} + \cdots$

$r=1$: $\quad g = m_1 n_1 + m_2 n_2 + \cdots$

$\frac{1}{g}\sum_{r=1}^{g} \tilde{\chi}_r\chi_r = g = m_1^2 + m_2^2 + \cdots$

$g \geq n_1^2 + n_2^2 + \cdots$

$(m_1-n_1)^2 + (m_2-n_2)^2 + \cdots \leq 0$

This is true only for

$m_1 = n_1$, $\quad m_2 = n_2$, $\cdots$

and $\quad n_1^2 + n_2^2 + \cdots = g$.

In regular representation, all irred. repres. appear as many times as their dimensions.

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Regular representation for arbitrary finite group G_f:

Waerden, p. 50.

$$R_r R_s = \sum_t (C_r)_{ts} R_t$$

$(C_r)_{ts} = 1$ for $R_t^{-1} R_r R_s = 1$.

$= 0$ for any other comb. of R_t and R_s for a given R_r

$(C_1)_{ts} = \delta_{ts}$ if $R_1 = 1$

$(C_r)_{ss} = 0$ if for $R_2, R_3, \ldots R_g$.

$(C_r)_{ts}$ are the matrices of regular repres. of G_f. We conclude in exactly the same way that

$$m_1 = n_1, \quad m_2 = n_2, \ldots$$

and

$$g^2 = n_1^2 + n_2^2 + \ldots$$

The theorem that there are exactly f inequivalent irreducible representations can be proved for any finite group of g operators

$S_r = \frac{1}{g} \sum_t R_t R_r R_t^{-1}$

commute with all R_t and there are as many indep. S_r as the number of classes f.

and so on.

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belonging to the same Energy level

Reduction of Simultaneous Representation of Two Groups.

Let us take a set of eigenfunctions of an N-electron atom. Ignore first the spin term in Hamiltonian;

$$\Psi_{j,\mu}(x^{(1)} \cdots x^{(N)}, s_1 \cdots s_N) = \varphi_j(x^{(1)} \cdots x^{(N)})\, \chi_\mu(s_1 \cdots s_N)$$

They transform linear under

(i) Rotation of $x^{(i)}$ coordinates
(ii) Rotation of spins
(iii) Permutation of coordinates
(iV) Permutation of spins

More generally, if we take into account spin term in Hamiltonian, the eigenfns

$$\Psi_{i,\mu}(x^{(1)} \cdots x^{(N)}, s_1 \cdots s_N)$$

transform linearly under simultaneous
(i) Rotation of Coord. and Spins
(ii) Permutation of Coord. and spins

In this sense, we have a representation of two groups $\mathfrak{d}_3$, π_N. simultaneous

Since any two elements, one of $\mathfrak{d}_3$ and the other of π_N, are mutually commutative, the corresponding matrices must be commutative. First one can reduce the representation with respect to $\mathfrak{d}_3$, so that all the matrices corresponding to the rotations have the form and each submatrix represents one of the irred. repr. D_J.

$$\begin{pmatrix} \{\ \} & 0 \\ 0 & \{\ \} \end{pmatrix}$$

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~~Now, since all elements of π_N is commutative~~
let, ~~the~~ eigenfunctions corresponding to one
(a set of) of them, D_J say be

$$\psi_{J,M}(x^{(1)} \cdots x^{(N)}; s_1 \cdots s_N).$$

The same (or equivalent) representation D_J
may appear several times, so they can
be discriminated by

$$\psi^{(\mu)}_{J,M}(x^{(1)} \cdots x^{(N)}; s_1 \cdots s_N)$$

Now, since all matrices corresp to the
elements of π_N are commutative
with those corresponding the elements of
d_3, we have

$$(J,M;\mu|D|J',M';\mu')(J',M';\mu|P|J'',M'',\mu')$$
$$= (J,M,\mu|P|J',M',\mu')(J',M',\mu'|D|J'',M'',\mu')$$

In particular, if we take as D the operator
$\mathfrak{M}^2$, we have

$$J(J+1)(J,M;\mu|P|J',M';\mu')$$
$$= (J,M,\mu|P|J',M',\mu')J'(J'+1)$$

Thus,

$$(J,M;\mu|P|J',M';\mu') \neq 0$$

only if $J = J'$

Further, for $D = \mathfrak{M}_3$, we have

$$M(J,M;\mu|P|J',M';\mu')$$
$$= (J,M;\mu|P|J',M';\mu')M'$$

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or

$$(J, M; \mu | P | J', M'; \mu') \neq 0$$

only if $\mu\mu' = M'$.

Also, all the matrices

$$(J, M; \mu | P | J', M'; \mu')$$

are already diagonal with respect to J, M, while all the matrices

$$(J, M; \mu | D | J', M'; \mu')$$

were diagonal with respect to μ.

~~In order to~~ One can, therefore, reduce the ~~matrices~~ representation of P's by taking suitable set of eigenfunctions for each value of J, M, each being a certain linear combination $\psi^{(\mu)}_{J,M}$ with the same J, M.

$$P = \begin{pmatrix} J,M & & 0 \\ & J,M' & \\ 0 & & \ddots \\ & & & \ddots \end{pmatrix}$$

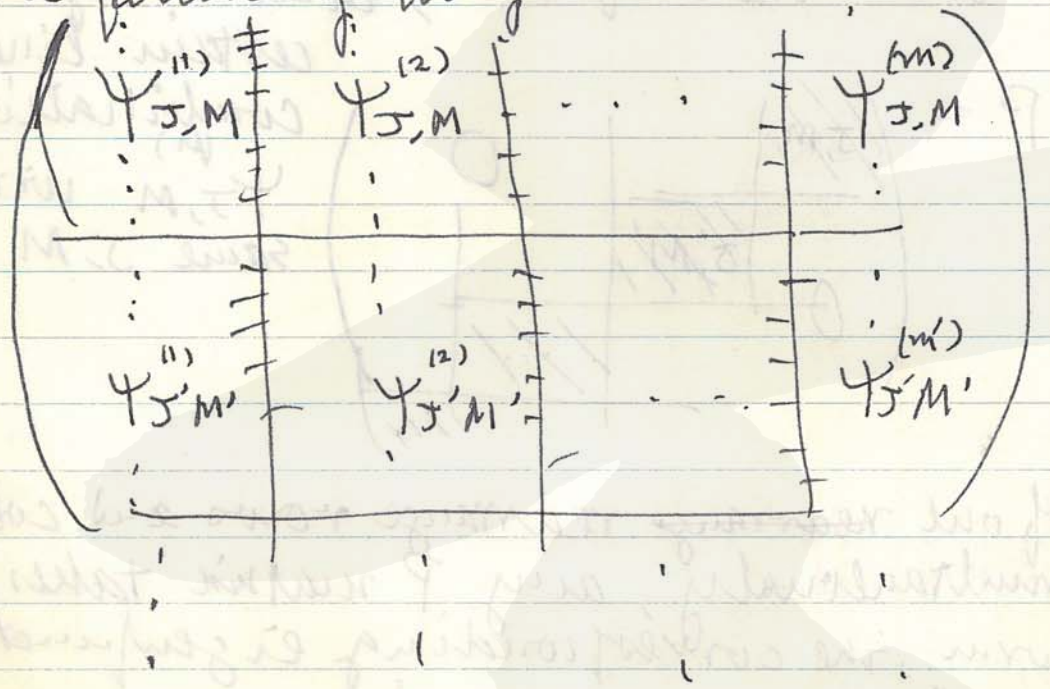
If one ~~rearrange~~ rearrange rows and columns simultaneously, any P matrix takes the above form, the corresponding eigenfunctions ~~are~~ being rearranged.

$$\begin{pmatrix} \psi^{(1)}_{J,M} \\ \psi^{(2)}_{J,M} \\ \vdots \\ \hline \psi^{(1)}_{J,M'} \\ \vdots \\ \hline \vdots \end{pmatrix}$$

Now, the reductions of ~~the~~ submatrices of P for the same J and different M's, can be carried

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out simultaneously, because
$(J, M; \mu | P | J', M'; \mu')$
depends only on J and μ, but not on M.
(This is because P must be commutative with $M_1 \pm iM_2$. More generally, since P is commutative with all D, it must be a multiple of 1 with respect to J, M (if one ignores μ)).
Thus, we can write the totality of eigenfunctions which are the basic vectors for the simultaneous representation of D_3 and Π_N in the following way



where the eigenvectors of the same column transform among themselves under the rotations, while those of the same row transform among themselves under

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the permutations.
The matrices of this representation have the form

$$(J, M; \mu | P_\pi | J', M'; \mu')$$

Now, according to Pauli principle, only the ~~representation~~ antisymmetric repres. of Π_N is realized in nature, we are actually left with <u>one column</u> which corresponds to the antisymmetric combination of ~~$\psi_{J,M}$~~ original $\psi^{(\mu)}_{J,M}$ for each J, M.

~~If~~ What will happen, if we ~~look the~~ proceed ~~situation~~ for the starting point that the eigenfunctions have the product form

$$\varphi(r^{(1)}, \ldots r^{(N)})\, U_\mu(s_1 \ldots s_N)$$

As for the rotation group, these ~~are~~ can be the basic vectors for the representation

$$\mathbb{D}_L \times \mathbb{D}_S,$$

or more generally, a representation of ~~which can~~ the form

$$\sum_{L,S} (\mathbb{D}_L \times \mathbb{D}_S)$$

where the same L and the same S ~~may~~ appear several times.
As for the permutation group, one can reduce the representation separately for the coordinates and spins. Namely,

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one can take those eigenfunctions with the same spin part and proceed as before, by taking into the permutation of coordinates alone. One arrives at a set of coordinate eigenfunctions which can be rearranged as before in a ~~rego~~ rectangular form. They have common spin factor. Now, one can take any one of them and combine ~~the~~ it with others with the same coordinate part and different spin factors and can obtain suitable combinations which are the basic vectors for the irreducible representation of spin rotations, simultaneous and spin permutations.

Now, ~~if we operate~~ the simultaneous rotations of coord. and spins and permutations of coord. and spins, a certain set of the above basic vectors transforms ~~among themselves~~ in itself. The important question is whether ~~there~~ this ~~are~~ is a physical realizable set of states. What we have obtained was a product representation

$$\Delta \times \Delta'$$

of ~~the~~ N, where Δ is the repres. of π_N (~~due to~~ with) the coord fns as basic vectors and

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Δ' is met with the spin fns as basic vectors, the question can thus be formulated in the following way:

(Wigner, p. 277)

Suppose that we have a correct linear combination

$$\Psi = \sum_{j,\mu,\lambda,s} a^{j,\mu}_{\lambda,s}\, \varphi^{(j)}_{L,\lambda M_L}\, u^{(\mu)}_{S,s M_S}$$

which is antisymmetric. Then, we have

$$P_r \Psi = \sum_{j,\mu,\lambda,s} a^{j,\mu}_{\lambda,s}\, \varphi^{(j)}_{L,\lambda}(j'|P^{(x)}_r|j) \times \sum_{j',\mu'} u^{(\mu')}_{S,s}(\mu'|P^{(\sigma)}_r|\mu)$$

or $$\sum_{j',\mu'} (\quad) = \varepsilon(P_r) \sum_{j,\mu,\lambda,s} a^{j,\mu}_{\lambda,s}\, \varphi^{(j)}_{L,\lambda}\, u^{(\mu)}_{S,s}$$

Hence

$$\sum_{j',\mu'} (L,j|P^{(x)}_r|L,j')(S\mu|P^{(\sigma)}_r|S\mu')\, a^{j',\mu'}_{\lambda,s} = \varepsilon(P_r)\, a^{j,\mu}_{\lambda,s}$$

One may introduce the representation $\bar{\Delta}'$ as given by

$$\varepsilon(P_r)(S\mu|P^{(\sigma)}_r|S\mu') \equiv (\mu|\bar{P}^{(\sigma)}_r|\mu')$$

of Π_N instead of (Δ' as given by $(S\mu|P^{(\sigma)}_r|S\mu')$. Let us call this the associated (dual) representation then we have

$$\sum_{j',\mu'} (j|P^{(x)}_r|j')(\mu|\bar{P}^{(\sigma)}_r|\mu')\, a^{j',\mu'}_{\lambda s} = a^{j,\mu}_{\lambda,s}$$

If we sum up over all elements P_r ($r = 1 \cdots N!$)

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of Π_N, then from the orthogonality relations one concludes that the left dual side is identically zero, if the representations Δ and $\bar{\Delta}'$ are inequivalent.

Now for the ~~whole~~ N-electron ~~system~~ atom, the spin ~~eigenfu~~ states can be

$$S = \frac{N}{2}, \quad \frac{N}{2}-1, \quad \cdots$$

For each S, we reduce the representation with respect to Π_N. However, in the spin space, there is a very simple relation between the rotations and the permutations, both being expressed in terms of $\sigma^{(\nu)}$ $(\nu = 1, \cdots N)$.

If we reduce 2^N-dimensional space with respect to d_3,

$$(D_{\frac{1}{2}})^N = D_{\frac{N}{2}} + (N-1) D_{\frac{N}{2}-1} + \left\{\frac{N(N-1)}{2} - N\right\} D_{\frac{N}{2}-2} + \cdots$$

$$2^N = (N+1) + (N-1)^2 + \frac{N(N-3)}{2}(N-3) + \cdots$$

$$N=5: \quad (D_{1/2})^5 = D_{\frac{5}{2}} + 4 D_{\frac{3}{2}} + 5 D_{\frac{1}{2}}$$

$$2^5 = 32 = 6 + 16 + 10.$$

~~The~~ Each part is an irreducible representation (appearing once or several times) of d_3, but, at the same time, the f-operations

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$Q_1, Q_2, \ldots Q_f$ ~~have~~ are defined as functions of $\textcircled{1}^{(n)}$, so that it cannot depend on the parameters other than S, s (and N). It cannot, of course, depend on s, because it is rotationally invariant. Thus, they are unique values as determined by S and N. This shows that $\mathbb{D}_S$ is already an irreducible repres. of π_N, since $\chi_1, \chi_2, \ldots \chi_f$ are uniquely determined ~~and~~ (except for the discussion in) and further reduction is impossible. (otherwise, $Q_1, Q_2, \ldots Q_f$ will take ~~an~~ some other set of values, which is impossible. They are already multiples of unit matrix.)

Only thing we ~~do not~~ ~~did~~ do not know is ~~whether~~ $\mathbb{D}_S$ ~~may~~ is as a whole one irred. rep. of π_N or there are in $\mathbb{D}_S$ several equivalent irred. rep. of π_N.

One can say generally that $\mathbb{D}_{S=\frac{N}{2}}$ is at the same time the symmetric repres. of π_N. Other $\mathbb{D}_S$'s cannot be the sym. repr. of π_N, because the symmetric functions are all included in the eigenfys for $\mathbb{D}_{\frac{N}{2}}$. Anti-symmetric ~~funct~~ representations can not appear for $N > 2$.

For $N=3$, ~~the~~ $\mathbb{D}_{\frac{3}{2}}$: Sym.

$\mathbb{D}_{\frac{1}{2}}$: $\chi^{(3)}_{(1)} = 2$, $\chi^{(3)}_{(12)} = 0$, $\chi^{(3)}_{(123)} = -1$

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For $N=4$. $\mathfrak{D}_{\frac{1}{2}}$: sym.

$\mathfrak{D}_{\frac{3}{2}}$: Either two and one dimensional repres. of π_N

or three dim. repres. of π_N

The first case is impossible, because of the irreducibility and the exclusion of sym. and antisym. repres.

Thus $\mathfrak{D}_{\frac{3}{2}}$ is either

$\chi^{(4)} = 3, +1, 0, -1, -1$

or $\chi^{(5)} = 3, -1, 0, -1, +1$

Each of these is to be combined with the irreducible representation of π_N $\chi^{(5)}$ or $\chi^{(4)}$ with respect to coordinate functions. This is possible only when ~~$\varphi_1, \varphi_2, \varphi_3$ are different orbital functions.~~

$$\left.\begin{array}{l} \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3) \\ \varphi_1(r_1)\varphi_3(r_2)\varphi_1(r_3) \\ \varphi_3(r_1)\varphi_1(r_2)\varphi_1(r_3) \end{array}\right\}$$

Remark:

In general, there are $\overset{\text{irred}}{\text{repres.}}$ of π_N, other than sym. and anti. repr., which are dual to each other. If the characters of all odd permutations vanish, the two representations are equivalent. If some of the characters of odd permutations are not zero, they are equal and opposite in two representations. Sym. and anti. repres. are dual to each other.

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Slater's Method (Waerden, p. 120)

Antisymmetric combination

$$\Psi = \sum_{r=1}^{N!} \varepsilon(P_r) P_r \psi_1(1)\psi_2(2)\cdots\psi_N(N)$$

Under the rotations of coordinates alone or spins alone, this changes into antisym. fn, because the operators for coord. rotations and spin rotations are both commutative with all P_r's. ($P_r = P_r^{(x)} P_r^{(\sigma)}$). Thus the Ψ changes into another anti-sym. fn

$$\Psi' = \sum_r \varepsilon(P_r) P_r D\, \psi_1(1)\psi_2(2)\cdots\psi_N(N)$$

where D is the operator, which is commutative with all P_r. D can be coord. rot. or spin rotation or any operator which is symmetric with respect to N-particles.

Thus, in this space of with antisym. eigenfunct. as basis vectors, we can find the represent. of rotation group, separately for coord. and spin. Then, from

$$\mathbb{D}_L \times \mathbb{D}_S = \sum_J \mathbb{D}_J$$

one can obtain the repres. of simultaneous rotations.

Now let us specify the individual state by take orthogonal eigenfns

$$\varphi_{n,l,m}(r)\,\alpha(s) \quad \text{or} \quad \varphi_{n,l,m}(r)\,\beta(s)$$

or $(n, l, m; \frac{1}{2})$ or $(n, l, m; -\frac{1}{2})$

simply. ~~Then, for a~~

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Problem:
Let us take an example of three electrons in the same shell 2p: (Nitrogen)
$n=2$, $l=1$, ($m=+1, 0,$ or -1)

	$M_L=\lambda$	$M_S=s$
$(211;\frac{1}{2})$ $(210;\frac{1}{2})$ $(21-1;\frac{1}{2})$	0	$\frac{3}{2}$
$(211;\frac{1}{2})$ $(210;\frac{1}{2})$ $(211;-\frac{1}{2})$	2	$\frac{1}{2}$
$(211;\frac{1}{2})$ $(210;\frac{1}{2})$ $(210;-\frac{1}{2})$	1	$\frac{1}{2}$
$(211;\frac{1}{2})$ $(210;\frac{1}{2})$ $(21-1;-\frac{1}{2})$	0	$\frac{1}{2}$
$(211;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(211;-\frac{1}{2})$	1	$\frac{1}{2}$
$(211;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(210;-\frac{1}{2})$	0	$\frac{1}{2}$
$(211;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(21-1;-\frac{1}{2})$	-1	$\frac{1}{2}$
$(210;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(211;-\frac{1}{2})$	0	$\frac{1}{2}$
$(210;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(210;-\frac{1}{2})$	-1	$\frac{1}{2}$
$(210;\frac{1}{2})$ $(21-1;\frac{1}{2})$ $(21-1;-\frac{1}{2})$	-2	$\frac{1}{2}$

and as many other states as above with negative M_S. (20 in all, 10 in the above)
Now we look for the largest possible values of S and L.
For $S=\frac{3}{2}$, the only possibility is $L=0$. (4 states, or 2 from the above table)
For $S=\frac{1}{2}$, the largest L is $L=2$ ($2\times 5=10$ or 5 from the above)
We are left with 6 states (or 3 in the table) which correspond to $S=\frac{1}{2}$, $L=1$.
Thus we have
^4S ($L=0$, $S=\frac{3}{2}$); ^2D ($L=2$, $S=\frac{1}{2}$); ^2P ($L=1$; $S=\frac{1}{2}$)

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Plus, equivalent three p-electrons give
^4S, ^2D, ^2P.
(If we have ignored Pauli principle,
we had obtained from
$L = 2, 1, 0$; $S = \frac{1}{2}, \frac{3}{2}$
^4D, ^4P, ^4S, ^2D, ^2P, ^2S
from altogether
$3 \times 3 \times 3 \times 2 \times 2 \times 2 = 6 \times 6 \times 6 = 216$
states ;
$5 \times 4 + 3 \times 4 + 4 = 36$
$5 \times 2 + 3 \times 2 + 2 = 18$,
they must appear several times.)

~~The above consideration can be~~
Relative positions of these levels
are is governed by Hund's rules: one is
(i) of terms given by equivalent
electrons, those with greatest
multiplicity lie deepest, and of
these the lowest is that with the
greatest L.

^2P ——
^2D ——
^4S ——

Problem. ~~Show that~~ both in the cases of Carbon and Oxygen ($2p^2$ and $2p^4$), the lowest energy levels are ^3P, ^1D, ^1S. (just the same as Carbon.

^1S —
^1P —
^3P —

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The other Hund rule is
(ii) Multiplets formed from equivalent electrons are regular when less than half the shell is occupied, but inverted when more than half the shell is occupied.

Derivation of terms can be ~~done~~ simplified by drawing the following diagram: (Carbon)

m_ℓ			M_L	M_S
+1	0	−1		
↑↓			+2	0
↑	↑		+1	1
↑	↓		+1	0
↓	↑		+1	0
↓	↓		+1	−1
↑		↑	0	+1
↑		↓	0	0
↓		↑	0	0
↓		↓	0	−1
↑↓			0	0
↑	↑		−1	+1
↑	↓		−1	0
↓	↑		−1	0
↓	↓		−1	−1
	↑↓		−2	0

$15 = {}^1D(5) + {}^3P(9) + {}^1S(1)$

($6 \times 6 = 36$. ${}^1D(5) + {}^3P(9) + {}^1S(1)$
$+ {}^3D(15) + {}^1P(3) + {}^3S(3)$)

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In the case of two electron atoms, we have the rule:

Two equivalent electrons (n, l), $(n, l^{\prime\prime})$ give sym. singlet term $L = 2l, 2l-2, \ldots 0$ and anti-sym. triplet term $L = 2l-1, 2l-3, \ldots 1$. Triplet is deeper than singlet in general.*

(For (n'', l'') and (n', l'), we have both sing. and triplet for $L = l+l', \ldots |l-l'|$)

Closed Shell:

If all the states of a shell is filled up by each one electron, ~~the state~~ a closed shell is formed and the corresponding eigenfn is uniquely determined

$$\mathcal{E}(P_r)\, P_r\, \psi_1(1)\psi_2(2)\cdots\psi_N(N)$$

~~so that it is lea~~ This is a scalar in the sense that it is invariant with respect to rotations of coord. as well as those of spins, so we conclude that

$$L = 0, \quad S = 0.$$

If we add one more electron, the eigenfn will be

$$\mathcal{E}(P_r)\, P_r\, \psi_1(1)\psi_2(2)\cdots\psi_N(N)\cdot\psi_{N+1}(N+1)$$

where the state ψ_{N+1} is outside the shell. This is a linear combination of fns, each

* This will can be derived from the fact that if there are $M_L = 2l-1$ states which must corresp. to only 4-states with one singlet and one triplet, etc.

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of which is uniquely determined for one M_{L+1}, so that the ~~structure of the~~ nature of the state for the whole system is already determined by the extra-~~electron~~ state of the extra-electron, i.e. valence electron.

If the number of the electrons in the open shell is equal to

$$N = N_0' - N',$$

where N_0' is the number of states of an a ~~closed~~ shell, then the ~~same~~ number and types of the possible states are the same as in the case of N'-electrons.

Proof: In the above example, there are 4- vacant states and could be filled up in a unique way:

$$\overset{+1}{\uparrow}\quad \overset{0}{(\uparrow\downarrow)}\quad \overset{-1}{(\uparrow\downarrow)} \longrightarrow \overset{+1}{(\uparrow\uparrow)}\quad \overset{0}{\uparrow\downarrow}\quad \overset{-1}{\uparrow\downarrow}$$

The corresponding eigenfunctions are

$$\sum_{r=1}^{2} \varepsilon(P_r) P_r \psi_1(1)\psi_2(2) \longrightarrow \sum_{r=1}^{4!} \varepsilon(P_r') P_r' \psi_3(3)\psi_4(4)\psi_5(5)\psi_6(6)$$

~~They can be~~ The correspondence is unique and

$$M_L = -M_L', \qquad M_S = -M_S',$$

because if ~~the N+N'~~ this shell is filled up, $M_L + M_L' = 0$, $M_S + M_S' = 0$.

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As for the relative position of various terms, one can see immediately from Dirac's estimation of ~~one~~ perturbation energy that Hund rule (i) is correct for the multiplicity.

Problem: Show that ~~Mg^{12}~~ Ti^{22} and Ni^{28} have the same scheme of energy levels in the following order:
$3d^2$ and $3d^8$: ^3F, ^3P, ^1G, ^1D, ^1S.
(Herzberg, Atomic spectra and atomic structure)
1s 2s 2p 3s 3p 3d 4s
2+2+6+2+6+2+2=22
2+2+6+2+6+8+2=28
~~Hund rule~~
As for the relative position of levels, for different L with the same S, there is no very simple qualitative arguments. In order to actually ~~calculate~~ estimate in any specific case the energy levels, one can work out the matrix elements of the perturbation energy, which consist of ~~two~~ classical Coulomb terms and exchange terms by using the anti-symm. combination. Then take a particular set of M_L and M_S

* The levels with larger S lie deeper usually, because the electrostatic repulsion between electrons will be smaller, if due to the fact more electrons be anti sym. with respect to coord.

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and take the diagonal sum. This gives the sum of the (average) energies of the levels with $L \geq M_L$, $S \geq M_S$. Thus, one starts from the largest values of M_L, M_S, one gets the (average) energy level for the largest L and S. Then, next highest values of M_L, M_S will give (in combination with the previous result) the energy level of the next highest values of L, S and so on.

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Relation between Darwin's method and Slater's method.

Slater determinant

$$\Psi = \frac{1}{\sqrt{N!}} \sum_{r=1}^{N!} \varepsilon(P_r) P_r \, \psi_{\alpha_1}(1) \cdots \psi_{\alpha_N}(N)$$

$$= \sum_{L,S} \Phi^{(L)(\kappa)}_{M_L}(\vec{r}^{(1)} \cdots \vec{r}^{(N)}) \, \mathcal{U}^{(S)(\lambda)}_{M_S}(s_1 \cdots s_N) \Big/ C^{(\kappa)(\lambda)}_{L,S}$$

where the same L, S may appear several times, but M_L, M_S is fixed for a ~~fixed~~ given Slater det. This always possible because one can first decompose a product of the form

$$\alpha(s_1)\beta(s_2) \cdots \beta(s_N)$$

into spin functions with the same M_S and different S. Then the coefficient of each term, which depend still on $\vec{r}^{(1)} \cdots \vec{r}^{(N)}$ is a linear combination of products of orbitals with the same M_L and can be decomposed into ~~the~~ a linear combinations of coordinate fcts belonging to ~~a~~ different values of L and the common M_L.

If we ignore the spin dependent term in the Hamiltonian H, the matrix element of H_{rs} is different from zero only if r, s refer to the same (L, S). Thus, one take the submatrix of H corresponding to ~~$(2L+1)(2S+1)$~~ eigenfcts $\Phi^{(L)(\kappa)}_{M_L} \mathcal{U}^{(S)(\lambda)}_{M_S}$ with different κ and λ.

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The dependent of H_{rs} on S, λ is trivial, because H is independent on spin variables, H_{rs} has always the factor $\delta_{SS'}\delta_{\lambda\lambda'}$. One can reduce further ~~The reduction of~~ the matrix (H_{rs}) with respect to the permutations $P_r^{(x)}$ $(r = 1, 2, \ldots g)$. However, since this is not easy for large values of N, we had to content ourselves with determining the average over levels belonging to ~~the~~ an exclusive set characterized by a set of primitive characters of $P_r^{(x)}$. ~~The~~ This primitive characters of $P_r^{(x)}$ are related to the corresponding primitive characters of $P_r^{(0)}$:

$$P_r^{(x)} = \varepsilon(P_r)\, P_r^{(0)}$$

on account of Pauli exclusion principle or more specifically, on account of the fact that ~~the~~ the coordinate and spin functions are derived from the original Slater determined.

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Part ~~II~~ III P III, 0

Program

Molecular Structure, Crystals

Theory of Valence

Nuclear Structure – Isotopic spin

Application to Collision Problems

Application to Field Theories.

(Lorentz group)

Fundamental groups in Q.M.

(i) Permutation Group (Change in the total number of particles)

(ii) Lorentz group

Non-relativistic Q.M.

(i) Permutation group

(ii) Rotation group

(a) atom → whole R.G.

(b) Molecules etc. → restricted rotation group.

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Molecular Structure

M. I

van der Waerden, p. 135.

Electron coordinates r_k $(k = 1, 2, \cdots n)$

Nuclear coordinates R_α $(\alpha = 1, 2, \cdots N)$

$$r_{kl} = |r_k - r_l|$$
$$\rho_{\alpha k} = |r_k - R_\alpha|$$
$$R_{\alpha\beta} = |R_\alpha - R_\beta|$$

$$H = -\sum_{\alpha=1}^{N} \frac{1}{2M_\alpha} \Delta_\alpha - \sum_{k=1}^{n} \frac{1}{2m} \Delta_k + V(r_k, R_\alpha)$$

$$V = \sum_{k<l} \frac{e^2}{r_{kl}} + \sum_{\alpha<\lambda} \frac{Z_\alpha Z_\beta e^2}{R_{\alpha\beta}} - \sum_{k,\alpha} \frac{Z_\alpha e^2}{r_{k\alpha}}$$

1. Adiabatic approximation:
Since the average kinetic energy of the nuclei is much smaller than the average kinetic energy of the electrons and the average potential energy in any molecule in the centre of mass system, in the first approx. one can ignore the kinetic energy of the nuclei in H. Then we have

$$\left\{ -\frac{1}{2m} \sum \Delta_k + V(r_k, R_\alpha) - W_n(R_{\alpha\beta}) \right\} \Phi_n(r_k, R_\alpha) = 0,$$

where Φ_n from which the wave $\sum_n \Phi_n$ belonging to the eigenvalues W_n can be obtained for any set of fixed values of $R_{\alpha\beta}$ and W_n will depend on the mutual distances $R_{\alpha\beta}$ between nuclei.

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Then, arbitrary function of $\Psi(r_k, R_\alpha)$ can be expanded

$$\Psi(r_k, R_\alpha) = \sum_n \chi_n(R_\alpha) \Psi_n(r_k, R_\alpha)$$

Thus, ~~In particular~~, the solutions of ~~eigenfn~~ for the Schrödinger equation

$$H\Psi = E\Psi$$

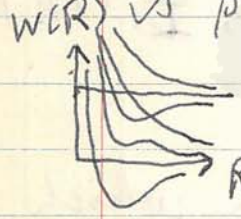
can be written in the above form, in general. In particular, if we assume that the ~~pot~~ eigenfns can be written approximately as

$$\Psi(r_k, R_\alpha) = \chi(R_\alpha)\,\Phi(r_k, R_\alpha),$$

$$\left\{-\sum \frac{1}{2M_\alpha}\Delta_\alpha + W(R_{\alpha\beta}) - E\right\}\chi(R_\alpha)$$

$$= \frac{1}{\Phi}\sum_\alpha \left\{\frac{1}{2M_\alpha}\left(\chi(R_\alpha)\cdot\Delta_\alpha\Phi + 2(\nabla_\alpha\chi\cdot\nabla_\alpha\Phi)\right)\right\}$$

The left-hand-side is a function of R_α alone, while the right-hand-side depends, in general, both on R_α and r_k. Thus Our approxim. is justified, only if the right-hand-side is small. Then, we have



$$\left\{\sum_\alpha \frac{1}{2M_\alpha}\Delta_\alpha + W(R_{\alpha\beta}) - E\right\}\chi(R_\alpha) = 0$$

approximately, where $W(R_{\alpha\beta})$ plays the rôle of the potential energy for the motion of nuclei, W is called the adiabatic potential associated with each of the electronic distribution as given by $\Psi(r_k, R_\alpha)$.

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If $W(R_{\alpha\beta})$ turns out to have a minimum for a certain set of values of $R_{\alpha\beta}$ and furthermore, a sufficient large negative value, we are sure that a stable or metastable molecule is formed, while $W(R_{\alpha\beta})$ decreases increases steadily with decreasing distances $R_{\alpha\lambda}$, the whole system cannot form a stable molecule.

2. Born-Oppenheimer's approximation.

Annalen der Phys. 84 (1927), 457.

Usually, energy levels of a molecule are classified in the following way

(i) electronic terms

(ii) nuclear vibration terms

(iii) nuclear rotation terms

according to the order of average separation of levels. This was justified q.m. by Born and Oppenheimer.

M: average nuclear mass, $M_\alpha = \frac{M}{\mu_\alpha}$

$\varkappa = (m)^{1/4}$

In a certain sense, $\varkappa^0$: electronic energy, $\varkappa^2$: vibrational energy, $\varkappa^4$: rotational energy in the expansion of H in powers of $\varkappa$.

$$-\sum \frac{1}{2M_\alpha}\Delta_\alpha \simeq -\frac{\varkappa^4}{2}\sum_\alpha \mu_\alpha \Delta_\alpha = \varkappa^4 T$$

R_α: $3N$-coordinates

centre of mass (3), Eulerian angles around

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its principle axes of moment of inertia.

(3). θ_j $(j=1, 2, \cdots 6)$

$3N-6$ remaining coordinates

η_j $(j=1, 2, \cdots 3N-6)$

R_α depends only on η's.

$R_\alpha(\theta_1 \cdots \theta_6, \eta_1 \cdots \eta_{3N-6})$

Remark: Diatomic molecule is exceptional in that the orientation of nuclei is determined by two angles (θ, φ), and so that we are left with one variable R (the mutual distance between two nuclei) describing the vibration. ~~Situation is similar~~

In the case of general linear molecule, we need special care in discriminating between rotational and vibrational degrees of freedom.

~~Since~~

In equilibrium state, η_j have fixed values $\eta_i^{(0)}$, and, in general, we have

$$\eta_j = \eta_j^{(0)} + x\zeta_j$$

$$V = V^{(0)} + xV^{(1)} + x^2V^{(2)} + \cdots$$

$$V^{(0)} = V^{(0)}(R_\alpha^{(0)}, r_k) \qquad R_\alpha^{(0)} = R_\alpha(\theta, \eta^{(0)})$$

$$V^{(1)} = \sum_j \zeta_j \left(\frac{\partial V}{\partial \eta_j}\right)_{\eta_j=\eta_j^{(0)}} = \sum \zeta_j \frac{\partial V^{(0)}}{\partial \eta_j^{(0)} \partial \eta_j^{(0)}}$$

$$V^{(2)} = \frac{1}{2}\sum_{j,k} \zeta_j \zeta_k \frac{\partial^2 V^{(0)}}{\partial \eta_j^{(0)} \partial \eta_k^{(0)}}$$

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Similarly, the kinetic energy of nuclei T is written in the form:

$$T\left(\theta, \eta, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\eta}\right) = T\left(\theta, \eta^{(0)} + \kappa\zeta, \frac{\partial}{\partial\theta}, \frac{1}{\kappa}\frac{\partial}{\partial\zeta}\right)$$

~~$\frac{\partial}{\partial\eta} = T(\theta, \eta^{(0)}$~~

This consists of a number of terms which are

(i) independent of $\frac{\partial}{\partial\zeta}$: $T_{\theta\theta}$

(ii) linear in $\frac{1}{\kappa}\frac{\partial}{\partial\zeta}$: $\frac{1}{\kappa}T_{\zeta\theta}$

(ii) quadratic in $\frac{1}{\kappa}\frac{\partial}{\partial\zeta}$: $\frac{1}{\kappa^2}T_{\zeta\zeta}$

$$T = \frac{1}{\kappa}T_{\zeta\theta}\left(\theta, \eta^{(0)} + \kappa\zeta, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\zeta}\right) + \frac{1}{\kappa^2}T_{\zeta\zeta}\left(\theta, \eta^{(0)} + \kappa\zeta, \frac{\partial}{\partial\zeta}\right)$$

$$= \frac{1}{\kappa}T_{\zeta}\left(\theta, \eta^{(0)}, \frac{\partial}{\partial\theta}\right)$$

$$T_{\theta\theta}\left(\theta, \eta^{(0)} + \kappa\zeta, \frac{\partial}{\partial\theta}\right) +$$

$$= \frac{1}{\kappa^2}T_{\zeta\zeta}\left(\theta, \eta^{(0)}, \frac{\partial}{\partial\zeta}\right) + \frac{1}{\kappa}\sum_j \zeta_j \frac{\partial T^{(0)}_{\zeta\zeta}}{\partial \eta^{(0)}_j} + \cdots = T^{(1)}_{\zeta\zeta}$$

$$+ \frac{1}{\kappa}T_{\zeta\theta}\left(\theta, \eta^{(0)}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\zeta}\right)$$

$$+ \sum_j \zeta_j \frac{\partial T^{(0)}_{\zeta\theta}}{\partial \eta^{(0)}_j} \quad T^{(1)}_{\zeta\theta}$$

$$+ T_{\theta\theta}\left(\theta, \eta^{(0)}, \frac{\partial}{\partial\theta}\right) + \kappa\sum_j \zeta_j \frac{\partial T^{(0)}_{\theta\theta}}{\partial \eta^{(0)}_j} + \cdots$$

$$H = H_0 + \kappa V^{(1)} + \kappa^2\left(V^{(2)} + T^{(0)}_{\zeta\zeta}\right)$$
$$+ \kappa^3\left(V^{(3)} + T^{(0)}_{\zeta\theta} + T^{(1)}_{\zeta\zeta}\right)$$
$$+ \kappa^4\left(V^{(4)} + T^{(0)}_{\theta\theta} + T^{(1)}_{\zeta\theta} + T^{(2)}_{\zeta\zeta}\right) + \cdots$$

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where

$$H_0 = -\frac{1}{2}\sum_k \Delta_k + V(R_\alpha^{(0)}, r_k)$$

Zeroth approximation for the eigenfunction:

$$\Psi = \text{for (centre of mass)} \varphi_{n\nu r}(\textcircled{H}, \eta^{(0)}) \vartheta_{n,\nu}(\zeta, \eta^{(0)}) \Psi_n(r_k, \eta^{(0)}, \textcircled{H})$$

with respect to

$\textcircled{H}$: Eulerian angles ~~around~~ the ~~centre of mass~~ principal axes of moment of inertia.

$$E = W_n(\eta^{(0)}) + \pi^2 \varepsilon_{n\nu}(\eta^{(0)}) + \pi^4 W_{n\nu r}(\eta^{(0)}) + \pi^4 E_{kin}.$$

$$\left\{ -\frac{1}{2}\sum_k \Delta_k + V(r_k, \eta^{(0)}, \textcircled{H}) - W_n(\eta^{(0)}) \right\} \Psi_n \Phi_{n\nu}(r_k, \eta^{(0)}, \textcircled{H}) = 0$$

$$\left\{ T_{\zeta\zeta}^{(0)} + \frac{1}{2}\sum_{j,l} \zeta_j \zeta_l \frac{\partial^2 W_n(\eta^{(0)})}{\partial \eta_j^{(0)} \partial \eta_l^{(0)}} - \varepsilon_{n\nu}(\eta^{(0)}) \right\} \vartheta_{n\nu}(\zeta, \eta^{(0)}) = 0$$

$$\left\{ H_{rot}(\eta^{(0)}, \textcircled{H}, \frac{\partial}{\partial \textcircled{H}}) - W_{n\nu r}(\eta^{(0)}) \right\} \times \varphi_{n\nu r}(\textcircled{H}, \eta^{(0)}) = 0$$

Mulliken, R. M. P.

Z(1932) 3 (1933) 1, 4 (1934) 2

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Remark 1.

Argument for the smallness of

$$\sum_a \frac{1}{2M_a}\{\chi \Delta_a \psi + 2\nabla_a\chi\cdot\nabla_a\psi\}$$

If ψ: real and normalized

$$\int \psi^2 (dr_1\cdots r_N) = 1.$$

$$\int \psi \nabla_a \psi = 0$$

$$\int \psi^2 \{ \quad \} \chi = \sum_a \int \chi \psi \Delta_a \psi$$

$$\int \{(\nabla_a\psi)^2 + \psi\Delta_a\psi\} = 0$$

$$= \sum_a \int \chi (\nabla_a\psi)^2$$

Thus, if we take min of $W(R_{\alpha\beta})$, $\nabla_\alpha\psi$ is small for those values of α ~~where ψ^2 and is large~~ where χ is large,

$$\{ \quad \} \chi \doteqdot 0.$$

Variational approximation

$$\int\cdots\int \tilde{\Psi} H \Psi (d\mathbf{R})^N (d\tau)^n = \int\cdots\int \tilde{\chi}\{-\sum \frac{1}{2M_a}\Delta + W(R_{\alpha\beta})\}\chi \times (d\mathbf{R})^N$$

$$-\int\cdots\int \sum \frac{1}{2M_a}\{\tilde{\chi}\chi\,\tilde{\psi}\Delta_a\psi + 2\tilde{\chi}\nabla_a\chi\cdot\tilde{\psi}\nabla_a\psi\}\frac{(d\mathbf{R})^N}{(d\tau)^n}$$

$$= \min.$$

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$$\int d\tilde{\Psi} \nabla_\alpha \Psi + \nabla_\alpha \tilde{\Psi} \Psi \zeta (d\tau)^n = 0$$

$$\iint \tilde{\Psi} H \Psi = \iint + \sum \frac{1}{2M_\alpha} \nabla_\alpha \tilde{\Psi} \cdot \nabla_\alpha \Psi + \int \tilde{\Psi} H_0 \Psi$$

$$= \iint \sum \frac{1}{2M_\alpha} (\tilde{\chi} \nabla_\alpha \tilde{\Psi} + \tilde{\Psi} \nabla_\alpha \tilde{\chi})(\chi \nabla_\alpha \Psi + \Psi \nabla_\alpha \chi)$$

$$+ W(R) \int \tilde{\chi} \chi \cong \int \sum \frac{1}{2M_\alpha} \nabla_\alpha \tilde{\chi} \cdot \nabla_\alpha \chi$$

$$+ W(R) \int \tilde{\chi} \chi + \int f(\nabla_\alpha \Psi, \nabla_\alpha \tilde{\Psi})$$

The last term is small ~~if $W(R)$~~ Ra takes ~~is a minimum~~ and $\Psi, \tilde{\Psi}$ are ~~stationary for those~~ if χ is large only for those values of R_α which are close to the equilibrium position.
The rotation of nuclei do not affect $W(R)$ and so, if the amplitudes of vibrations are small, the above assumption is justified.

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Diatomic Molecule

In the particularly pimple case of the diatomic molecule, the above considerations hold more rigorously, because, in contrast to the case of polyatomic molecules, in any configuration, the diatomic molecule is linear.

The Schrödinger equation takes the form in the centre of mass system:

$$\left\{ -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right) - \sum_k \frac{1}{2m}\Delta_k + V \right\}\Psi = E\Psi$$

where $\mu = \frac{M_1 M_2}{M_1+M_2}$ X, Y, Z are relative coordinates between nuclei.

One can go over to the polar coord. and has

$$\left\{ -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\mathfrak{M}^2\right) - \sum_k \frac{1}{2m}\Delta_k + V \right\}\Psi = E\Psi .$$

where $\mathfrak{M}$ is the angular momentum vector for the relative motion of nuclei.

$$\mathfrak{M}^2 = \left(\mathfrak{M}^{(\text{total})} - \mathfrak{M}^{(\text{electron})}\right)^2$$

$$\mathfrak{M}^2 = (\mathfrak{M}^{(t)})^2 + (\mathfrak{M}^{(e)})^2 - 2\mathfrak{M}^{(t)}\mathfrak{M}^{(e)}$$

$$(\mathfrak{M}^{(t)})^2 = \hbar^2 K(K+1)$$

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Electronic states of a molecule

In the above approximation, the electronic, vibrational and rotational states can be investigated separately to a large extent.

Let us first consider the electronic states of a molecule. ~~In order to see the situation more easily, let us take the case of diatomic molecule~~ more things

$$\left\{-\frac{1}{2}\sum_{k=1}^{n}\Delta_k + V(r_k, R_\alpha) - W(R_\alpha)\right\}\psi(r_k, R_\alpha) = 0$$

In order that ψ represents a state of a stable (or metastable) molecule, the parameters R_α must take a set of values corresponding to ~~the~~ minimum of $W(R_\alpha)$.

Now the total kinetic energy of the electrons is invariant under

(i) rotations (including reflections)

(ii) permutations of electronic coordinates,

just as in the case of the atom.

The potential energy V is given by

$$V = \sum_{\alpha<\beta}\frac{Z_\alpha Z_\beta e^2}{R_{\alpha\beta}} + \sum_{k<l}\frac{e^2}{r_{kl}} - \sum_{k,\alpha}\frac{Z_\alpha e^2}{r_{k\alpha}}$$

and is invariant under the operations, in which, not only the electrons, but also all the nuclei are rotated by ~~in~~ simultaneously. ~~In other words~~, Namely,

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is ~~invariant under the~~ commutative with the operators which are the products of electronic rotation operators ~~and~~ $D^{(el)}(\alpha_1, \alpha_2, \alpha_3; \pm)$ and the corresponding nuclear rotation operators $D^{(n)}(\alpha_1 \alpha_2 \alpha_3; \pm)$. Hence, if V is commutative with a certain $D^{(n)}(\alpha_1, \alpha_2, \alpha_3; \pm)$, then it must be commutative with the corresp. $D^{(el)}(\alpha_1, \alpha_2, \alpha_3; \pm)$. In other words, $D^{(n)}$ is such an operation, that leaves the ~~nuclear configuration~~ positions of the nuclei exactly the same as before*, the corresponding electronic rotations will leave V invariant. The totality of such rotations including reflections ~~include~~ constitutes a subgroup of the whole group of proper and improper rotations in space. Such a group is called the symmetry group of the molecule.

Before going into general considerations of the symmetry group of an arbitrary polyatomic molecule, let us consider the particular case.

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* If there are two or more identical nuclei (in the same internal state), the movement of each into the original position of another will make no difference to V.

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linear Molecule

CO_2 O C O 1.16 A°

CS_2 1.6 A°

SCO O C S

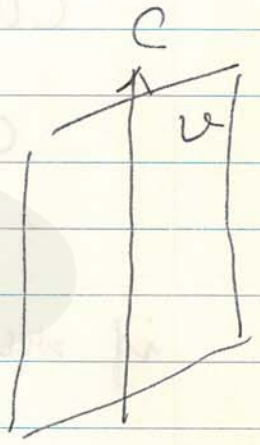
N_2O N N O 2.38 Å

HCN H C N 1.06 Å 1.15 Å

C_2H_2 H C C H 1.204 1.057

C_2N_2 N C C N 1.5 1.2

C_4H_2 H C C C C H 1.43 1.21



in addition to all diatomic molecules such as H_2, CO, HCl etc.

Now, the axis of nuclei becomes an axis of symmetry C and the rotation $C(\varphi)$ of any angle φ around this axis leaves the Hamiltonian H_0 invariant. Take any plane v containing C. The reflection σ_v with respect to v also leaves H_0 invariant. (vertical axis) Any symmetry operation which leaves H_0 invariant can be obtained by combining $C(\varphi)$ and σ_v. The totality of such operations is usually denoted by $C_{\infty v}$, where C_∞ means the limit $n \to \infty$ of C_n, C_n being the totality of rotations around C of angles $\varphi = \frac{2\pi}{n}$ $(n = 2, 3, \cdots)$

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Wallden, p. 135

$C_{\infty v}$ = ~~$C_\infty \otimes \sigma_v$~~ ; cyclical rotation-reflection group (A_1, A_0^+, A_0^-)

Since C_∞ constitutes an Abelian group, the irreducible representations is are all one dimensional

$$C(\varphi) \rightarrow f(\varphi)$$

$$C(\varphi_1)C(\varphi_2) = C(\varphi_1+\varphi_2)$$

$$\rightarrow f(\varphi_1)f(\varphi_2) = f(\varphi_1+\varphi_2)$$

$$C(2\pi n) = 1 \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\rightarrow f(2\pi n) = 1 \quad (n = 0, \pm 1, \pm 2, \cdots)$$

if the representation is one-valued.

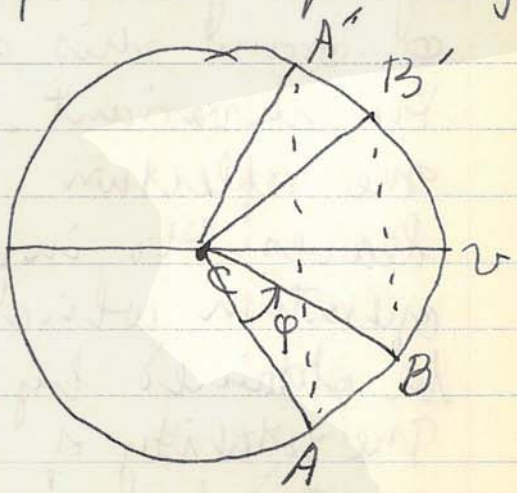
$$f(\varphi) = e^{i\Lambda\varphi} \qquad \Lambda = 0, \pm 1, \pm 2, \cdots$$

The eigenfunctions $\psi_\Lambda(\varphi)$ which are the basic vectors for the representations of C_∞ are thus

$$D(\varphi)\,\psi_\Lambda = e^{i\Lambda\varphi}\psi_\Lambda ,$$

where $D(\varphi)$ is the operator representing $C(\varphi)$.

Under the reflection σ_v, points A, B are transformed into A', B'.



$$C(\varphi)\sigma_v C(\varphi) = \sigma_v$$

or

$$\sigma_v C(\varphi) = C(-\varphi)\sigma_v$$

or

$$S \cdot R(\varphi) = R(-\varphi) S$$

where S is the operator repres. σ_v.

$$R(-\varphi)\{S\psi_\Lambda\} = e^{+i\Lambda\varphi}\{S\psi_\Lambda\}$$

$$R(\varphi)\{S\psi_\Lambda\} = e^{-i\Lambda\varphi}\{S\psi_\Lambda\}$$

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Thus, $S\psi_\Lambda$ is the eigenfn belonging to the the eigenvalue $-\Lambda$ since the representation of $C_{\infty v}$ is ~~reduced to one dimension~~ ~~one dimensional~~ and there is no degeneracy,

$$S\psi_\Lambda = \varepsilon(\Lambda)\psi_{-\Lambda}$$

Since $S^2 = 1$,

$$S^2\psi_\Lambda = S\cdot\varepsilon(\Lambda)\psi_{-\Lambda} = \varepsilon(\Lambda)\varepsilon(-\Lambda)\psi_\Lambda$$

~~S is unitary~~ One can confine oneself to unitary representation and one ~~may~~ has

$$\varepsilon(-\Lambda) = \tilde{\varepsilon}(\Lambda),$$

so that $|\varepsilon(\Lambda)|^2 = 1$, or

$$\varepsilon(\Lambda) = e^{i\alpha} \qquad (\varepsilon(-\Lambda) = e^{-i\alpha})$$

with real α.

For $\Lambda \neq 0$, one can replace ψ_Λ by $\psi_\Lambda e^{+i\alpha/2}$ and $\psi_{-\Lambda}$ by $\psi_{-\Lambda} e^{-i\alpha/2}$, we have

$$S\psi_\Lambda = \psi_{-\Lambda} \qquad (\Lambda \neq 0).$$

Thus, for $\Lambda \neq 0$, the irreducible representation of $C_{\infty v}$ ~~is a cons is~~ is obtained by taking ψ_Λ and $\psi_{-\Lambda}$ as basic vectors.

For $\Lambda = 0$, $\varepsilon^2(0) = 1$. $\varepsilon(0) = \pm 1$.

Thus, there are two possibilities

a) $S\psi_{\Lambda+} = +\psi_{\Lambda+}$

b) $S\psi_{\Lambda-} = -\psi_{\Lambda-}$,

each being the basic vector for the irreducible repres. of $C_{\infty v}$.

Since $C_{\infty v}$ is a subgroup of d_3 which leaves ~~to~~ the Hamiltonian

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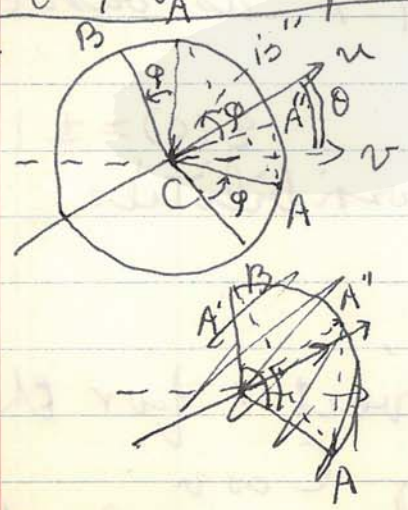
invariant, the stationary states of ~~a~~ linear molecule can be classified by the values of Λ and q:

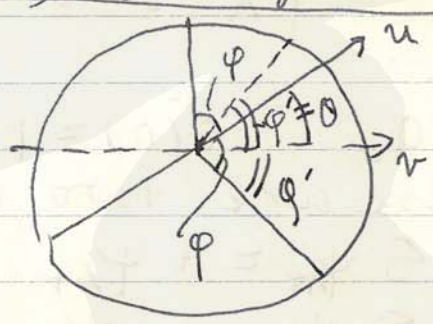
$\Lambda = 0$, $q = +1$: Σ^+ (S)
$\Lambda = 0$, $q = -1$: Σ^-
$\Lambda = \pm 1$ ($|\Lambda| = 1$): Π (P)
$\Lambda = \pm 2$ ($|\Lambda| = 2$): Δ (D)
$\Lambda = \pm 3$ ($|\Lambda| = 3$): Φ (F)
$\Lambda = \pm 4$ ($|\Lambda| = 4$): Γ (G)

Λ is called the axial quantum number. *

The above considerations hold for any linear molecule. In the case of ~~a~~ molecules such as CO_2, CS_2, C_2H_2 etc., there is an additional symmetry: the centre of symmetry and the plane of symmetry which passes through the centre and perpendicular to the axis C. (horizontal plane) (horizontal axis)

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* 

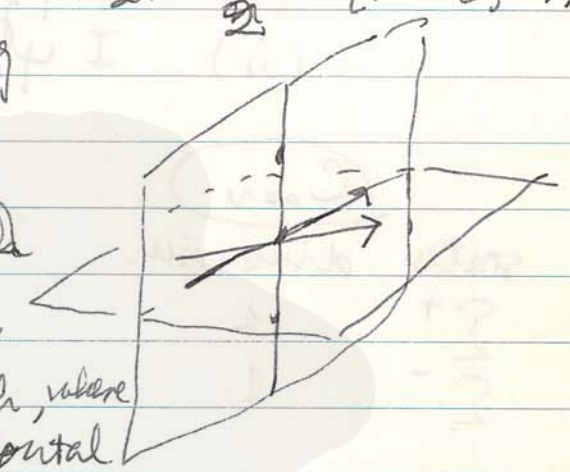


$\sigma_u = C(2\varphi - 2\varphi')\sigma_v$
$= C(2\theta)\sigma_v$

~~$\varphi' + \theta + \varphi = 2\varphi$~~
$\varphi' + \theta = \varphi$.
$\theta = \varphi - \varphi'$

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The Hamiltonian is invariant ~~with respect to~~ under the reflection with respect to the centre and the rotations of angles $\varphi = 2\pi \frac{m}{n}$ $(m = 0, \pm 1, \pm 2, \ldots)$ around any axis passing through the centre and is $\perp$ or $\parallel$ to C_n, all these



Totality of ~~all these such~~ operations ~~or~~ symmetry operation is denoted by D_{nh}, where D means that the horizontal axis is the symmetry axis for $n = 2$ and h means that the horizontal plane is a plane of symmetry. Any operation of D_{nh} can be constructed from $C(\varphi)$, σ_v and I, where I denotes the reflection with respect to the centre.

I is commutative with $C(\varphi)$ and σ_v, so that the representation of the group D_{nh} can be reduced with respect to C_{nv} and the groups consisting of 1 and I simultaneously, since

$$I^2 = 1,$$

there are two irred. representations of the latter

i) $I = +1$

ii) $I = -1$,

The corresponding eigenfunctions ~~are called~~ have even or odd parity and ~~the~~ discriminated by the indices g and u.

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thus $\Sigma_g^+, \Sigma_g^-, \Sigma_u^+, \Sigma_u^-, \Pi_g, \Pi_u$

etc.

(i) $I\,\psi_g = \psi_g$

(ii) $I\,\psi_u = -\psi_u$

$C_{\infty v}$

states	dimension	symmetry operation $C(\varphi)$	σ_v
Σ^+	1	+1	+1
Σ^-	1	+1	−1
Π	2	$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Δ	2	$\begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$D_{\infty h}$		$C(\varphi)$	σ_v	σ_h*	I
Σ_g^+	1	+1	+1	+1	+1
Σ_g^-	1	+1	−1	+1	+1
Σ_u^+	1	+1	+1	−1	−1
Σ_u^-	1	+1	−1	−1	−1
Π_g	2	$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	−1	+1
Π_u	2			+1	−1
Δ_g	2	$\begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	+1	+1
Δ_u	2			−1	−1

* σ_h is a product of I and a certain $C(\varphi)$.

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If we take into account the spins of the electrons, as the Hamiltonian in our approxim. is commutative with any of the spin variables, the electronic states can be characterized further in terms of (the total spin quantum numbers S and) M_S, where $\hbar M_S$ is the eigenvalue of the component of the total spin angular momentum in the direction of the symmetry axis.* Thus, we have the multiplet structure as the case of atoms, if we take into account the spin dependent terms in the Hamiltonian. Further, the exclusion principle restricts the possible electronic states.

See:
Kronig, Band Spectra and Molecular Structure

Example: Diatomic Molecules:
There are two methods of dealing with molecules in greater details:
(i) Method of Molecular Orbitals
(Mulliken, Rev. Mod. Phys. 4 (1939), 19)
(ii) Method of Atomic Orbitals
Heitler-London's theory.

---

*Only M_S is a good q.n. In many cases, nuclear vibrational levels are more widely seperated than the multiplet levels. Sometimes, this is reversed.

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Selection Rules for linear molecules.

For a dipole radiation, what we have to know is whether the matrix elements

$$\int \tilde{\psi}' \sum_k x_k \psi$$

would be different from zero for a transition between an electronic state ψ to another ψ'. Just as in the case of the atom, ~~it is only necessary to~~ one can argue in the following way. ψ be an eigenfunction belonging to a certain irreducible representation of the group $C_{\infty v}$ (or $D_{\infty h}$) and ψ' be another. Now, if we take the symmetry axis as z-axis, we have

$$\left.\begin{aligned} x_k + i y_k &= r_k e^{i\varphi_k} \\ x_k - i y_k &= r_k e^{-i\varphi_k} \\ z_k &= z_k \end{aligned}\right\}$$

This show that $\sum x_k, \sum y_k$ can be regarded as basic vectors for the representation Π_u of $D_{\infty h}$ and z_k as that for Σ_u^+.

Thus, as in the case of atoms, one can conclude that the matrix element will be different from zero, if only if ψ' belongs to ~~an irr~~ a representation R' which is contained in the product

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representation

$$\Pi_u \times R$$
$$\Sigma_u^+ \times R$$

where R is the irreducible representation to which ψ belongs.

Now we have the relations, for example,

$$\Delta_g \times \Pi_u = \Sigma_u + \Pi_u$$

Proof: Let $\Psi(\Delta_g)$ be any wave which belong to Δ_g etc.

$$\Psi(\Delta_g) = a\psi_{2,g} + b\psi_{-2,g}$$
$$\Psi(\Pi_u) = a'\psi_{1,u} + b'\psi_{-1,u}$$

where $\psi_{1,g}$ and $\psi_{1,u}$ denote the eigenfs of axial q.n. Λ and with even or odd parity.

If we operates a rotation $R(\varphi)$ of an angle φ around z-axis on the product:

$$R(\varphi)\,\Psi(\Delta_g)\,\Psi(\Pi_u)$$
$$= R(\varphi)\{aa'\psi_{2,g}\psi_{1,u} + bb'\psi_{-2,g}\psi_{-1,g} + ab'\psi_{2,g}\psi_{-1,u} + ba'\psi_{-2,g}\psi_{1,u}\}$$
$$= aa'e^{3i\varphi}\psi_{2,g}\psi_{1,u} + bb'e^{-3i\varphi}\psi_{-2,g}\psi_{-1,u} + ab'e^{i\varphi}\psi_{2,g}\psi_{-1,u} + ba'e^{-i\varphi}\psi_{-2,g}\psi_{1,u}$$

Thus, it is clear that $\Delta_g \times \Pi_u$ contains Σ and Π, either even or odd.

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Further, if we operate the reflection with respect to the center:

$$D\Psi(\Delta_g)\Psi(\Pi_u) = -\Psi(\Delta_g)\Psi(\Pi_u),$$

which shows that

$$\Delta_g \times \Pi_u = \Phi_u + \Pi_u$$

Problem: Show that

$$\Pi_g \times \Pi_g = \Delta_g + \Sigma_g^+ + \Sigma_g^-$$

In general, one can prove the following general rules:

(i) $R_g \times R'_g = R''_g + \cdots$

~~(i) $R(\text{even}) \times R'(\text{odd})$~~

$R_g \times R'_u = R''_u + \cdots$

$R_u \times R'_u = R''_g + \cdots$

(ii) $R(\Lambda) \times \Sigma_u^+ = R''(\Lambda) + \cdots$

$R(\Lambda) \times \Pi_u = R''(\Lambda+1) + R'''(\Lambda-1)$

$+ \cdots$

In particular, for Σ-states:

(iii) $\Sigma^+ \times \Sigma^+ = \Sigma^+$

$\Sigma^+ \times \Sigma^- = \Sigma^-$

Thus, we have the selection rules

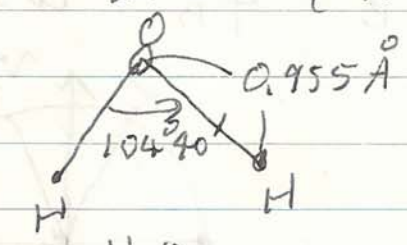
(a) $\Delta\Lambda = 0, \pm 1$

(b) $g \rightleftarrows u$

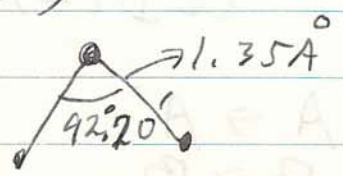
(c) $\Sigma^+ \rightleftarrows \Sigma^+$ $\quad \Sigma^- \rightleftarrows \Sigma^-$

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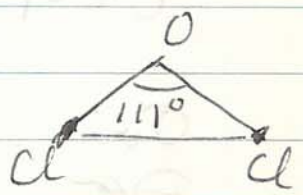
Molecules with double symmetry axis. ($n=2$)



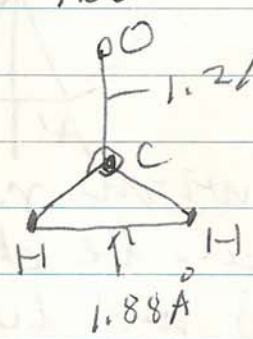
H_2O



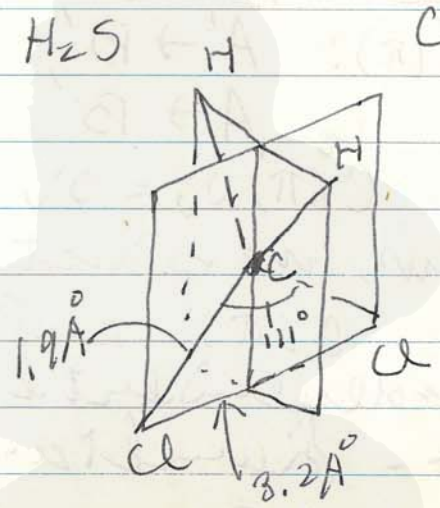
H_2S



Cl_2O



H_2CO



H_2CCl_2

as

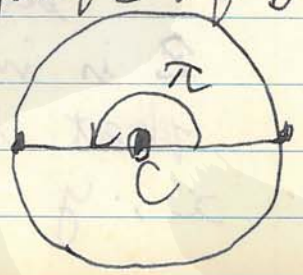
For instance, H_2CO has a double symmetry axis ~~CO~~ and H_2O has the line which bisects the angle $H\hat{O}H$. They have each two symmetry planes, one being the plane including all nuclei and the other perpendicular to it. (In the case of H_2CCl_2, one includes HCH and the other ClCCl.

The symmetry group is denoted by C_{2v}. The irreducible representations of C_{2v}: let ψ be an eigenfunction of $D(\pi)$

$$D(\pi)\,\psi = c\,\psi$$

$$D^2 = 1: \quad c^2 = 1$$

$$c = \pm 1.$$

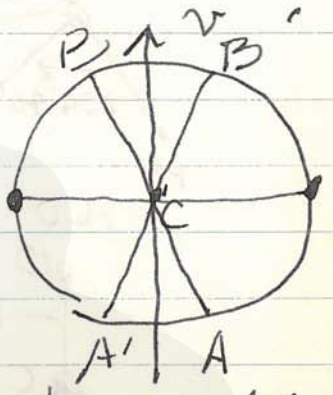


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C_2

$C=1$: $D(\pi)\psi_A = \psi_A$ (a)

$C=-1$: $C_2 D(\pi)\psi_B = -\psi_B$ (b)



σ_v : $A \to A'$,
$B \to B$

$C(\pi)$: $A' \to B'$,
$A \to B$. C_2

$C(\pi)\sigma_v = \sigma_v C(\pi)$.

Thus, the reductions with respect to $C(\pi)$ and σ_v can be done independently and we have 4-dimensional eigenfunctions

$$\left.\begin{aligned} S\psi_{A_1} &= \psi_{A_1} \\ S\psi_{A_2} &= -\psi_{A_2} \\ S\psi_{B_1} &= \psi_{B_1} \\ S\psi_{B_2} &= -\psi_{B_2} \end{aligned}\right\} \text{symmetry operation}$$

stationary states	$C(\pi)$	σ_v
A_1	+1	+1
A_2	+1	−1
B_1	−1	+1
B_2	−1	−1

Selection rules

$\mathcal{Z}$ is invariant under S and $D(\pi)$, so that it belongs to A_1.

x, y change sign under $D(\pi)$, but

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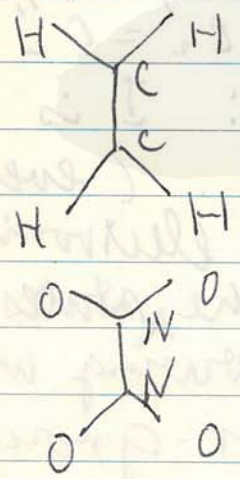
may or may not change sign under S, so that they belong to B_1.

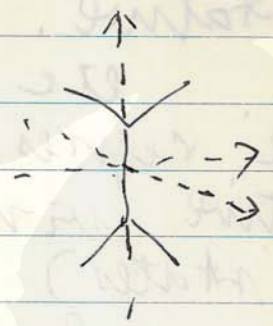
A_1	A_2	B_1	B_2	
(A_1)	A_2	(B_1)	(B_2)	A_1
(A_1)	(B_2)	(B_1)		A_2
(A_1)	A_2	B_1		
(A_1)	B_2			

As shown in the above table, for example ~~a state~~ A_1 ~~can~~ the transition between $A_1 \to A_1, B_1, B_2$ are allowed, but $A_1 \to A_2$ is forbidden

$A_1 \times A_1 = A_1$
$A_1 \times B_1 = B_1$
$A_1 \times B_2 = B_2$
~~$A_1 \times A_2 = A_2$~~

Problem:





the molecules like C_2H_4 have three symmetry axes (double), one centre of symmetry and three planes of symmetry. The group which leaves this configuration invariant ~~can~~ is constructed ~~for~~ first ...

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the rotations of angle 180° around two axes (the rotation around the third axis being the product of the first two rotations, because

$x \to -x$, $y \to -y$; $x \to -x$, $z \to -z$;
$\to (x \to x)$ $y \to -y$, $z \to -z$,

and ~~the~~ this is the group consisting of four elements *. Then, if one can add the reflection I with respect to the centre, one obtains the whole group V_h consisting of eight elements. (The reflections with respect to any of the symmetry planes can be obtained as the product I and the rotations: $x \to -x$, $y \to -y$; $x \to -x$, $y \to -y$, $z \to -z$; $\to x \to x$, $y \to y$, $z \to -z$)

Irreducible repres. of V: C_2, C_2', C_2'' are commutative.

$(C_2)^2 = 1$ etc $C_2 C_2' = C_2''$ etc.

Irreducible repres. of V_h: I is commutative with C's. (even and odd states)

Thus, we can classify the electronic states of the molecule in the following way.

* Vierer Gruppe (four-group)

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stationary states	Symmetry operation C_2	C_2'	(C_2'')	I
A_g	+1	+1	(+1)	+1
A_u	+1	+1	(+1)	−1
B_{1g}	1	−1	(−1)	+1
B_{1u}	1	−1	(−1)	−1
B_{2g}	−1	+1	(−1)	+1
B_{2u}	−1	+1	(−1)	−1
B_{3g}	−1	−1	(+1)	+1
B_{3u}	−1	−1	(+1)	−1

selection rules

$z \rightarrow B_{1u}$

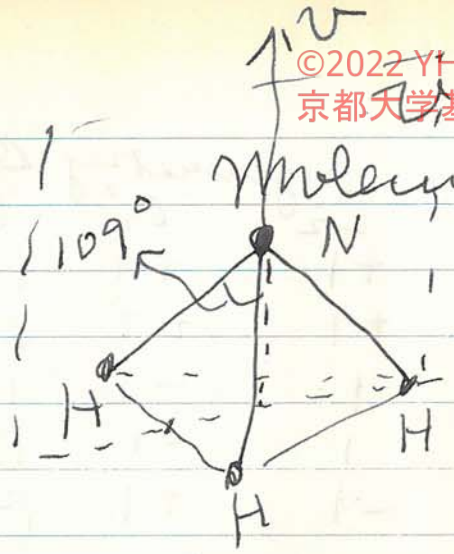
$x, y \rightarrow B_{3u}, B_{2u}$

$B_{1u} \times B_{2g} = B_{3u}$

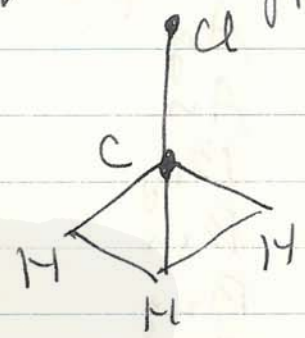
$B_{1u} \rightleftarrows B_{3u}$

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Molecules of the type NH_3.



NH_3



$ClCH_3$

Symmetry operations: $C_3 = C(\frac{2\pi}{3})$

σ_v

Group: C_{3v}

$D(\frac{2}{3}\pi)\psi = c\psi$

$(D(\frac{2}{3}\pi))^3 \psi = \psi$

$c^3 = 1$. $c = 1, e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}$

There are three irreducible repres.

C_3: A, E^+, E^-

$\psi(A), \psi(E^+), \psi(E^-)$

$D(\frac{2}{3}\pi)\psi(E^+) = e^{i\frac{2\pi}{3}}\psi(E^+)$

etc.

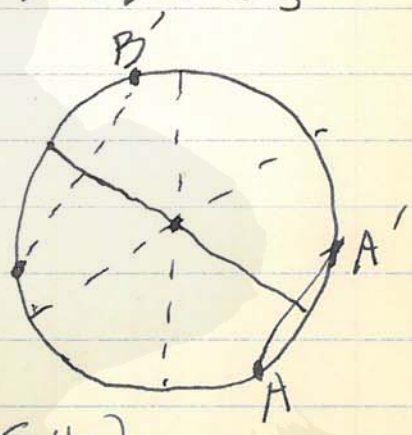
σ_v is not commutative with C_3

σ_v: $A \to A'$

C_3: $A' \to B'$

C_3^{-1}: $A \to B$

σ_v: $B \to B'$



$D(\frac{2}{3}\pi) S \psi = S D(-\frac{2\pi}{3})\psi$. B

$D(\frac{2}{3}\pi)(S\psi) = c^{-1}(S\psi)$

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Since, we have the one-dimensional irreducible repres. of C_3, we must have

$$S\,\psi(A) = \varepsilon\,\psi(A)$$
$$S\,\psi(E^{+}) = \varepsilon(+)\,\psi(E^{-})$$
$$S\,\psi(E^{-}) = \varepsilon(-)\,\psi(E^{+})$$

and since $S^2 = 1$.

$$\varepsilon = +1 \text{ or } -1.$$

A_1 $\quad$ A_2

As for $\varepsilon(+)$, $\varepsilon(-)$, the situation is the same as in the case of C_{3v}.

$$S\,\psi(A_1) = \psi(A_1)$$
$$S\,\psi(A_2) = -\psi(A_2)$$
$$S\,\psi(E^{+}) = \psi(E^{-})$$
$$S\,\psi(E^{-}) = \psi(E^{+})$$

Stationary states	Symmetry operation C_3	σ_v
A_1	$+1$	$+1$
A_2	$+1$	-1
E	$\begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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General Considerations on Symmetry Groups

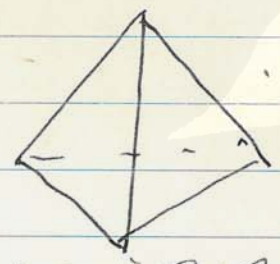
Murnaghan, Theory of Group Representation (1938)

Any subgroups of ds consists of finite and infinitely ~~many~~ number of proper and improper rotations. ~~If we have~~ a molecule consisting of a finite number of nuclei and, except for the case of linear molecules, the order of symmetry group is finite, because under any rotation (and reflection), any point outside the axis of rotation moves and so in the case of the molecule, the angle of rotation ~~cannot~~ must be finite. Moreover, the angle of any rotation φ must be a certain multiple of $\frac{2\pi}{n}$, where n is a positive integer, because otherwise $\varphi = 2\pi \cdot c$, where c is an irrational number and after repeating such a rotation N times, $N\varphi$ will become as nearly as possible an integral multiple of 2π, which certainly ~~contracts~~ ~~the~~ is a contradiction. *

Thus, the symmetry group is, except for the case of a linear molecule,

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a finite subgroup of rotation group.
The classification of such subgroups were known and can be found in book on group theory. There are infinitely many groups of the type C_n. In nature, only n up to 6 appears. There are also groups of the type D_n, if the molecule has an additional symmetry axis which is perpendicular to C_n and which is double ($n=2$): $C_2H_4Cl_2$ etc. each
There are groups which leave a ~~pot~~ regular polyhedron invariant.
~~Tetra~~hedral group
etc.
By combining this ~~and~~ the reflections, one can construct various types of symmetry groups*.
Irreducible representations of ~~all~~ these groups are known.



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* Also, the angles between symmetry axes must be finite, so that there are only finite number of symmetry axes.

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(N.1)

Nuclear structure

A nucleus consists of Z protons and N = A - Z neutrons. Each of the protons and neutrons can be described by the dynamical variables x_k, p_k and spin σ_k. It is convenient to introduce a variable τ_3 which discriminate between the proton and the neutron.

$\tau_3 = 1$: proton

$\tau_3 = -1$: neutron

Then, the Schrödinger eq. for the whole nucleus becomes of the form

$$\Psi(x_1^{(1)}, \sigma_z^{(1)}, \tau_3^{(1)}; \cdots; x^{(A)}, \sigma_z^{(A)}, \tau_3^{(A)})$$

where the condition that the atomic number is Z is expressed as

$$\sum_{k=1}^{A} \frac{1+\tau_3^{(k)}}{2} \Psi = Z\Psi .$$

The exclusion principle is satisfied for protons and neutrons separately. We can generalized it to the nucleons which contain protons and neutrons as their different states. Namely,

$$P_r \Psi = \varepsilon(P_r)\Psi$$

where P_r is a permutations of A - nucleons (including the variables $\tau_3^{(k)}$)

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As long as the neutrons and the protons retain their identity separately, the Schrödinger fn can be written as

$$\Psi = \Psi_p \cdot \Psi_N$$

where Ψ_p, Ψ_N are ~~each~~ antisym. fns of the proton variables and neutron variables.

However, we know that the neutron and proton are interchangeable in connection with the β-decay or mesonic reactions. Thus, we had better to accept the generalized exclusion principle which as above.

As for the Hamiltonian for a nucleus, ~~our knowledge is not very~~ we cannot presently claim that we know precisely what the nuclear forces ~~the proton~~ is, are. One can either try to deduce it from a more fundamental field theory or to determine it phenomenologically. In the former case, we were not very successful for many known and unknown reasons. In the latter case, the choice for the nuclear potential is so arbitrary that we cannot hope to ~~obtain~~ determine uniquely the nuclear

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General Theoretical Considerations (N.2)

potential. Thus, at the present stage, all we can hope is to derive as many conclusions as possible by starting from a number of plausible assumptions which are more or less well-established. First of all, the Hamiltonian, if it ever exist seems to satisfy a number of fundamental symmetry conditions.

(i) Invariance with respect to rotation (this must be true permutation, translation and without any doubt)

(ii) Symmetry with respect to the proton and the neutron (charge conservation and symmetry)

(iii) Nuclear potential is a mixture of ordinary and exchange forces (Position Change)

(iv) It is also a mixture of central and non-central potentials.

(v) Nuclear forces are short range forces with range of the order of 10^{-13} cm.

(vi) charge independence

~~(this last assumption~~

(vii) Nuclear forces are spin dependent (at least for non-central forces), but velocity independent,

the last two assumptions are not so well-established as the other five, and they are, at most, approximately true. In fact, we need velocity

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dependent force such the spin-orbit coupling in the current shell structure model, but, nevertheless, it seems that the velocity dependent and ~~spin~~ dep charge dependent forces are much smaller than the main forces.

Now, if the forces are spin independent, ~~only positivity~~ the potential must contain the coordinates $r^{(1)}$, $r^{(2)}$ of two nucleons in the form

$$V(r)$$

where $r = |r^{(1)} - r^{(2)}|$ due to (i)

If we include the spin dependent forces, there are a number of rotationally invariant operators, in addition to $\sigma^{(1)}\sigma^{(2)}$, r, such as

$r^{(1)}\sigma^{(1)}$ $r^{(2)}\sigma^{(2)}$, $r^{(1)}\sigma^{(2)}$, $r^{(2)}\sigma^{(1)}$, $(r^{(1)} - r^{(2)})\sigma^{(1)}$ etc.

which, however, change sign under reflection, so the ~~invariant~~ operators reduce to which ~~satisfy (ii)~~ are invariant with respect to permutations

$(r^{(1)}\sigma^{(1)})(r^{(2)}\sigma^{(2)})$ etc.

$(r^{(1)}\sigma^{(1)})^2 = \sum_j x_j^{(1)2} = (r^{(1)})^2$, ~~etc~~

$+ (r^{(2)}\sigma^{(2)})^2 = \sum_j x_j^{(2)2} = (r^{(2)})^2$

these, however, ~~do not~~ are not invariant with transition, so that, finally, we are left with the combination

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(N.3)

$$(\mathbf{r}\times\boldsymbol{\sigma}^{(1)})(\mathbf{r}\times\boldsymbol{\sigma}^{(2)}) = \sum (x\sigma_y^{(1)} - y\sigma_x^{(1)})(x\sigma_y^{(2)} - y\sigma_x^{(2)})$$
$$= \sum (x^2\sigma_y^{(1)}\sigma_y^{(2)} + y^2\sigma_x^{(1)}\sigma_x^{(2)} - xy\sigma_y^{(1)}\sigma_x^{(2)} - yx\sigma_x^{(1)}\sigma_y^{(2)})$$

$$(\mathbf{r}\boldsymbol{\sigma}^{(1)})(\mathbf{r}\boldsymbol{\sigma}^{(2)}) = (\mathbf{r}^{(1)}\boldsymbol{\sigma}^{(1)})(\mathbf{r}^{(1)}\boldsymbol{\sigma}^{(2)}) - 2(\mathbf{r}^{(1)}\boldsymbol{\sigma}^{(1)})(\mathbf{r}^{(2)}\boldsymbol{\sigma}^{(2)}) + (\mathbf{r}^{(2)}\boldsymbol{\sigma}^{(1)})(\mathbf{r}^{(2)}\boldsymbol{\sigma}^{(2)})$$

all other combinations being reduced to a function of r and $\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}$. ~~the most~~; thus, the most general form of the velocity-independent ~~ordinary~~ forces is

$$V(r) + (\boldsymbol{\sigma}_1^{(1)}\boldsymbol{\sigma}^{(2)})\,V'(r) + (\mathbf{r}\boldsymbol{\sigma}^{(1)})(\mathbf{r}\boldsymbol{\sigma}^{(2)})\,V''(r),$$

because

$$\left\{\frac{1+(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})}{2}\right\}^2 = 1 \quad \text{or} \quad \frac{1}{4} + \frac{\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2} + \frac{(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})^2}{2} = 1.$$

$$\left\{(\mathbf{r}\boldsymbol{\sigma}^{(1)})(\mathbf{r}\boldsymbol{\sigma}^{(2)})\right\}^2 = (x\sigma_x^{(1)} + y\sigma_y^{(1)} + z\sigma_z^{(1)})^2 \times (x\sigma_x^{(2)} + y\sigma_y^{(2)} + z\sigma_z^{(2)})^2$$
$$= (x^2+y^2+z^2)(x^2+y^2+z^2) = r^4.$$

It is more convenient to use the combinations

$$P_{12}^{(0)} = \frac{1+(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})}{2}$$

$$S_{12} = \frac{3(\mathbf{r}\boldsymbol{\sigma}^{(1)})(\mathbf{r}\boldsymbol{\sigma}^{(2)})}{r^2} - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$$

$$V_{12} = J(r) + P_{12}^{(0)}K(r) + S_{12}L(r)$$

The ~~last term is~~ second term is the spin exchange force and the last term

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Problem: Show that
is the tensor force, because it can
be regarded as the product of two
tensors: either (pseudo)

$$t_{ij}^{(1)} = x_i \sigma_j^{(1)} - \frac{x_k \sigma_k^{(1)}}{3} \delta_{ij}$$

$$t_{ij}^{(2)} = x_i \sigma_j^{(2)} - \frac{x_k \sigma_k^{(2)}}{3} \delta_{ij}$$

$$\sum_{ij} t_{ij}^{(1)} t_{ij}^{(2)} = \sigma_j^{(1)} \sigma_j^{(2)} x_i^2 - \frac{(x_i \sigma_i^{(1)})(x_k \sigma_k^{(2)})}{3}$$

$$- \frac{(x_i \sigma_i^{(1)})(x_k \sigma_k^{(2)})}{3} + \frac{(x_k \sigma_k^{(1)})(x_k \sigma_k^{(2)})}{3}$$

$$= (\sigma_j^{(1)} \sigma_j^{(2)}) x_i^2 - \frac{(x_i \sigma_i^{(1)})(x_k \sigma_k^{(2)})}{3}$$

$$= -\frac{1}{3} S_{12}$$

or the product of two tensors.

$$t_{ij} = \frac{1}{2}(\sigma_i^{(1)} \sigma_j^{(2)} + \sigma_j^{(1)} \sigma_i^{(2)}) - \frac{\sigma_k^{(1)} \sigma_k^{(2)}}{3} \delta_{ij}$$

$$\sum_i t_{ii} = 0.$$

$$T_{ij} = x_i x_j - \frac{x_k x_k}{3} \delta_{ij}$$

$$S_{12} = \frac{3}{2} \sum_{ij} t_{ij} T_{ij}$$

$$\delta_{ij}' = a_{ik} a_{jl} \delta_{kl}$$

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Remarks:
There are other combinations such as
$(r \times \sigma^{(1)})(r \times \sigma^{(2)})$.

$$= \sum_{xyz} (x\sigma_y^{(1)} - y\sigma_x^{(1)})(x\sigma_y^{(2)} - y\sigma_x^{(2)})$$

$$= \sum (x^2 \sigma_y^{(1)}\sigma_y^{(2)} + y^2 \sigma_x^{(1)}\sigma_x^{(2)}) - \sum xy(\sigma_x^{(1)}\sigma_y^{(2)} + \sigma_y^{(1)}\sigma_x^{(2)})$$

$$= \sum_{xyz} (r^2 - z^2)\sigma_x^{(1)}\sigma_x^{(2)} - \sum xy(\sigma_x^{(1)}\sigma_y^{(2)} + \sigma_y^{(1)}\sigma_x^{(2)})$$

$$= r^2 \sigma^{(1)}\sigma^{(2)} - \sum (r\sigma^{(1)})(r\sigma^{(2)})$$

which reduces to ~~the~~ $\sigma^{(1)}\sigma^{(2)}$ and $\mathbb{1}_{12}$.
a combination of

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Now, as for the dependence on the charge, ~~each~~ of the nucleons, since any operator operating on the ~~variable~~ τ_3 function $\psi(\tau_3)$ can be written as a linear combination of

$$1, \quad \tau_1, \quad \tau_2, \quad \tau_3$$

where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

any function $\tau_j^{(1)}, \tau_j^{(2)}$ which is symmetric with respect to the permutation of 1 and 2 takes the form

$$a_0 + \sum_j c_j (\tau_j^{(1)} + \tau_j^{(2)}) + \frac{1}{2} \sum_{jk} c_{jk} (\tau_j^{(1)} \tau_k^{(2)} + \tau_k^{(1)} \tau_j^{(1)}),$$

where a_0, c_j, c_{jk} can still depend on coordinates and ~~among others~~, the terms of the form ~~$\tau_j^{(1)} + \tau_j^{(2)}$~~ Spins

Now, if we postulate the charge symmetry, the linear terms like $\tau_3^{(1)} + \tau_3^{(2)}$ drops, because it ~~has the value~~ changes sign for $p-p$ and $n-n$.

As for the terms like

$$\tau_1^{(1)} + \tau_1^{(2)} \qquad \tau_2^{(1)} + \tau_2^{(2)},$$

one can exclude them simply on the ground that they are not commutative with $\tau_3^{(1)} + \tau_3^{(2)}$, so that the total charge is not conserved, ~~in order that~~ Similarly, the terms like

$$c_{13}(\tau_1^{(1)} \tau_3^{(2)} + \tau_3^{(1)} \tau_1^{(2)}) + c_{23}(\tau_2^{(1)} \tau_3^{(2)} + \tau_3^{(1)} \tau_2^{(2)})$$

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are excluded from the, because

$$\{ C_{13}(\tau_1^{(1)}\tau_3^{(2)} + \tau_3^{(1)}\tau_1^{(2)}) + C_{23}(\tau_2^{(1)}\tau_3^{(2)} + \tau_3^{(1)}\tau_2^{(2)}),\ \tau_3^{(1)} + \tau_3^{(2)} \}$$

$$= -i\{C_{13}(\tau_2^{(1)}\tau_3^{(2)} + \tau_3^{(1)}\tau_2^{(2)}) - C_{23}(\tau_1^{(1)}\tau_3^{(2)} + \tau_3^{(1)}\tau_1^{(2)})\}$$

$\neq 0$. unless $C_{13} = C_{23} = 0$.

Further,

$$[\tau_1^{(1)}\tau_2^{(2)} + \tau_2^{(1)}\tau_1^{(2)},\ \tau_3^{(1)} + \tau_3^{(2)}]$$

$$= -i(\tau_2^{(1)}\tau_3^{(2)} + \tau_1^{(1)}\tau_3^{(2)} + \tau_1^{(1)}\tau_2^{(2)} + \tau_2^{(1)}\tau_2^{(2)}) \neq 0$$

Then consider

$$[C_{11}\tau_1^{(1)}\tau_1^{(2)} + C_{22}\tau_2^{(1)}\tau_2^{(2)},\ \tau_3^{(1)} + \tau_3^{(2)}]$$

$$= -i\{C_{11}(\tau_2^{(1)}\tau_1^{(2)} + \tau_1^{(1)}\tau_2^{(2)}) - C_{22}(\tau_1^{(1)}\tau_2^{(2)} + \tau_2^{(1)}\tau_1^{(2)})\}$$

$= 0$ only for $C_{11} = C_{22}$

Thus, the most general form of the operator which satisfies the conservation of charge and the invariance with respect to the permutation is

$$C_0 + C_{12}(\tau_1^{(1)}\tau_2^{(2)} + \tau_2^{(1)}\tau_1^{(2)}) + C_{33}\tau_3^{(1)}\tau_3^{(2)}.$$

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Normal vibrations ~~Physically~~, the three operators ~~is~~ $1, \frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})$, $\tau_3^{(1)}\tau_3^{(2)}$ (each of) have simple physical meaning:

$\tau_3^{(1)}\tau_3^{(2)} = 1$ for n-n and p-p

$= -1$ for n-p,

while

$\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})\,\psi_{pn}(\cdots\tau_3^{(1)}, \tau_3^{(2)}) = \psi_{np}(\cdots\tau_3^{(1)}\tau_3^{(2)})$

$\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})\,\psi_{np}(\cdots\tau_3^{(1)}\tau_3^{(2)}) = \psi_{pn}(\cdots\tau_3^{(1)}\tau_3^{(2)})$

$\psi_{nn}(\cdots) = 0$ etc.

more precisely

$\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})\,\alpha(1)\beta(2) = \beta(1)\alpha(2)$

$\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})\,\beta(1)\alpha(2) = \alpha(1)\beta(2)$

$\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})\,\alpha(1)\alpha(2) = 0$

etc.

Thus, $\tau_3^{(1)}\tau_3^{(2)}$ gives rise to the ordinary forces between two nucleons, but the forces have opposite signs for n-n, p-p and n-p. The operator $\frac{1}{2}(\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)})$ gives rise to the <u>exchange</u> forces between a neutron and a proton. Now, if we take the combination

$\vec{\tau}^{(1)}\vec{\tau}^{(2)} = \tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)} + \tau_3^{(1)}\tau_3^{(2)}$

$\vec{\tau}^{(1)}\vec{\tau}^{(2)}\cdot\alpha(1)\alpha(2) = 1$

$\vec{\tau}^{(1)}\vec{\tau}^{(2)}\cdot\alpha(1)\beta(2) = 2\beta(1)\alpha(2) - \alpha(1)\beta(2)$

$\vec{\tau}^{(1)}\vec{\tau}^{(2)}\cdot\beta(1)\alpha(2) = 2\alpha(1)\beta(2) - \beta(1)\alpha(2)$

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Hence,

$$(\vec{\tau}^{(1)}\vec{\tau}^{(2)})\frac{1}{\sqrt{2}}\{\alpha(1)\beta(2)+\beta(1)\alpha(2)\} = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2)+\beta(1)\alpha(2)\}$$

$$(\vec{\tau}^{(1)}\vec{\tau}^{(2)})\frac{1}{\sqrt{2}}\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\} = \frac{-3}{\sqrt{2}}\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\}$$

~~Although the vector~~ τ-spin space has
Now we can imagine the $\vec{\tau}$ is a vector
in the isobaric spin space, although
this space is fictitious and has nothing
to do with the ordinary space or ordinary
spin space. Then, one can talk about just
as in the case of ordinary space,
isobaric spin multiplets. Namely,
$\vec{\tau}^{(1)}\vec{\tau}^{(2)}$ has two eigenvalues +1 and -3
and the ~~single~~ corresponding
eigenfunctions belong to the triplet
and singlet ~~of the~~ states with respect
to total isobaric spins of the two nucleons.
The important thing is that, in triplet
states, $\vec{\tau}^{(1)}\vec{\tau}^{(2)}$ has the same value independent
of whether nucleons they are neutron or proton
states, ~~In the~~ while, in singlet state,
only n-p combination is possible.
In that sense, $(\vec{\tau}^{(1)}\vec{\tau}^{(2)})$ is an operator
which is independent of the charges
of the nucleons and gives rise to the
charge independent nuclear forces.
Of course, the operator 1 itself

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is charge independent in a more rigorously exact sense.*

Thus, if we accept the postulate of charge independence of the specific nuclear forces is true, at least, approximately, then we are left with the most general possible form for the nuclear potential between two nucleons:

$$V_{12}(r^{(1)}, \sigma^{(1)}, \tau^{(1)}; r^{(2)}, \sigma^{(2)}, \tau^{(2)})$$

$$= U'(r^{(1)}, \sigma^{(1)}; r^{(2)}, \sigma^{(2)})$$

$$+ \tau^{(1)}\tau^{(2)} U(r^{(1)}, \sigma^{(1)}, r^{(2)}, \sigma^{(2)})$$

where

$$U' = J'(r_{12}) + P_{12}^{(\sigma)} K'(r_{12}) + S_{12} L'(r_{12})$$

$$U = J(r_{12}) + P_{12}^{(\sigma)} K(r_{12}) + S_{12} L(r_{12})$$

(exact)

As for the form of the functions $J(r)$ etc., one can say very little except from the above phenomenological arguments alone, except that they must decrease very rapidly with the distance r.

---

* Of course, the (exact) charge symmetry is destroyed due to the presence of Coulomb repulsion between two protons, which has the form $\frac{(1+\tau_3^{(1)})}{2}\frac{(1+\tau_3^{(2)})}{2}\frac{e^2}{r_{12}}$.

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Conclusions from the Meson Theory

According to the field theory of nuclear forces, ~~in which~~ the ~~interactions~~ effective potential between nucleons can be deduced from the more fundamental interaction between the nucleons and the mesons. Since we know now that ~~the π-~~ both the ~~ch~~ charged and neutral π-mesons exist with about the same mass (265 ~ 278 me) and interact with the nucleons, ~~about equally~~ strongly we can conclude that, at least partly and certainly for comparatively large distance, the nuclear ~~interaction~~ potential is due the π-mesons. ~~Charged π-mesons~~ Both There are a number of evidences that both the charged and neutral π-mesons have spin zero and very likely to be pseudoscalar*. In order that the nuclear forces as deduced from the meson theory is in conformity with the charge independence hypothesis, we have to choose the interaction between π-mesons and nucleons in the following way:

$$H_{int} = g\left(\frac{1}{2}(\tau_1 + i\tau_2)\,\varphi_{p\to n} + \frac{1}{2}(\tau_1 - i\tau_2)\,\varphi^*_{n\to p} + \frac{1}{2}\tau_3\,\varphi_0\right)$$

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where $\varphi^{(+)}$ ~~includes~~ consists of the operators which ~~create the~~ annihilates ~~the~~ a positive meson and ~~those~~ that which creat a negative meson, while φ_0 consists of the operator which annihilate a neutral meson and that which create a neutral meson.

$$\frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\left.\begin{aligned} \frac{1}{2}(\tau_1 - i\tau_2)\alpha &= 0 \\ \frac{1}{2}(\tau_1 - i\tau_2)\beta &= \alpha \end{aligned}\right\}$$

Thus, $\frac{1}{2}(\tau_1 - i\tau_2)$ is an operator which changes a neutron into a proton, while $\frac{1}{2}(\tau_1 + i\tau_2)$ is that which changes a proton into a neutron.

One may write also

$$H_{int} = g \tau \varphi$$

where

$$\left.\begin{aligned} \varphi_1 &= \frac{1}{2}(\varphi + \varphi^*) \\ \varphi_2 &= \frac{-i}{2}(\varphi - \varphi^*) \\ \varphi_3 &= \frac{1}{2}\varphi_0 \end{aligned}\right\}$$

are Hermitian operators.

~~In ord~~ If the mesons are scalar, the above interaction results in ~~what~~ is called the symmetric scalar theory.

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If the mesons are pseudoscalar,

$$H_{int} = \frac{f}{\kappa}\,\tau_j \cdot \sigma\,\mathrm{grad}\,\varphi_j \qquad \kappa \simeq \frac{mc}{\hbar}$$

which is called the pseudovector coupling in the non-relativistic approximation for the nucleon. Relativistically, there is one more possibility of pseudoscalar coupling with

$$H_{int} = g\,\gamma_5 \cdot \tau_j \varphi_j .$$

In the second order perturbation calculation or the first approximation in Fock's configuration space method, we obtain the well-known potential

$$V^{(ps)} = f^2 \frac{(\sigma^{(1)}\mathrm{grad})(\sigma^{(2)}\mathrm{grad})}{\kappa^2} \frac{e^{-\kappa r}}{r} \frac{(\tau^{(1)}\tau^{(2)})}{2}$$

~~* There are always other things, the arguments about the detailed balance~~

$$= f^2 \frac{\tau^{(1)}\tau^{(2)}}{2} \cdot \frac{1}{3} \cdot \left\{ \sigma^{(1)}\sigma^{(2)} + S_{12}\left(1 + \frac{3}{\kappa r} + \frac{3}{\kappa^2 r^2}\right) \right\} \times \frac{e^{-\kappa r}}{r}$$

which consists of both ordinary and exchange force on the one hand and central and tensor forces on the other hand.

There are a number of reasons that

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this potential cannot be accepted:
(i) The singularity $\propto \frac{1}{r^3}$ is not admissible, because the ~~lowest energy of the~~ lowest eigenvalue for the energy of n-p system would be $-\infty$. Thus, the above potential is to be modified for small values of r.
(ii) Since the value of the coupling constant f is comparatively large ($f^2 > \frac{1}{10}$), and, in particular, g^2 ($g = f\left(\frac{2M}{\mu}\right)$) is larger than one, the validity of second order perturbation is dubious.
(iii) If we take into account higher order terms, the forces with shorter ranges will appear one after another and one knows for sure what will be the net effect of all these terms at short distances. One can only say that at larger distances, the main contribution will be from the second order with the form $e^{-\kappa r}/r$.

~~(iv)~~ In spite of these objections, the above potential which was obtained ~~as~~ from ~~ps~~ symmetric ps. theory is interesting, at least, qualitatively: In the ^3S states of the ~~two~~ nucleon system (i.e. n-p system such as deuteron), which is the singlet ~~state~~

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(Because of the antisymmetry of total
of with respect to τ-spins, wave fn,
$\frac{1}{3}\frac{\tau^{(1)}\tau^{(2)}}{2} = -\frac{3}{2}\cdot\frac{1}{3} = -\frac{1}{2}$

$\sigma^{(1)}\sigma^{(2)} = 1$,
so that we have central attractive
potential plus the tensor force.
The tensor force potential with
the factor

$-\frac{1}{2}\{3(\sigma^{(1)}r)(\sigma^{(2)}r) - (\sigma^{(1)}\sigma^{(2)})\}$

is deeper for the directions of r ||el to
the resultant spin $\sigma^{(1)}+\sigma^{(2)}$ than those
⊥ar to $\sigma^{(1)}+\sigma^{(2)}$, so that the electric
quadrupole moment of the deuteron
becomes positive.

$\frac{1}{r^2}(\sigma^{(1)}r)(\sigma^{(2)}r)\cdot\alpha(1)\alpha(2) = ((x+iy)\beta(1) + z\alpha(1))$
$\times((x+iy)\beta(2) + z\alpha(2))\frac{1}{r^2}$
$(\sigma^{(1)}\sigma^{(2)})\alpha(1)\alpha(2) = \beta(1)\beta(2) - \beta(1)\beta(2)$
$+ \alpha(1)\alpha(2)$

$\{3(\quad)-(\quad)\}\alpha(1)\alpha(2) = 2\alpha(1)\alpha(2)$
$=$ for $z=r$, $x=y=0$.
$= 3\beta(1)\beta(2) - \alpha(1)\alpha(2)$
for $z=0$, $x^2+y^2=r^2$

Problem: Show that
$S_{12}\alpha(1)\alpha(2) = 2\alpha(1)\alpha(2)$ for $z=r$
$= -\alpha(1)\alpha(2)$ for $z=0$
$x=r$ $y=0$
$3\beta(1)\beta(2)$

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$$S_{12}\,\beta(1)\beta(2) = 3\,\alpha(1)\alpha(2) - \beta(1)\beta(2) \quad \text{for } z=r$$
$$= 2\,\beta(1)\beta(2) \quad \text{for } z=0$$

$$S_{12}\{\alpha(1)\alpha(2)+\beta(1)\beta(2)\}$$
$$= 2\{\alpha(1)\alpha(2)+\beta(1)\beta(2)\} \quad \text{for } z=r$$
$$= 2\{\quad,\quad\} \quad \text{for } z=0$$

In singlet 'S state of the two nucleon system (n-n, p-p, n-p), or triplet with respect to τ-spins

$$\frac{1}{3}\,\frac{\tau^{(1)}\tau^{(2)}}{2}\,\sigma^{(1)}\sigma^{(2)} = -\frac{1}{2}$$

again, so that we have the central attractive force, while there is no tensor force in 'S states. singlet

$$\therefore\quad S_{12}\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\}$$
$$= \frac{3}{r^2}\Big[\{(x+iy)\beta(1)+z\alpha(1)\}\{(x-iy)\alpha(2)-z\beta(2)\}$$
$$-\{(x-iy)\alpha(1)-z\beta(1)\}\{(x+iy)\beta(2)+z\alpha(2)\}$$
$$+3\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\}$$
$$= 3\{\quad\} = 0 \quad *$$

Problem: show that *

$$S_{12}\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\} = 0.$$

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In the triplet states, the total orbital ang. ~~quantum~~ momentum is not the good q.n. and; for instance, the ~~the~~ normal state of the deuteron is a mixture of ^3S and ^3D. The total ang. momentum including the spin ang. momentum is commutative with S_z, so that the normal state of the deuteron has $J=1$, $M=1, 0, -1$.

Problem: Show that

Thus, the pseudoscalar system. theory gives rise to qualitatively acceptable nuclear potential. However, if we compare it with the phenomenological calculations

(See, for example
Feshbach and Schwinger,
P. R.)

we find that the central potential ~~should~~ have shorter range and a little deeper than predicted by pseudoscalar meson theory.

For this purpose, the ~~the~~ central scalar ~~meson theory with~~ the existence of mesons of mass larger than that of π-mesons ~~could be pos~~ could be postulated.

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$$-g^2 \frac{e^{-\kappa' r}}{r}$$

with the interaction

~~int~~ $H_{int} = g \cdot \varphi_0$

$\rightarrow g \gamma_4 \varphi_0$

this satisfies obviously the postulate of charge independence.

Now according to F. and S. (H. Feshbach and J. Schwinger, P.R. 84 (1951), 194), if we take the nuclear potential

$$V(r) = -\left[V_c f(r/r_c) + V_t g(r/r_t) S_{12}\right]$$

for a n.-p system, where

$f(x) = g(x) = e^{-x}/x$,

one can determine V_c, V_t, r_c, r_t from

ε = binding energy of the deuteron
2.23 ± 0.007 MeV

Q = q. m. of d. $(2.766 \pm 0.02) \times 10^{-27}$ cm^2

and further, by accepting the charge independence hypothesis, one can make use of the data of n-p scattering for 'S state. This leaves one parameter free. Then mag. moment of H_2^2, n-p scattering, photo-disinteg. of H^2 can be

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taken into account. general trend is that both conclusion

$$\gamma = V_t / V_c$$

$$\tau = r_c / r_t$$

are preferably smaller than one.
(See later)

$$* \quad S_{12} = \frac{3(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} - \sigma^{(1)}\sigma^{(2)}$$

For singlet,

$$(\sigma^{(1)} + \sigma^{(2)})\psi = 0$$

$\therefore (\sigma_z^{(1)} + \sigma_z^{(2)})\psi = 0$ and hence $(\sigma_x^{(1)} + \sigma_x^{(2)})\psi = 0$ etc because of invariance of ψ with respect to rotation.

Hence

$$S_{12}\psi = \left(-\frac{3(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} - \sigma^{(1)}\sigma^{(2)}\right)\psi$$

$$= (-3 + 3)\psi = 0$$

For triplet

$$S_{12} = 6\frac{(Sr)^2}{r^2} - 4$$

where $S = \frac{1}{2}(\sigma^{(1)} + \sigma^{(2)})$

$S^2 = S(S+1) = 2$, $S = 1$.

$\therefore$

$$2\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} = 4\frac{(Sr)^2}{r^2} - 2$$

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N.11

Two Nucleon System

~~Nature of~~ Tensor forces

$$S_{12} = \frac{3(\sigma^{(1)}\hat{r})(\sigma^{(2)}\hat{r})}{r^2} - \sigma^{(1)}\sigma^{(2)}$$

$m = [r \times p]$ $\quad m_z = x p_y - y p_x$

$$[(\sigma_x^{(1)} x + \sigma_y^{(1)} y + \sigma_z^{(1)} z), (x p_y - y p_x)]$$
$$= i\hbar (\sigma_y^{(1)} x - \sigma_x^{(1)} y)$$
$$[(\sigma^{(2)} x + \sigma_y^{(2)} y + \sigma_z^{(2)} z), (x p_y - y p_x)]$$
$$= i\hbar (\sigma_y^{(2)} x - \sigma_x^{(2)} y)$$

$$[S_{12}, x p_y - y p_x] = \frac{3}{r^2}\{ i\hbar (\sigma_y^{(1)} x - \sigma_x^{(1)} y)(\sigma^{(2)} r) + (\sigma^{(1)} r)(\sigma_y^{(2)} x - \sigma_x^{(2)} y) \}$$

$$[(\sigma_x^{(1)} x + \sigma_y^{(1)} y + \sigma_z^{(1)} z), \sigma_z^{(1)} + \sigma_z^{(2)}]$$
$$= 2 i\hbar (\sigma_y^{(1)} x - \sigma_x^{(1)} y)$$

$$[S_{12}, \sigma_z^{(1)} + \sigma_z^{(2)}] = \frac{3}{r^2}(-2i\hbar)\{(\sigma_y^{(1)} x - \sigma_x^{(1)} y) \times (\sigma^{(2)} r) + (\sigma^{(1)} r)(\sigma_y^{(2)} x - \sigma_x^{(2)} y)\}$$

Thus, the total angular momentum

$$\mathcal{M} = m + \frac{\hbar}{2}(\sigma^{(1)} + \sigma^{(2)})$$

is commutative with S_{12}:

$$[S_{12}, \mathcal{M}] = 0.$$

For a two nucleon system,

$$\mathcal{M}^2 = \hbar^2 J(J+1)$$
$$\mathcal{M}_z = \hbar M$$

are constants of motion and the stationary states are characterised by ~~two~~ the quantum number J (and M).

Now, if we start from the irreducible

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representations of with respect to
M^2, M_z; Σ, Σ_z separately with
eigenvalues

$$\psi_{L,M_L;S,M_S}$$

and then go over to the representation
in which S_{12} is diagonal, M^2, M_z
~~can be~~ being diagonalized simultaneously:
since

$$D_L \times D_S = D_{L+S} + \cdots + D_{|L-S|},$$

and ~~ψ~~ the eigenfunctions belonging
to a certain J ~~can be~~ are constructed
from those ~~ψ~~ belonging to L, S
which satisfy

$$J = L+S, \cdots |L-S|.$$

Thus, for the triplet state $S=1$,

$$L = J-1, J, J+1.$$

The case $L=J$ is to be rejected,
because S_{12} is invariant with
respect to reflection, so that the
matrix elements of ~~S_{12}~~:

$$(L' \cdots | S_{12} | L'' \cdots)$$

is ~~zero~~ different from zero only,
for L', L'': both even or
both odd.
Hence, ~~if~~ either

$$L = J-1, J+1 \text{ ~~(for } J=0, \pm 1\text{)}},$$

or $L=J$ is possible.
~~However, the latter case is impossible,~~

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In other words, one can further classify the states with a definite J into substates according to as the parity is even or odd.

J	even	odd
0		$^3P_0$
1	$^3S_1 + {}^3D_1$	$^3P_1$
2	$^3D_2$	$^3P_2 + {}^3F_2$
3	$^3D_3 + {}^3G_3$	$^3F_3$

As for the singlet states, since

$$S_{12}\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\} = 0, \quad *$$

we can ignore tensor forces altogether:

J	even	odd
0	$^1S_0$	
1		$^1P_1$
2	$^1D_2$	
3		$^1F_2$

As for the dependence on τ-spins:

~~In the case of the neutron-proton system in the triplet states,~~

$$\vec{\tau}^{(1)}\vec{\tau}^{(2)} = 1 \quad \text{for } {}^1S_0, {}^3P_0, {}^3P_1, {}^1D_2, \ldots$$
$$= -3 \quad \text{for } {}^3S_1, {}^3D_1, {}^1P_1, \ldots$$

~~Thus, we have four cases.~~ If we take the same dependence $J(r)$ for all types of forces:

$$V_{12} = \{1 + \vec{\tau}^{(1)}\vec{\tau}^{(2)}\}(a + b(\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}) + cS_{12}) + \vec{\tau}^{(1)}\vec{\tau}^{(2)}(a' + b'(\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}) + c'S_{12})\}J(r)$$

Thus, in general, we have different potentials for the four cases:

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$^3S_1 + {}^3D_1$, $^3D_2$, ...:

$$\{(a-3a') + (b-3b')(\sigma^{(1)}\sigma^{(2)}) + (c-3c')S_{12}\}J(r)$$

$^3P_0$, $^3P_1$, $^3P_2 + {}^3F_2$, ...;

$$\{(a+a') + (b+b')(\sigma^{(1)}\sigma^{(2)}) + (c+c')S_{12}\}J(r)$$

$^1S_0$, $^1D_2$ etc:

$$\{(a+a') + (b+b')(\sigma^{(1)}\sigma^{(2)})\}$$

$^1P_1$, $^1F_3$, etc:

$$\{(a-3a') + (b-3b')(\sigma^{(1)}\sigma^{(2)})\}$$

or one may normalize $J(r)$ what
~~write~~ according to
W. Rarita and J. Schwinger,
P. R. 59 (1941), 436; 556

$$V_{12}^{(c)} = \{1 - \tfrac{1}{2}g + \tfrac{1}{2}g(\sigma^{(1)}\sigma^{(2)}) + \gamma S_{12}\}J(r)$$

for $^3S_1 + {}^3D_1$, $^3D_2$ etc

and

$$V_{12}^{(c)} = -\{1 + \gamma S_{12}\}J(r)$$

for $^3S_1 + {}^3D_1$, $^3D_2$ etc.

$$V_{12}^{(s)} = -gJ(r)$$ with $S_{12} = 3\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} - 1$,

for $^1S_0$, $^1D_2$ etc.

The corresponding Schrödinger equations
for the relative motion are

$$\left(-\frac{\hbar^2}{M}\Delta - gJ(r)\right)\psi = E\psi$$ for singlet, even.

and

$$\left(-\frac{\hbar^2}{M}\Delta - \left(1 + \gamma\left(\frac{3(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} - 1\right)\right)\right)\psi = E\psi$$

for triplet even.

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Let us now consider a function ~~$\chi(r)$~~

$$\frac{u(r)}{r}\chi_1^m(\sigma_2^{(1)},\sigma_2^{(2)})$$

which are supposed to describe ${}^3S_1$ states.

Then $$\frac{w(r)}{r}S_{12}\chi_1^m(\sigma_2^{(1)},\sigma_2^{(2)})$$

will correspond to ${}^3D_1$ states, because

$$\Delta\{r^2 S_{12}\chi_1^m\} = \left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right\}$$
$$\times 3\{\sigma_x^{(1)}\sigma_x^{(2)}x^2+\sigma_y^{(1)}\sigma_y^{(2)}y^2+\sigma_z^{(1)}\sigma_z^{(2)}z^2+\sigma_x^{(1)}\sigma_y^{(2)}xy+$$
$$+\sigma_y^{(1)}\sigma_x^{(2)}xy+\cdots-\frac{(\sigma^{(1)}\cdot\sigma^{(2)})r^2}{3}\}\chi_1^m$$
$$=\{6(\sigma_x^{(1)}\sigma_x^{(2)}+\sigma_y^{(1)}\sigma_y^{(2)}+\sigma_z^{(1)}\sigma_z^{(2)})-6(\quad,\quad)\}\chi_1^m$$
$$=0.$$

On the other hand,

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)(r^2 S_{12}\chi_1^m)=\frac{1}{r^2}\frac{\partial}{\partial r}(2r^3 S_{12}\chi_1^m)$$
$$=6S_{12}\chi_1^m$$

or

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)S_{12}\chi_1^m$$
$$=-6S_{12}\chi_1^m=-l(l+1)S_{12}\chi_1^m$$
$$\left(=-\frac{1}{\hbar^2}m^2 S_{12}\chi_1^m \quad \text{with } l=2\right)$$

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normalized eigenf.ⁿ for ${}^3S_1 + {}^3D_1$:

$$\psi = \frac{1}{\sqrt{4\pi}}\left\{\frac{u(r)}{r} + \frac{1}{\sqrt{8}}\frac{w(r)}{r}S_{12}\right\}\chi_1^m(\sigma_z^{(1)},\sigma_z^{(2)})$$

χ_1^m : normalized

$$\int\{|u(r)|^2 + |w(r)|^2\}dr = 1$$

Problem: $\int \tilde{\chi}_1^m S_{12}\cdot S_{12}\chi_1^m \sin\theta\, d\theta\, d\varphi = 8(4\pi)$

$$= \int\tilde{\chi}_1^m\left\{9\left(\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2}\right)^2 - 6\,\sigma^{(1)}\sigma^{(2)}\left(\frac{(\sigma^{(1)}r)(\sigma^{(1)}r)}{r^2}\right)^{\#}\right.$$
$$\left. + (\sigma^{(1)}\sigma^{(2)})^2\right\}\chi_1^m \sin\vartheta\, d\theta\, d\varphi$$

$$\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} = \frac{\sigma_x^{(1)}\sigma_x^{(2)}x^2 + \sigma_y^{(1)}\sigma_y^{(2)}y^2 + \sigma_z^{(1)}\sigma_z^{(2)}z^2}{r^2} + \frac{\sigma_x^{(1)}\sigma_y^{(2)}xy + \cdots}{r^2}$$

$$\to \frac{1}{3}(\sigma^{(1)}\sigma^{(2)})$$

$$\left\{\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2}\right\}^2 = \left(\frac{\sigma_x^{(1)}\sigma_x^{(2)}\cdot x^2 + \sigma_y^{(1)}\sigma_y^{(2)}y^2 + \sigma_z^{(1)}\sigma_z^{(2)}z^2}{r^2}\right)^2$$
$$+ \left\{\frac{(\sigma_x^{(1)}\sigma_y^{(2)}xy + \sigma_y^{(1)}\sigma_z^{(2)}yz + \cdots)^2}{r^2}\right\} + \cdots$$

$$= \sigma_x^{(1)2}\sigma_x^{(2)2}\frac{x^4}{r^4} + 2\sigma_x^{(1)2}\sigma_y^{(2)}$$

$$S_{12}^2 = 9\left\{\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2}\right\}^2 - 6(\sigma^{(1)}\sigma^{(2)})\left(\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2}\right)$$
$$+ (\sigma^{(1)}\sigma^{(2)})^2$$
$$= 9\left\{\frac{(\sigma_1 r)^2(\sigma_2 r)^2}{r^4}\right\} = 9 - 6(\sigma^{(1)}\sigma^{(2)})\left(\frac{(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2}\right)$$
$$= 9 - (\sigma^{(1)}\sigma^{(2)})^2 - 2(\sigma^{(1)}\sigma^{(2)})S_{12}$$
$$= 8 - 2S_{12} \text{ for triplet states}$$

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$$\oint \chi_1^{m} S_{12}^2 \chi_1^{m} \sin\theta d\theta d\varphi$$

$$= 8\cdot(4\pi)$$

$$\left\{-\frac{\hbar^2}{M}\Delta - (1+\gamma S_{12})\right\}\psi = E\psi$$

$$\psi = \frac{1}{\sqrt{4\pi}}\left\{\frac{u(r)}{r} + \frac{1}{\sqrt{8}}\frac{w(r)}{r} S_{12}\right\}\chi_1^{m}$$

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}(E+J)u + \gamma\frac{M}{\hbar^2} J S_{12} u$$

$$+ \frac{S_{12}}{\sqrt{8}}\left[\frac{d^2w}{dr^2} + \frac{6w}{r^2} + \frac{M}{\hbar^2}(E+J)w\right.$$

$$\left. + \gamma S_{12} J w\right\} = 0 \quad *$$

or

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}(E+J)u = -\sqrt{8}\gamma\frac{M}{\hbar^2} w$$

$$\frac{d^2w}{dr^2} = \frac{6w}{r^2} + \frac{M}{\hbar^2}[E + (1-2\gamma)J]w$$

$$= -\sqrt{8}\gamma\frac{M}{\hbar^2} J u$$

$$* \quad \Delta\left\{\frac{w(r)}{r} S_{12}\chi_1^{m}\right\} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{w(r)}{r}\right) * S_{12}\chi_1^{m}$$

$$- 6\frac{w(r)}{r}\frac{S_{12}\chi_1^{m}}{r^2}$$

Quadrupole moment for $m = +1$.

$$Q = \frac{1}{4}\oint (3z^2 - r^2)\bar{\psi}\psi\, r^2 dr \sin\theta d\theta d\varphi$$

$$= \frac{1}{10}\int r^2 dr\left(uv\sqrt{2} - \frac{1}{2}v^2\right)$$

(lecture course Meson Theory)

D. G.

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Definition of moments:

electric dipole moment of two nucleon system:

$$\mathbf{P} = e \sum_{i=1}^{2} \int \tilde{\psi} \frac{1+\tau_3^{(i)}}{2} (r_k^{(i)} - R) \psi \, dV_1 dV_2$$

$$R = \frac{r^{(1)} + r^{(2)}}{2}$$

$$= e \int \tilde{\psi} \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} \frac{(r^{(1)} - r^{(2)})}{2} \psi \, dV_1 dV_2$$

$$= e \int \tilde{\psi} \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} r \psi \, dV,$$

el. quadr. moment

$$Q_{jk} = e \sum_{i=1}^{2} \int \tilde{\psi}(r^{(1)}, r^{(2)} \ldots) \frac{1+\tau_3^{(i)}}{2} \Big(\frac{3}{2} \{ (x_j^{(i)} - X_j)(x_k^{(i)} - X_k) \} - \sum_l (x_k^{(i)} - X)^2 \delta_{jk} \Big)$$

$$\times \psi(r^{(1)}, r^{(2)}) \, dV_1 dV_2$$

$$= e \sum_i \int \sum \tilde{\psi}(R, r) \frac{1+\tau_3^{(i)}}{2} \frac{1}{4} \begin{pmatrix} 3x^2 - r^2 & 3xy & \cdot \\ 3xy & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\psi(R, r) \, dV dV$$

$$= e \int \tilde{\psi}(R, r) \frac{1}{4} \begin{pmatrix} 3x^2 - r^2 & 3xy & \cdot \\ 3xy & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \psi(R, r)$$

$$\times dV dV$$

for the deuteron.

mag. moment

$$M = \mu_p (g_N \sigma_N + g_p \sigma_p + \tfrac{1}{2} L)$$

$$m = \tfrac{e}{2} L = \tfrac{e}{2} [r \times p]$$

$$\therefore \frac{1+\tau_3^{(1)}}{2} \left[\left(r^{(1)} - \frac{r^{(1)} + r^{(2)}}{2} \right) \times p^{(1)} \right] + \frac{1+\tau_3^{(2)}}{2} \left[\left(r^{(2)} - \frac{}{2} \right) \right.$$

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Rarita and Schwinger assumed a rectangular potential

$$J(r) = +V_0 \quad \text{for } r_0 < r_0$$

$r > r_0$: $u = A e^{-\alpha(r-r_0)}$ $\quad \alpha = \sqrt{M\mathcal{E}}/\hbar$

~~$r < r_0$~~

$r > r_0$: $w = \beta e^{-\alpha(r-r_0)}\left(1 + \frac{3}{\alpha r_0} + \frac{3}{\alpha^2 r^2}\right)$

($w = e^{-\alpha r} \sum_{n=0}^{\infty} a_n r^{n+\nu}$

$$\frac{dw}{dr} = -\alpha e^{-\alpha r}\sum_{n=0}^{\infty} a_n r^{n+\nu} + e^{-\alpha r}\sum_n (n+\nu) a_n r^{n+\nu-1}$$

$$\frac{d^2w}{dr^2} = \alpha^2 e^{-\alpha r}\sum_{n=0}^{\infty} a_n r^{n+\nu} - 2\alpha e^{-\alpha r}\sum_{n=0}^{\infty}(n+\nu) a_n r^{n+\nu-1} + e^{-\alpha r}\sum_{n=0}^{\infty}(n+\nu)(n+\nu-1) a_n r^{n+\nu-2}$$

$$\frac{d^2w}{dr^2} + \frac{6w}{r^2} - \alpha^2 w = 0$$

$r^{\nu-2}$: $\quad \nu(\nu-1)a_0 - 6a_0 = 0$

$\nu = -2$ (or $+3$.)

$r^{\nu-1}$: $\quad (\nu+1)\nu \cdot a_{n+1} - 2\alpha\nu a_0 - 6a_1 = 0$

$\nu = -2$: $a_1 = +\alpha a_0$ $\quad (\nu = 3 \quad a_1 = \alpha a_0)$

r^{ν}: $\quad \alpha^2 a_0 - 2\alpha(\nu+1)a_1 + (\nu+1)(\nu+2)a_2 - 6a_2 - \alpha^2 a_0 = 0$

$\nu = -2$; $\quad a_2 = \frac{1}{3}\alpha a_1$

($\nu = 3$; $\quad a_2 \neq 0$)

$r^{\nu+1}$: $\nu = -2$: $\quad a_3 = 0$

($\nu = 3 \quad a_3 \neq 0$)

($\nu \gg 1$: $\nu = 3$: $\quad a_{\nu+1} \sim \frac{2}{\nu} a_\nu$)

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$r \lesssim r_0$:

$$\left(\frac{d^2}{dr^2} + \varkappa^2\right) u(r) = -\lambda^2 w(r)$$

$$\left(\frac{d^2}{dr^2} - \frac{6}{r^2} + \varkappa'^2\right) w(r) = -\lambda^2 u(r)$$

$$\varkappa^2 = \frac{M}{\hbar^2}(V_0 - E)$$

$$\varkappa'^2 = \frac{M}{\hbar^2}((1-2\gamma)V_0 - E)$$

$$\lambda^2 = \sqrt{8}\cdot\gamma\frac{M}{\hbar^2}V_0$$

$$u(r) = \sum_0^\infty A_n x^{n+1} + \log x \sum_0^\infty C_n x^{n+2}$$

$$w(r) = \sum_0^\infty B_n x^{n+3} + \log x \sum_0^\infty D_n x^{n+3}$$

$$x = r/r_0$$

$$\left(\frac{r_0}{u}\frac{du}{dr}\right)_{r\lesssim r_0} = -\alpha r_0$$

$$\left(\frac{r_0}{w}\frac{dw}{dr}\right)_{r=r_0-} = -\left(2 + \frac{(\alpha r_0)^2(1+\alpha r_0)}{(\alpha r_0)^2 + 3\alpha r_0 + 3}\right)$$

These conditions determine B_0/A_0 and V_0 for a given γ.

Normalization condition:

$$\int_0^{r_0}(u^2+w^2)\,dr + \frac{A^2}{2\alpha} + \frac{B^2}{2\alpha}\left(1 + 6\frac{(1+\alpha r_0)^2}{(\alpha r_0)^3}\right) = 1$$

In this way, one obtains a relation between V_0 and γ.

Problem: Prove that

$$Q = \frac{1}{4}\int \tilde{\psi}(3z^2 - r^2)\psi$$

$$= \frac{1}{10}\int r^2(\sqrt{2}uw - \frac{1}{2}w^2)\,dr$$

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$$\psi = \frac{1}{\sqrt{4\pi}}\left\{\frac{u(r)}{r} + \frac{1}{\sqrt{8}}\frac{w(r)}{r}S_{12}\right\}\chi_1^m$$

$m=1$: $\chi_1' = \alpha(1)\alpha(2)$

$$S_{12}\chi_1' = \frac{3}{r^2}\{(x+iy)\beta(1) + z\alpha(1)\}\{(x+iy)\beta(2) + z\alpha(2)\} - \alpha(1)\alpha(2)$$

$x+iy = r\sin\theta e^{i\varphi}$ $z = r\cos\theta$

$$S_{12}\chi_1' = 3\{\sin\theta e^{i\varphi}\beta(1) + \cos\theta\alpha(1)\}\times\{\sin\theta e^{i\varphi}\beta(2) + \cos\theta\alpha(2)\} - \alpha(1)\alpha(2)$$

$$= 3\sin^2\theta e^{2i\varphi}\beta(1)\beta(2) + 3\sin\theta\cos\theta e^{i\varphi}(\alpha(1)\beta(2) + \beta(1)\alpha(2)) + (3\cos^2\theta - 1)\alpha(1)\alpha(2)$$

$$\psi = \frac{1}{\sqrt{4\pi}}\frac{1}{r}\left\{\left(u(r) + \frac{(3\cos^2\theta-1)}{\sqrt{8}}w(r)\right)\alpha(1)\alpha(2) + \frac{3w}{\sqrt{8}}\sin\theta\cos\theta e^{i\varphi}(\alpha(1)\beta(2) + \beta(1)\alpha(2)) + \frac{3w}{\sqrt{8}}\sin^2\theta e^{2i\varphi}\beta(1)\beta(2)\right\}$$

$$\sum_{sm}\tilde{\psi}\psi = \frac{1}{4\pi}\frac{1}{r^2}\left\{\left|u(r) + \frac{(3\cos^2\theta-1)}{\sqrt{8}}w(r)\right|^2 + \frac{9}{4}|w|^2\sin^2\theta\cos^2\theta + \frac{9|w|^2}{8}\sin^4\theta\right\}$$

$$Q = \iiint \frac{1}{4\pi}\left[(3\cos^2\theta-1)u^2 + \frac{(3\cos^2\theta-1)^2}{\sqrt{2}}uw + \left\{\frac{(3\cos^2\theta-1)^3}{8} + \frac{9}{4}\sin^2\theta\cos^2\theta(3\cos^2\theta-1) + \frac{9}{8}\sin^4\theta(3\cos^2\theta-1)\right\}w^2\right] r^2 dr\sin\theta d\theta d\varphi$$

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$$Q = \frac{1}{8}\int_0^{\infty}\int_{-1}^{+1}\left\{\frac{(3x^2-1)^2}{\sqrt{2}}uw + \frac{(3x^2-1)(3x^2-5)}{4}w^2\right\} \times r^2\,dr\,dx$$

$$= \frac{1}{8}\left[\frac{1}{\sqrt{2}}\frac{8}{5}\int uw\,r^2dr + \frac{2}{5}\int w^2r^2dr\right]$$

$$= \frac{1}{10}\left[\sqrt{2}\int uw\,r^2dr - \frac{1}{2}\int w^2r^2dr\right]$$

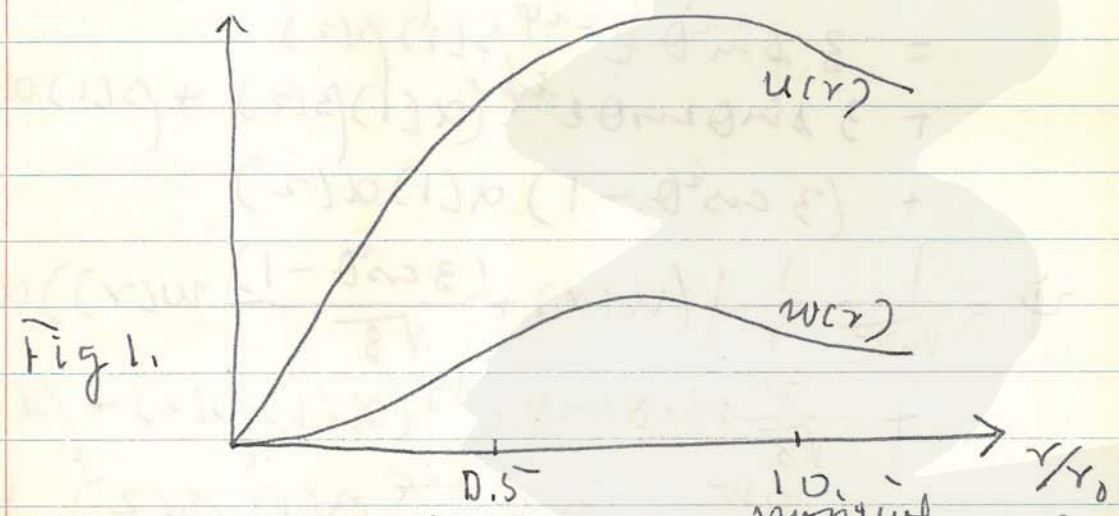


Fig 1.

In order to have the ~~correct~~ value of Q, γV_0 must be positive and in order to have correct value of Q, ~~on~~ a relation between γ and V_0 is to be satisfied.

Thus, one get a set of values of γ, V for a given value of r_0. For instance,

$r_0 = 2.8 \times 10^{-13}$ cm

$V_0 = 6.40$ Mc, $\gamma = 0.775$

and the S, D - radiations become as shown in ~~the diagr~~ Fig. 1.

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There is a test of the above assignment of the constants:
The magnetic moment of the deuteron is given by

$$\mathbf{M} = \frac{e\hbar}{2Mc}\left(g_p \boldsymbol{\sigma}_p + g_n \boldsymbol{\sigma}_n + \frac{1}{2}\mathbf{L}\right)$$

$\therefore$ $M_{orbital} = \frac{e\hbar}{2Mc}\left\{\frac{1+\tau_3^{(1)}}{2}\left[\left(\mathbf{r}^{(1)} - \frac{\mathbf{r}^{(1)}+\mathbf{r}^{(2)}}{2}\right)\times \mathbf{p}^{(1)}\right]\right.$

$\left. + \frac{1+\tau_3^{(2)}}{2}\left[\left(\mathbf{r}^{(2)} - \frac{\mathbf{r}^{(1)}+\mathbf{r}^{(2)}}{2}\right)\times \mathbf{p}^{(2)}\right]\right\}$

$\frac{\mathbf{r}^{(1)} - \mathbf{r}^{(2)}}{2}$

$-\frac{(\mathbf{r}^{(1)} - \mathbf{r}^{(2)})}{2}$

$\mathbf{p} = \frac{1}{2}(\mathbf{p}^{(1)} - \mathbf{p}^{(2)})$ $\quad \mathbf{p}^{(1)} + \mathbf{p}^{(2)} = 0$

For the deuteron $\tau_3^{(1)} + \tau_3^{(2)} = 0$, so that

$$M_{orbital} = \frac{e}{2Mc}\left\{\frac{1}{2}[\mathbf{r}\times\mathbf{p}]\right.$$

$$= \frac{e\hbar}{2Mc}\,\frac{1}{2}\mathbf{L}$$

$$\mathbf{M} = \frac{e\hbar}{2Mc}\left\{\frac{1}{2}\mathbf{J} - \frac{1}{2}\mathbf{S} + \frac{1}{2}(g_p+g_n)\mathbf{S} + \frac{1}{2}(g_p-g_n)\right.$$
$$\left.\times(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n)\right\} \qquad \mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)$$

$$= \frac{e\hbar}{2Mc}\left\{\frac{1}{2}\left\{\mathbf{J} + (2(g_p+g_n)-1)\mathbf{S}\right.\right.$$
$$\left.\left. + \frac{1}{2}(g_p-g_n)(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n)\right\}\right.$$

For a triplet state,

$$(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n)\,\Psi = 0,$$

$$(\sigma_x^{(1)} - \sigma_x^{(2)})\,\alpha(1)\alpha(2) = \beta(1)\alpha(2) - \alpha(1)\beta(2)$$

Thus, triplet-triplet matrix elements of (is zero) $\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n$ are all zero, so that the expectation value of $\mathbf{S}$ can be replaced by its component in the direction of $\mathbf{J}$, since $\mathbf{S}$ precess rapidly around $\mathbf{J}$-direction:

$$\mathbf{S} \to \mathbf{J}\,\frac{\mathbf{S}\cdot\mathbf{J}}{J^2} = \mathbf{J}\,\frac{J^2+S^2-L^2}{2J^2} = \mathbf{J}\,\frac{4-L^2}{4}$$

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$$M = \frac{e\hbar}{2Mc} g_d J = \frac{e\hbar}{2Mc} J \left(g_p + g_n - \frac{1}{4}(g_p + g_n - 1)L^2 \right)$$

$$\left(\because \ \frac{1}{2}\hbar J \left\{ 1 + 2(g_p + g_n)\frac{4 - L^2}{4} - \frac{4 - L^2}{4} \right\} \right.$$

$$= J \left\{ (g_p + g_n) - \frac{1}{4}(g_p + g_n)L^2 + \frac{1}{8}L^2 \right\} \Bigg)$$

3D; $L^2 = 6$

$$g_d = g_p + g_n - \frac{3}{2}(g_p + g_n - 1)\int_0^\infty w^2 dr$$

From the known values of g_p, g_n, g_d,
($g_p = 2.785 \pm 0.02$, $g_n = -1.908 \pm 0.02$
$g_d = 0.855 \pm 0.006$)
we obtain $\int w^2 dr = 0.039$.
a very small value, which is consistent with the figure

$$\int_0^a w^2 dr \simeq 0.039,$$

the latter being obtained from the wave functions in figure 1.

Present values of the magnetic moments
(Weisskopf and Blatt, Theoretical Nuclear Physics. 1952)

$\mu_p = \frac{e\hbar}{2M_p c} \cdot (2.7934 \pm 0.0003)$

$\mu_n = \quad " \quad (-1.9135 \pm 0.0003)$

$\mu_d =$

$p_D = \int_D w^2 dr \sim 0.04$

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In order to have complete agreement with all ~~various~~ of the experiments including proton-~~p~~ neutron scattering and proton-proton scattering at low energies, one must start from a more general type of force, which have different ranges and depths for central and ~~tensor~~ Tensor forces. If we further assume the charge independence and the shape of yukawa potential $e^{-\kappa r}/r$ both for central and Tensor forces, the proper choice is

$$V = -22.7 \frac{\exp(-2.12 r/b_c)}{r/b_c} - 10.9 \frac{\exp(-2.12 r/b_\pi)}{r/b_\pi} S_{12}$$

where the central force is the same for all cases $n-p$, $p-p$, $n-p$ in singlet states (with additional Tensor forces) and for triplet states. (spin independence of central forces)

$b_c = 2.47 \times 10^{-13}$ cm

$b_t = 3.68 \times 10^{-13}$ cm

$\int w^2 dr = 0.033$

(Platt and Weisskopf, p. 103.)

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how energy scattering between two nucleons. (Blatt-Weisskopf, p. 56)

For low energy of the order of 10 MeV or less, the effect of tensor forces is negligible; and for the n-p scattering we look for the solution

$$\psi(r) \simeq e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

asymptotically with

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{\sqrt{mE}}{\hbar}.$$

One can further restrict ourselves to S-scattering:

$$\psi(r) \simeq \sum_l \psi_l(r) Y_{l,0}(\theta) P_l(\cos\theta)$$

$$\psi_0(r) = \sqrt{4\pi}\int Y_{00}(\theta)\psi(r,\theta)\sin\theta\, d\theta\, d\varphi,$$

$$Y_{00}(\theta) = \frac{1}{\sqrt{4\pi}}$$

asymptotically

$$\simeq \frac{1}{\sqrt{4\pi}}\left\{\int e^{ikr\cos\theta}\sin\theta\, d\theta\, d\varphi + \sqrt{4\pi}\frac{e^{ikr}}{r}\bar{f}\right\}$$

$$\bar{f} = \frac{1}{4\pi}\int f(\theta)\sin\theta\, d\theta\, d\varphi$$

$$\psi_0(r) \simeq \frac{1}{\sqrt{4\pi}}\sqrt{4\pi}\frac{e^{ikr} - e^{-ikr}}{2ikr} + \sqrt{4\pi}\frac{e^{ikr}}{r}\bar{f}$$

$$= \frac{1}{\sqrt{4\pi}}\sqrt{4\pi}\frac{\sin kr}{kr} + \sqrt{4\pi}\frac{e^{ikr}}{r}\bar{f}$$

$$= \frac{\sqrt{4\pi}}{\sqrt{4\pi}}\cdot\frac{1}{2ikr}\left\{\left(1 + 2ik\bar{f}\right)e^{ikr} - e^{-ikr}\right\}$$

$$k\bar{f} = e^{i\delta}\sin\delta = \frac{1}{2i}\left\{e^{2i\delta} - 1\right\}$$

$$\psi_0(r) \simeq \frac{e^{i\delta}}{2ikr}\left\{e^{i(kr+\delta)} - e^{-i(kr+\delta)}\right\}$$

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$$\psi_0(r) \cong \frac{e^{i\delta}}{kr}\sin(kr+\delta) \Rightarrow \frac{e^{i\delta}\sin\delta}{kr}v(r)$$

$v(0)=1$

$$v(r)=\frac{\sin(kr+\delta)}{\sin\delta}$$

$$d\sigma = |f(\theta)|^2 \sin\theta\, d\theta\, d\varphi$$

$$= \frac{|f|^2}{k^2} = \frac{\sin^2\delta}{k}\, d\Omega$$

$$\psi(r) = \psi_0(r) = \frac{u(r)}{r}$$

$$u_0 : \left(-\frac{\hbar^2}{M}\right)\frac{d^2u}{dr^2} + V(r)u = E u$$

For $E=0$, $u = u_0$:

$$-u : \left(-\frac{\hbar^2}{M}\right)\frac{d^2u_0}{dr^2} + V(r)u_0 = 0$$

$$\frac{d}{dr}\left(u\frac{du_0}{dr} - u_0\frac{du}{dr}\right) = k^2 u u_0$$

Now, if the corresponding asymptotic forms $u(r) \cong v$ be

$$u(r) \to v(r) \quad ; \quad u_0(r) \to v_0(r),$$

we have

$$\frac{d}{dr}\left(v\frac{dv_0}{dr} - v_0\frac{dv}{dr}\right) = k^2 v v_0$$

or

$$\left|u\frac{du_0}{dr} - u_0\frac{du}{dr}\right|_0^\infty - \left|v\frac{dv_0}{dr} - v_0\frac{dv}{dr}\right|_0^\infty = \hbar^2 \int \frac{(uu_0 - vv_0)}{dr}$$

However, $u(0)=0$ $\quad u_0(0)\simeq 0$.

$v(\infty)\simeq u(\infty)$, $\quad v_0(\infty)\simeq v(\infty)$

$v(0)\simeq$ |, $\quad \left.\frac{dv}{dr}\right|_{r=0} = k\cot\delta$

$v_0(0)\simeq 1$, $\quad \left.\frac{dv_0}{dr}\right|_{r=0} = -\frac{1}{a}$.

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$$k\cot\delta \simeq -\frac{1}{a} + k^2 \int_0^\infty (vv_0 - uu_0)\, dr$$

v and u and v_0 and u_0 differ from each other, only inside the range of nuclear forces, where, however, the potential is large so that there is little difference between v and v_0 and u and u_0. Thus, we have an approximate relation

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2$$

where

$$r_0 = 2\int_0^\infty (v_0^2 - u_0^2)\, dr,$$

which is independent of the energy or k. (shape independent approximation)

r_0 : effective range

(because $v_0^2 - u_0^2$ has of the order of one inside the range of forces)

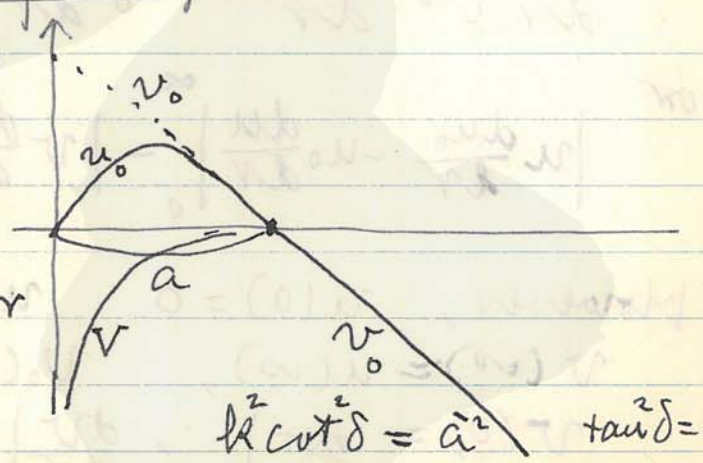
a : scattering length

$$v_0(r) = \frac{\sin kr\cos\delta}{\sin\delta} + \frac{\cos kr \cdot \sin\delta}{\sin\delta}$$

kr very small

$$\simeq v_0(r) \simeq 1 + (k\cot\delta)_0 r$$

$$= 1 - \frac{1}{a} r$$



$$k^2\cot^2\delta = \frac{1}{a^2} \qquad \tan^2\delta = k^2 a^2 \ll 1.$$

$$\sigma_0 = \frac{4\pi}{k^2}(\sin^2\delta)_0 = 4\pi a^2$$

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In the case of p-p scattering, we have to take into account the Coulomb repulsion in addition to the nuclear forces.

$$K' = -a^{-1} + \frac{1}{2} r_0 k^2 + \cdots$$

where

$$K' \equiv C^2 k \cot \delta + D^{-1} h(\eta)$$

$$C^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \qquad \eta = \frac{e^2}{\hbar v}$$

$$D = \frac{\hbar^2}{M e^2} = 2.88 \times 10^{-13} \text{ cm}$$

$$h(\eta) = \mathrm{Re}\, \Psi(-i\eta) - \log \eta$$

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M.N.1

General Many Nucleon system

$$\Psi(x^{(1)}\sigma_z^{(1)}\tau_3^{(1)};\ \cdots\ x^{(A)}\sigma_z^{(A)}\tau_3^{(A)})$$

$$H = \sum_{k=1}^{A} \frac{1}{2M}(p^{(k)})^2 + \sum_{j<k=1}^{A} V_{jk}$$

Total angular momentum can defined in connection with the ~~rotation~~ infinitesimal rotation operators

$$\mathbf{m} = \sum_{k=1}^{A}\left(\mathbf{m}^{(k)} + \frac{\hbar}{2}\sigma^{(k)}\right)$$

in the coordinate system in which the centre of mass is at rest. (the difference between the nucleon mass and the reduced mass is small ~~for~~ except for very light nuclei with small A)

Remark:

$$\mathbf{P} = \frac{1}{A}\sum_k p^{(k)}$$

$$\pi^{(1)} = p^{(1)} - p^{(2)}$$

$$\pi^{(2)} = p^{(2)} - p^{(3)}$$

$$\vdots$$

$$\pi^{(A-1)} = p^{(A-1)} - p^{(A)}$$

$$\left.\begin{aligned}
&\pi^{(1)} + \pi^{(2)} = p^{(1)} - p^{(3)}\\
&\pi^{(1)} + \pi^{(2)} + \pi^{(3)} = p^{(1)} - p^{(4)}\\
&\cdots\\
&\pi^{(1)} + \cdots + \pi^{(A-1)} = p^{(1)} - p^{(A)}
\end{aligned}\right\}$$

$$A^2\mathbf{P}^2 = \left(\sum_k p^{(k)}\right)^2 + (p^{(1)} - p^{(2)})^2 + (p^{(2)} - p^{(3)})^2 + \cdots$$

$$p^{(1)} = (A-1)p^{(1)} + (A-2)\pi^{(2)} + \cdots + \pi^{(A-1)}$$

$$= (A-1)p^{(1)} - (p^{(2)} + p^{(3)} + \cdots + p^{(A)})$$

$$= A p^{(1)} - A\mathbf{P}$$

$$p^{(1)} = \mathbf{P} + \frac{A-1}{A}\pi^{(1)} + \frac{A-2}{A}\pi^{(2)} + \cdots + \frac{A!}{A}\pi^{(A-1)}$$

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$$\mathcal{P} = \sum_{k=1}^{A} p^{(k)}$$

$$\pi^{(1)} = p^{(2)} - p^{(1)}$$

$$\vdots$$

$$\pi^{(A-1)} = p^{(A)} - p^{(1)}$$

$$\pi^{(1)} + \cdots + \pi^{(A-1)} = A p^{(1)} - \mathcal{P}$$

$$p^{(1)} = \frac{1}{A}\mathcal{P} - \frac{1}{A}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)$$

$$p^{(2)} = \pi^{(1)} + p^{(1)}$$

$$= \frac{1}{A}\mathcal{P} + \frac{A-1}{A}\pi^{(1)} - \frac{1}{A}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)$$

$$\vdots$$

$$p^{(A)} = \frac{1}{A}\mathcal{P} + \pi^{(A-1)} - \frac{1}{A}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)$$

$$\sum (p^{(k)})^2 = \frac{1}{A}\mathcal{P}^2 + \frac{A-1}{A^2}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)^2$$

$$+ \frac{2}{A}\pi^{(1)2} + \cdots + \left(\pi^{(A-1)}\right)^2$$

$$- \frac{2A}{A^2}\mathcal{P}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)$$

$$+ \frac{2}{A}\mathcal{P}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)$$

$$- \frac{2}{A}\left(\pi^{(1)} + \cdots + \pi^{(A-1)}\right)^2$$

$$= \frac{1}{A}\mathcal{P}^2 + \left(1 - \frac{1}{A}\right)\left(\pi^{(1)2} + \cdots + \pi^{(A-1)2}\right)$$

$$- \frac{2}{A}\sum_{\substack{j<k \\ 1}}^{A-1} \pi^{(j)}\pi^{(k)}$$

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Ph. N. 2

$\mathcal{M}$ is certainly commutative with each of V_{jk}, so that one can classify the stationary states of the nucleus by J:

$$\mathcal{M}^2 = \hbar^2 J(J+1).$$

J : integer for even A
half-integer for odd A.

Further, if one construct ~~the~~

$$\textcircled{T}_j = \frac{\hbar}{2} \sum_{k=1}^{A} \tau_j^{(k)},$$

the total isobaric spin vector, then if the nuclear forces are charge independent

$$[\tau^{(k)}\tau^{(k)}, T_j] = 0$$

$$\therefore [\tau^{(1)}\tau^{(2)}, \tau_j^{(1)} + \tau_j^{(2)}] = 0$$

$$\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)} + \tau_3^{(1)}\tau_3^{(2)}$$

Thus, the stationary states of a nucleus can be further classified by

$$\textcircled{T}^2 = \hbar^2 T(T+1)$$

or ~~spin~~ isobaric spin multiplet,

where T : integer for even A $(\leq \frac{A}{2})$
half-integer for odd A $(\leq \frac{A}{2})$

$$\textcircled{T}_3 = \hbar T_3$$

Different T_3 belonging to the same T discriminate between isobars with same A.

$A = 3$

$T = \frac{3}{2}$

He_3	$T_3 = \frac{1}{2}$	$z = 2$
H_3	$T_3 = -\frac{1}{2}$	$z = 1$
$(n_3$	$T_3 = -\frac{3}{2}$	$z = 0$)
$(Li_3$	$T_3 = \frac{3}{2}$	$z = 3$)

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$T = \frac{1}{2}$　　$T_3 = \frac{1}{2}$　　H^{B3}

$T_3 = -\frac{1}{2}$　　$H^{\not B 3}$

($T = \frac{3}{2}$; must be a very highly excited states.)

Individual Particle (Orbit) Model.

In order to simplify the problem, various models were proposed. Among them, the individual particle orbit model made a remarkable success, in spite of the fact that there are a number of doubts as to the validity of the model. There is a great difference between the nuclear structure and atomic structure:

(i) Density is larger and interaction is stronger.

(ii) Short range forces interacts between individual nucleons and there is no common centre of force in contrast to the atom.

If we are interested in the internal structure of the nucleus, we must first of all eliminate the motion of the centre of mass and take up only the relative motion. However, such a procedure is

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inconvenient except for the case $A=2$.
We had better imagine that we have chosen the coordinate system in which the centre of mass is at rest. This means that the Schrödinger fn must satisfy

$$P_j \Psi = 0 \qquad j=1,2,3 \qquad (1)$$

where

$$P_j = p_j = \sum_{k=1}^{A} p_j^{(k)}$$

this S.C. is compatible, of course, with the Schrödinger equation

$$H\Psi = E\Psi$$

The S.C. (1) or a more general S.C

$$P_j \Psi = P_j' \Psi \qquad (2)$$

can be more easily handled in momentum space than in the ordinary coordinate space. Namely

$$\Psi(x^{(k)} y^{(k)} z^{(k)}; \ldots) = \int e^{\Sigma i p_j^{(k)} x_j^{(k)}} \cdot \bar{\Psi}(p_x^{(k)}, p_y^{(k)}, p_z^{(k)}; \ldots) \times \prod_k (d p^{(k)})$$

$$\sum_k p_j^{(k)} \cdot \bar{\Psi}(p_j^{(k)}; \ldots) = P_j' \Psi$$

or

$$\bar{\Psi}(p_j^{(k)}; \ldots) = \Phi'(p_j^{(k)}; \ldots)\, \delta(\textstyle\sum_k p_x^{(k)})\, \delta(\sum_k p_y^{(k)})\, \delta(\sum_k p_z^{(k)})$$

,

$$H \Phi = E \Phi$$

$$H(p_j^{(k)}, q_j^{(kl)}, \ldots)$$

$$r^{(kl)} \to i\hbar\,(\mathrm{grad}_k - \mathrm{grad}_l) \equiv q^{(kl)}$$

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In the individual orbit approx.

$$\Phi' \propto \sum_r \varepsilon(P_r) P_r \prod_{k=1}^{A} \varphi_k(p_j^{(k)}, \ldots)$$

~~Then, Φ has also the form~~

~~$$\Phi = \varepsilon(P_r) P_r \prod_{k=1}^{A} \psi_k(x_j^{(k)} \ldots)$$~~

~~but the S.C. is automatically satisfied by the factor $\delta(\cdot,\cdot)$ in Φ,~~

since

$$\delta\Big(\sum_k p_j^{(k)} - P_j\Big) = \frac{1}{(2\pi\hbar)^3} \int e^{+i(\sum p_j^{(k)} - P_j')X_j/\hbar}\, dX,$$

$$\Phi = \frac{1}{(2\pi\hbar)^3} \int \Phi'(p_j^{(k)})\, e^{-i(\sum p_j^{(k)} - P_j')X_j/\hbar}\, dX$$

$$\Psi = \frac{1}{(2\pi\hbar)^3} \int \Phi'(p_j^{(k)})\, e^{\,i\sum (p_j^{(k)}(x_j^{(k)} - X_j)} (dp_j^{(k)}) \times e^{\,i P_j X_j}\, dX$$

$$= \frac{1}{(2\pi\hbar)^3} \int \Psi'(x_j^{(k)} - X_j;\ldots)\, e^{\,i P_j' X_j}\, dX$$

where

$$\Psi'(x_j^{(k)};\ldots) = \int \Phi'(p_j^{(k)})\, e^{\,i\sum p_j^{(k)} x_j^{(k)}} (dp_j^{(k)}).$$

Thus, if

$$\Phi' \propto \sum_r \varepsilon(P_r) P_r \prod_{k=1}^{A} \varphi_k(p_j^{(k)}, \ldots),$$

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H, N. 4

$$\Psi'(x_j^{(k)}\ldots) \propto \sum \varepsilon(P_r) P_r \prod_{k=1}^{A} \varphi_k(p_j^{(k)}\ldots),$$

where φ_k is the Fourier transform of ψ_k.
In other words, the solution of the type Ψ' above does not, in general, correspond to the state of the whole system, in which the centre of mass is at rest or moving with a constant velocity. However, one can construct from such a $\Psi'(x_j^{(k)}\ldots)$ the required function Ψ as shown above.

(Remark: Since the Hamiltonian is invariant with respect to the translation, $\Psi'(x_j^{(k)} - X_j)$ must be also the solution of the Schrödinger equation belonging to the same energy as $\Psi'(x_j^{(k)})$)

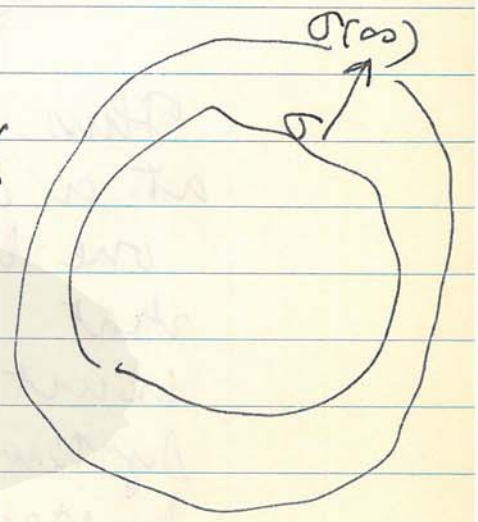
Proof:

$$-i\sum_k \frac{\partial \Psi}{\partial x_j^{(k)}} \Rightarrow i\int \sum_k \frac{\partial \Psi'(x-X)}{\partial x_j^{(k)}} e^{iP_j \cdot X_j} dX$$

$$= i\int \frac{\partial \Psi'}{\partial X} e^{iP_j \cdot X_j} dX$$

$$= i\left[\Psi'(x-X) e^{iP_j \cdot X_j}\right]_{\sigma(\infty)}$$

$$+ P_j \int \Psi' e^{iP_j \cdot X_j} dX$$



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thus, if $\Psi'(x_j^{(k)}\cdots)$ ~~were~~ was so ~~const~~ chosen that it was zero ~~at infinity for any~~ of coordinates $x_j^{(k)}$ when ~~one~~ becomes infinity, then Ψ' satisfies the supplementary condition. However, this is impossible ~~for the following obvious reason:~~

Actually, if

$$H\Psi'(x_j^{(k)}) = E\Psi'(x_j^{(k)}),$$

and if there is no degeneracy,

$$\Psi'(x_j^{(k)} - X_j) = \text{const.}\ \Psi(x_j^{(k)})$$

~~so~~ that Ψ' will not change, however, far one may go from the origin.

If there is degeneracy, ~~and~~

$$\Psi_n'(x_j^{(k)} - X_j) = \sum_m C_{nm}\Psi_m'(x_j^{(k)}),$$

but in this case, too, one can choose suitable set of eigenfunctions in such a way that

$$\Psi_n''(x_j^{(k)} - X_j) = \text{const.}\ \Psi_n''(x_j^{(k)}).$$

Thus, ~~again, one cannot~~ one arrive at a conclusion that one has to give up the assumption that Ψ' is a solution of the exact Hamiltonian H of the system.

Instead, one has to adopt the assumption

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(M, N, S)

that Ψ' is a solution of Schrödinger eq.

$$H'\Psi' = E'\Psi'$$

where H' is different from H and depends explicitly on $\sum x_j^{(k)}$, so that it is not strictly invariant under the translation. Then, we have the relation

$$H\Psi = (\cancel{H'+H''}\ H) \int \Psi'(x_j^{(k)} - X_j)\, e^{i p_j' X_j}\, dX$$

$$= \int H\Psi'(x_j^{(k)} - X_j)\, e^{i p_j' X_j}\, dX$$

$$= \int \{H'(p, x-X) + H''(p, x-X)\}\Psi'(x-X)\, e^{ip'X}\, dX$$

$$= E'\Psi + \int H''(p, x-X)\Psi'(x-X)\, e^{ip'X}\, dX$$

Thus, if H'' is sufficiently small, Ψ is an approximate solution of

$$H\Psi = E\Psi \quad (\text{with } E' \simeq E)$$

In any case, it is difficult to have a clear-cut picture of the whole problem. The elimination of centre of mass part is incompatible with the simple assumption of individual particle model.

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There was no such difficulty in the case of atoms, because the centre of mass was very close to the centre of force. always

Alternatively, one can say that ~~the~~ a solution of the ~~fo~~ determinant form of the individual cop particle model ~~or~~ results in a complicated motion of the centre of mass so that three conditions

(i') $H\Psi = E\Psi$

(ii) Ψ or Ψ' takes a determinant form.

(iii) S, C.

are not compatible with each other.

For a very large A, the nuclear dimension itself will be very large and H'' can be quite small compared with H'.

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M, W. 6

Shell Structure

In the individual orbit model, if we assume a common central potential $V(r)$, the energy levels for the individual particle can be classified

$s_{1/2}$, $p_{1/2}$, $d_{3/2}$ $f_{5/2}$ . .
$p_{3/2}$, $d_{5/2}$, $f_{7/2}$ . .

The relative position of these levels depends on the form of the potential.
(Blatt and Weisskopf, p. 279)

1. Screened Inverse Square Force

$$V = -\text{const}\,\frac{e^{-ar}}{r}$$

(1s), (2s), (2p), (3s), (3p), (3d), (4s) . . .

2. Oscillator model $V = \text{const}\cdot r^2$

(1s) (2p) ((3d), (2s)) ((4f), (3p))
((5g), (4d), (1s)) . . .

$E = h\nu\left(n_x + n_y + n_z + \frac{3}{2}\right)$ $\quad n = n_x + n_y + n_z$

$n_x = n_y = n_z$: (1s) $\quad n = 0$

$n_x = 1$, $n_y = n_z = 0$ etc : (2p) $\quad n = 1$

$n = 2$: $n_x = 2$, $n_y = n_z = 0$ etc (3)
$n_x = n_y = 1$ $n_z = 0$ etc (3) 6 levels.

$n_x = 2$, x^2 : d-states (5)
+ s-state (1)

$n = 3$: $n_x = 3$ $n_y = n_z = 0$ etc (3) f (7)
$n_x = 2$, $n_y = 1$, $n_z = 0$ etc (6) p (3)
$n_x = n_y = n_z = 1$ (1)

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3. Square Well:

$V = 0$ for $r < r_0$

$V = V_0$ for $r > r_0$ $\quad V_0 \to \infty$

The solution is given in terms of Bessell-function ($J(kr_0) = 0$)

(1s), (2p), (3d), (2s), (4f)

(3p), (5g), (4d), (3s), ...

Among them, the first case, which ~~was~~ is valid for atoms, can be applied to nuclear structure, because there is no concentrated source in the centre. The case 2 and 3 do not differ much from each other, only difference being that ~~there~~ degeneracy occurs in the case 2. The character difference between case 1 and 2.3 is that the ~~states~~ levels with higher angular momentum ~~lie~~ are much higher than those with lower ang. momentum, because the former is inhibited from the region near the centre where ~~the attract~~ the potential V is very deep, while ~~for~~ it decreases very rapidly with the distance.

Let us now take the case 2, because the potential is smoother and the exact solution of the problem is much simpler.

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M, N, Z

If we ignore the Coulomb repulsion between protons, we have the same scheme for the energy levels for protons and neutrons, on account of the charge symmetry of nuclear forces. Thus, for a nucleus (A, Z) ~~with A and Z~~ or (N, Z) where N = A - Z, N-lowest neutron states are occupied each by one neutron according to the exclusion principle. Similarly Z-lowest states are occupied by protons. The closed shell will be form by neutrons and protons separately for

N and Z: 2, 2+6=8, 8+12=20

20+20=40, 70, 112

However, these numbers do not agree with famous magic numbers *

2, 8, (14), 20, 28, 50, 82, 126,

except ~~that~~ 2 and the first three.

He^4_2 Z = N = 2

O^{16}_8 Z = N = 8

are well known very stable nuclei.

Mayer, Haxel, Jensen and Suess * tried to explain this in terms of spin-orbit coupling as follows.

---

* M. G. Mayer, P. R. 78 (1950), 16, 22.
Haxel, Jensen and Suess. 75 (1949); 1766.
Blatt and Weisskopf, p. 761.

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If we assume spin-orbit coupling of appropriate sign such that the levels with $j = l + \frac{1}{2}$ are lower than those with $j = l - \frac{1}{2}$ for the same l. More precisely, the former are lowered and the latter are raised.

Let us take for definiteness sake, the oscillator model:

n			Nor. of states	shell	Total nor.
0	1s	$1s_{1/2}$	2	2	2
1	1p	$1p_{3/2}$	4	6	8
		$1p_{1/2}$	2		
2	1d	$1d_{5/2}$	6	12	20
		$1d_{3/2}$	4		
	2s	$2s_{1/2}$	2		
3	1f	$1f_{7/2}$	8	8	28
		$1f_{5/2}$	6		
	2p	$2p_{3/2}$	4	22	50
		$2p_{1/2}$	2		
4	1g	$1g_{9/2}$	10	32	82
		$1g_{7/2}$	8		
	2d	$2d_{5/2}$	6		
		$2d_{3/2}$	4		
	3s	$3s_{1/2}$	2		
5	1h	$1h_{11/2}$	12	44	126
		$1h_{9/2}$			
	2f	$2f_{7/2}$			
		$2f_{5/2}$			
	3p	$3p_{3/2}$	$3p_{1/2}$	$1i_{13/2}$	

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M.N. 8

It is not easy to understand the so-called spin-orbit coupling because it refers to the interaction between the orbital motion of a particle and its own spin. Some of the meson theories simply such a force, (Rosenfeld,) but these are not the one which is commonly used.

Mayer made further assumptions such as

a. Pairs of spin levels within a shell tend to make the levels closer; ($d_{3/2}$ and $s_{1/2}$; $f_{5/2}$, $p_{3/2}$;) so that ~~these~~ order of these levels may be reversed.
(However, this situation can be understood by considering the effect of perturbation on degenerated states, ~~and may be regarded as forbidden~~ is already included in the fundamental assumption)

b. Even number of ~~identical nucleons~~ protons or neutrons in any orbit in any j couple to give a spin zero ($J=0$) and no contribution to the magnetic moment.

c. Odd number of identical protons or neutrons with the same j couple to give a total spin $J=j$ and the

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magnetic moment equal to that of a single particle in that state.

d. For a given nucleus, the pairing energy of nucleons in the same orbit is greater for orbits with larger j. (Example: If $3s_{1/2}$ is slightly lower than $h_{11/2}$, but the pairing energy of $h_{11/2}$ may be larger, so that $J=1/2$ may appear in the odd nuclei rather than $J=11/2$.)

Major test of the above model is the spins and magnetic moments of nuclei with odd A. The magnetic moment of an odd A nucleus can be computed just as in atoms* and the theoretical values are plotted as Schmidt lines. There are two possible cases $l=j-\frac{1}{2}$, $j+\frac{1}{2}$, so that Schmidt lines are given by

$l=j-\frac{1}{2}$: $\mu_l = g_l l + g_s$ for $j = l+\frac{1}{2}$

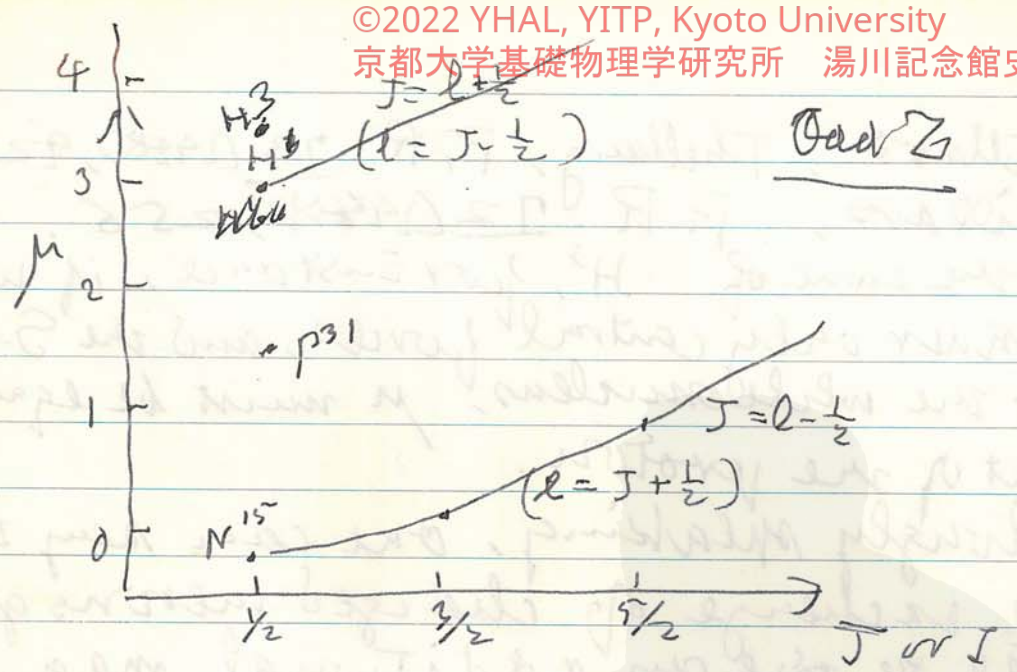
$l=j+\frac{1}{2}$: $\mu_l = g_l \frac{(l+1)(2l-1)}{2l+1} - g_s \frac{2l-1}{2l+1}$ $j = l-\frac{1}{2}$

$g_l = 1$ or 0 $\quad g_s = 2.85$ or -1.91

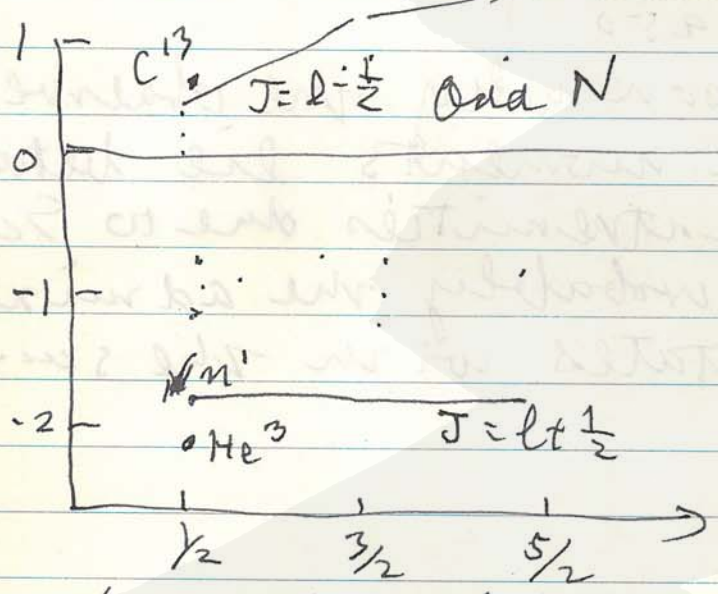
* We have to take into account the fact that the intrinsic magnetic moments of protons and neutrons and the orbital m. m. of the proton.

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M, N. 9



Wrathall, P. R. 75 (1949), 1894



E. Segrè, Experimental Nuclear Physics, I, 449.
H. I. Poss, Properties of Atomic Nuclei (1949), Brookhaven

- H^3 and He^3 (and C^{13}) are outside the Schmidt lines. However, in the cases of H^3, He^3, one can find an interpretation in terms of the exchange of mesons between nucleons.

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Villars, Thelling, P. R. 73 (1948), 924
Villars, P. R. 72 (1948), 256.
In the case of H^3, for instance, if we consider only central forces, and the S-state for the whole nucleus, μ must be equal to that of the proton.

Roughly speaking, one can say that the exchange of charged mesons give rise to an additional mag. moment due to the meson current. (Sachs, 1950)

The reason why the observed magnetic moments lie between the two extremities due to Schmidt lines is probably the admixture of other states with the same J.

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There are only very few odd-odd nuclei. All of them are very light:

H_1^2 Li_3^6 B_5^{10}, N_7^{14}.

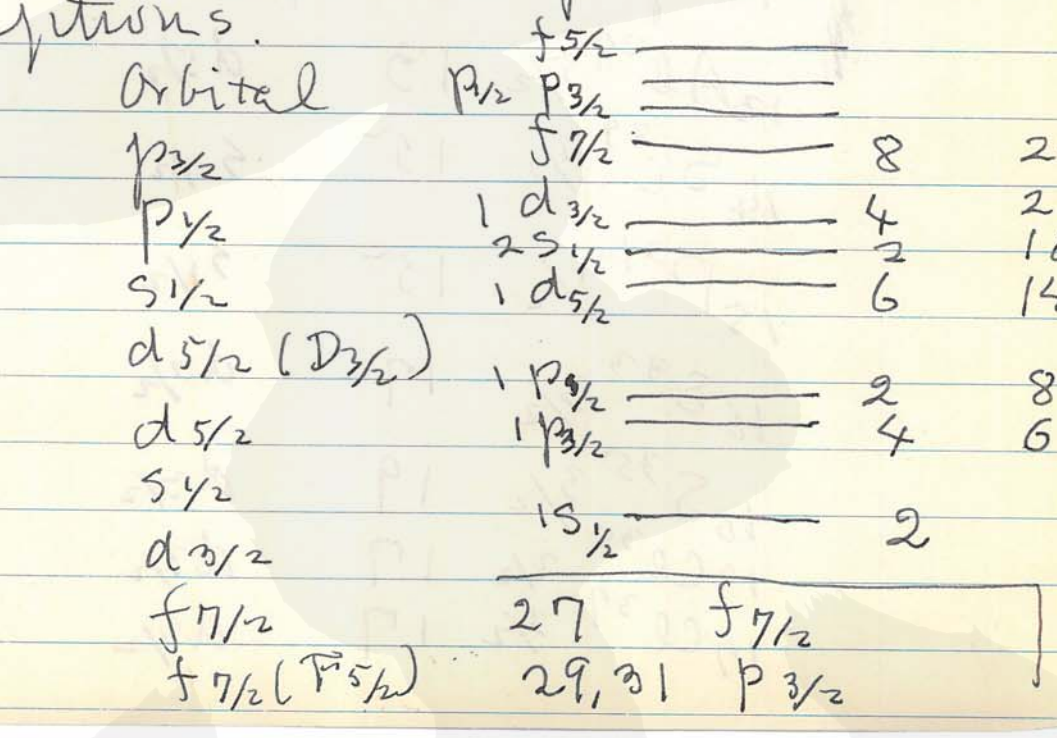
This shows that there is a strong tendency for nuclei to have a pair of neutrons and protons, in addition to the fact that the spin (and magnetic moment) is always zero for even-even nuclei.

Nuclear orbitals as functions of N and Z

(Rev. Mod. Phys. 23 (1951), 315, 322
I. Mayer and Moszkowski;
II. Nordheim)

For a nucleus with odd A, the state of the whole nucleus is determined by the orbital of the unpaired proton or neutron according to the above assumptions.

N or Z	Orbital
3, 5	$p_{3/2}$
7	$p_{1/2}$
9	$s_{1/2}$
11	$d_{5/2}$ ($D_{3/2}$)
13	$d_{5/2}$
15	$s_{1/2}$
17, 19	$d_{3/2}$
21, 23	$f_{7/2}$
25	$f_{7/2}$ ($F_{5/2}$)
27	$f_{7/2}$
29, 31	$p_{3/2}$



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Odd A Nuclei

Let us compare it with the observed spins of nuclei with odd A.

	(most stable)	N or P	state	μ theo. (Schmidt)	μ exp.
$_0n^1$	1/2				
$_1H^1$	1/2				
$_1H^3$	1/2	1	$S_{1/2}$	2.79	2.98
$_2He^3$	(1/2)	1	$S_{1/2}$	-1.91	-2.13
$_3Li^7$	3/2	3	$P_{3/2}$	3.79	3.25
$_4Be^9$	(3/2)	5	$P_{3/2}$	-1.91	-1.18
$_5B^{11}$	3/2	5	$P_{3/2}$	3.79	2.69
$_6C^{13}$	1/2	7	$P_{1/2}$	+0.64 ?	0.70
$_8O^{17}$	1/2	9	$S_{1/2}$		
$_9F^{19}$	1/2	9	$S_{1/2}$	2.79	2.62
$_{11}Na^{23}$	3/2	11	$d_{5/2}$!!!	0.12 !!!	2.21
$_{12}Mg^{25}$	5/2	13	$d_{5/2}$		
$_{13}Al^{27}$	5/2	13	$d_{5/2}$	4.79	3.64
$_{14}Si^{29}$	1/2	15	$S_{1/2}$		
$_{15}P^{31}$	1/2	15	$S_{1/2}$	2.79	1.13
$_{16}S^{33}$	3/2	17	$d_{3/2}$		
$_{16}S^{35}$	3/2	19	$d_{3/2}$	0.12 ?	0.82
$_{17}Cl^{35}$	3/2	17	$d_{3/2}$	~~3.79~~	
$_{17}Cl^{37}$	3/2	17	$d_{3/2}$	0.12 ?	0.68

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M, N. 11

$_{19}K^{39}$	3/2	19	$d_{3/2}$
$_{19}K^{41}$	3/2	19	$d_{3/2}$
$_{21}Sc^{45}$	7/2	21	$f_{7/2}$
$_{23}V^{51}$	7/2	23	$f_{7/2}$
○ $_{25}Mn^{55}$	5/2	25	$f_{7/2}$?
$_{26}Fe^{57}$			
$_{27}Co^{59}$	7/2	27	$f_{7/2}$
$_{29}Cu^{63}$	3/2	29	$p_{3/2}$
$_{29}Cu^{65}$	3/2	29	$p_{3/2}$

Thus, except for a very few exceptions, the ~~pred~~ above simple model predicts ~~the ground state~~ the spins of the nuclei in their ground states in a surprising manner in the case of odd A.

So, it is interesting to see whether this is true for nuclei which are unstable with respect to β-decay.

Now the selection rules for the β-decay refer primarily to the change of spin and parity of the nucleus.

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Brief discussions on Fermi's theory of β-decay.*

The process of β-decay

$$n \to p + e^- + \nu$$
$$p \to n + (e^+) + \nu$$

is due the interaction between two types of spinor fields, each having eight components. In Fermi's theory, the interaction term in the Hamiltonian density is to be a certain invariant combination of four field operators of the form

$$U_j^* O_{jk} U_k(x_\mu) \cdot \psi_l^*(x_\mu) O'_{lm} \psi_m(x_\mu)$$

(proton annih. operator)

where
U_1, U_2, U_3, U_4 : proton components
U_5, U_6, U_7, U_8 : neutron components
$\psi_1, \psi_2, \psi_3, \psi_4$: neutrino components
$\psi_5, \psi_6, \psi_7, \psi_8$: electron components

Since,

$$\tau_3 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$\frac{1}{2}(\tau_1 + i\tau_2) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right)$$

$$\frac{1}{2}(\tau_1 - i\tau_2) = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right),$$

Konopinski, R.M.P. 15 (1943), 209.

* Konopinski and Uhlenbeck, 60 (1941), 308;

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M, N, ...

$U^* \frac{1}{2}(\tau_1 + i\tau_2) U$: $p \to n$

$U^* \frac{1}{2}(\tau_1 - i\tau_2) U$: $n \to p$

$\psi^* \frac{1}{2}(\tau_1 + i\tau_2) \psi$: ~~$\nu \to e^-$~~ $e^- \to \nu$ } or $0 \to e^- + (\nu)$

$\psi^* \frac{1}{2}(\tau_1 - i\tau_2) \psi$: ~~$e^- \to \nu$~~ $\nu \to e^-$ } or $0 \to (e^+) + \nu$

Thus, only combinations

$U^* \frac{1}{2}(\tau_1 + i\tau_2) U \cdot \psi^* \frac{1}{2}(\tau_1 + i\tau_2) \psi$ ($0 \to 0$)

$U^* \frac{1}{2}(\tau_1 - i\tau_2) U \cdot \psi^* \frac{1}{2}(\tau_1 - i\tau_2) \psi$ ($0 \to 0$)

are compatible with the conservation of electric charge.

One can write also

$V_j^* O_{jk} U_k \cdot \varphi_l^* O'_{lm} \psi_m$ $\quad p \to n + (e^+) + \nu$

$U_j^* O_{jk} V_k \cdot \psi_l^* O'_{lm} \varphi_m$ $\quad n \to p + e^- + (\nu)$

where U_k ($k = 1, 2, 3, 4$) : proton ann. op.

V_k^* : neutron

ψ_m : electron

φ_m : neutrino

Now, an arbitrary bilinear combination of ψ and φ^* (or $\bar{\varphi} = \varphi^* \gamma_4$) can be written as a linear combination of

$\bar{\varphi} \psi = \varphi^* \gamma_4 \varphi$ scalar

$\bar{\varphi} \gamma_\mu \psi = \varphi^* \gamma_4 \gamma_\mu \varphi$ vector

$\frac{1}{2i} \bar{\varphi} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \psi = \cdots$ tensor

$\frac{1}{2} \bar{\varphi} (\varepsilon_{\kappa\lambda\mu\nu} \gamma_\lambda \gamma_\mu \gamma_\nu) \psi = \cdots$ pseudovector

$\frac{\kappa}{24} \bar{\varphi} \varepsilon_{\kappa\lambda\mu\nu} \gamma_\kappa \gamma_\lambda \gamma_\mu \gamma_\nu \psi = \cdots$ pseudoscalar

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$\rho_3\sigma_1 \cdot \rho_2\sigma_2 \cdot \rho_1\sigma_3 \cdot \rho_3$
$= -\rho_1\rho_2\rho_3 = -\rho_1$

$\gamma_4 = \rho_3 = \beta \qquad \gamma_j = \rho_2\sigma_j$

$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = -\rho_1$

$\varphi^*\gamma_4\psi = \varphi^*\rho_3\varphi$ scalar

$\varphi^*\gamma_4\gamma_\mu\psi = (\varphi^*\psi, -i\varphi^*\boldsymbol{\sigma}\psi)$ vector

$\frac{1}{2}i\varphi^*\gamma_4(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi$

$= (\varphi^*\rho_3\boldsymbol{\sigma}\psi, \; i\varphi^*\rho_2\boldsymbol{\sigma}\psi)$ tensor

$\rho_3 \quad i\rho_2$

$+\frac{i}{6}\varphi^*\gamma_4(\varepsilon_{\kappa\lambda\mu\nu}\gamma_\lambda\gamma_\mu\gamma_\nu)\psi$

$4\,i\,2$
$\gamma_4 \cdot \sigma_1\sigma_2$

$= (i\varphi^*\rho_1\psi, \; -\varphi^*\boldsymbol{\sigma}\psi)$

pseudovector

$\frac{i}{24}\varphi^*\gamma_4\varepsilon_{\kappa\lambda\mu\nu}\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu\psi = \varphi^*\rho_2\psi$

pseudoscalar

For the nucleon field, we have similar expressions. However, in non-relativistic approximation for nucleons with $v \ll c$, they reduce to

$V^*_\beta U \cong V^*U$ scalar (scalar)

$V^*\gamma_4\gamma_\mu\psi \cong (V^*U, \; 0)$ vector (scalar)

$(\alpha V^*\boldsymbol{\sigma}U, \; 0)$ tensor (axial vector)

$(0, \; -V^*\boldsymbol{\sigma}U)$ pseudovector (axial vector)

(0) pseudoscalar.

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M.IV.13

Thus, in the first approximation, we have four types of independent invariant interaction:

$V^* U \cdot \varphi^* \beta \psi$	scalar
$V^* U \cdot \varphi^* \psi$	vector
$V^* \sigma U \cdot \varphi^* \beta \sigma \psi$	tensor
$V^* \sigma U \cdot \varphi^* \sigma \psi$	pseudovector

and β^* (pseudoscalar) *

This gives rise to the so-called allowed transitions satisfying the following selection rules ~~for the nuclear transitions~~ between the initial and final nuclear states.

type	matrix	ΔJ	Parity change
scalar	$\int 1$	0	no
vector (polar)	$\int 1$	0	no
tensor	$\int \sigma$	$0, \pm 1$ ($^{no}_{0 \to 0}$)	no
axial vector	$\int \sigma$	$0, \pm 1$ ($^{no}_{0 \to 0}$)	no
(pseudoscalar	$\int \gamma_5$	0	yes)

* One can go over to the representation in configuration space of Annulleus with the interaction Hamiltonian

$$\sum_{k=1}^{A} \int \tilde{\Psi}_f^* \, \mathcal{H}(\dots r_k \dots) \Psi_i \cdot \varphi^*(r_A) \psi(r_A) \, dr_A)^A$$

$$= \int v_f(r) \, u(r) \, \varphi(r) \psi(r) \, dr)$$

$$= \text{const} \, \varphi(r_0), \psi(r_0) \int 1 \quad \text{etc}$$

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In order to find the selection rules for forbidden transitions, one has to take into account, on the one hand, the matrix ~~terms which~~ elements ~~which~~ for the nucleons which is of the higher orders ~~in~~ in v/c and, on the other hand, due to the change of the electron-neutrino wave functions inside the nucleus. For instance

$$\varphi^*_{(r_k)}\psi_{(r_k)} = A^* e^{-i q\cdot r_k} \cdot B e^{i p\cdot r_k}$$
$$\simeq A^* B \cdot \left(1 + i(p-q)r_k + \cdots - \frac{1}{2}(p-q\cdot r_k)^2 + \cdots\right)$$

Thus, for the first forbidden transitions, the matrix elements such as $\int \alpha$ which is proportional to v/c as well as $\int r$ are to be taken into account. (The latter is of the order of magnitude $\left(\frac{\text{average wave length of el-mag. waves}}{\text{nuclear radius}}\right)^{-1}$ which is not much smaller than v/c) Similarly, for second forbidden transitions, a number of other terms are to be considered. (See K. and L.)

Thus we arrive at selection rules as follows:

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M.N.14

First forbidden transitions

type	matrix	ΔJ	parity change
scalar	$\int r$	$0, \pm 1$	yes
polar vector	$\int r, \int \alpha$	$0, \pm 1$	yes
~~tensor~~	$\int \Sigma \cdot r; \int \Sigma \times r, \int \alpha$	$0; 0, \pm 1$ (n_0 $0 \to 0$)	yes, yes
~~axial~~	~~$\int B_{ij} =$~~	$0, \pm 1, \pm 2$	yes
~~(p.s.~~	$\int (\sigma_i x_j + \sigma_j x_i - \frac{2}{3}\delta_{ij}(\sigma r))$	(n_0 $\frac{1}{2} \to \frac{1}{2}$; $0 \to 0$, $1 \leftrightarrow 0$)	
axial vector	$\int \Sigma r$	0	yes
	$\int \Sigma \times r$	$0, \pm 1$ (n_0 $0 \to 0$)	yes
	B_{ij}	$0, \pm 1, \pm 2$ (n_0 $0 \to 0$, $\frac{1}{2} \to 0$, $1 \leftrightarrow 0$)	yes
(p.s.	$\int \gamma_5 r$	$0, \pm 1$ (n_0 $0 \to 0$)	n_0)

As for the shape of the spectrum, all of the allowed transitions have the same shape as given by

$$P(W)dW \sim q^2 [W/p\ dW$$

with p, W : momentum and energy of e^- or e^+
q : momentum (= energy) of ν or (ν).

In the case of forbidden transitions, the shape depends on the type of interaction and ~~give~~ can ~~be~~ serve as a criterion of correct choice of interaction Hamiltonian. However, we do not go into details here. The above brief survey would be sufficient for discussing the spin and parity of β-radioactive nuclei.

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In order to decide whether β-decay is allowed, first forbidden, second forbidden etc., it is customary to ~~indicate~~ calculate the ft-value. (Blatt and Weisskopf, p. 688)

In the case of β^--decay, the probability per unit time p^- is given by *

$$p^- = \frac{\log 2}{t_-} = C f_-(Z, E_0)$$

$$f_-(Z, E_0) = \int_0^{p_0} F(Z, E) p^2 (E_0 - E)^2 dp$$

where the function $F(Z, E)$ is determined from the wave functions of the electron and the neutrino with the energies E, $E_0 - E$ on the surface of the nucleus with the atomic number ~~of~~ Z.

C is the quantity which is proportional to the square of the nuclear matrix element.

In the case of β^+-decay, we have to take into account the orbital electron capture in addition to the positron emission.

In any case, ft is inversely proportional to the square of the nuclear matrix.

---

* $e^{-p^- t_-} = \frac{1}{2}$ $\quad p^- t_- = \log 2$

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M, N, 15

or log ft

no that ft is the smallest for allowed transitions and the larger ~~and larger as~~ the higher the degree of forbiddenness.

In the following is given a table of β-radioactive nuclei with odd A:*

allowed

	odd number of odd neutron or proton		log ft
${}_{10}Ne^{23} (\rightarrow {}_{11}Na^{23} + e^-)$	13 → 11	$d_{5/2} \rightarrow D_{3/2}$	4.9
${}_{11}Na^{25} (\rightarrow {}_{12}Mg^{25} + e^-)$	11 → 13	$D_{3/2} \rightarrow d_{5/2}$	5.2
${}_{16}S^{35} (\rightarrow {}_{17}Cl^{35} + e^-)$	19 → 17	$d_{3/2} \rightarrow d_{3/2}$	5.0
${}_{25}Mn^{51} (\rightarrow {}_{24}Cr^{51} + e^+)$	25 → 27	$F_{5/2} - f_{7/2}$	5.1

In all these cases,

$\Delta J = 0, 1,$ parity change no.

(~~Fermi or~~ Gamow-Teller alone, or Gamow-Teller plus Fermi

F. (or A.) or T.A. + S.V.

log ft - values are all around 5.

There are only very few exceptions such as Na and Mn.

~~first~~

There are several groups of forbidden

* They decay into ~~stable~~ the ground state of stable nuclei.

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transitions with
$\Delta J = 0, 1$ parity change. yes
with log ft values around 6~7. So
that ~~this can~~ they can be accounted
for as first forbidden, ~~transi~~ (T.A.;
S.V.) such as

$^{87}_{35}Br \rightarrow {}^{87}_{36}Kr + e^-$ 35~51 $p_{3/2} \sim d_{5/2}$ 7.3

⋮

The third group with
$\Delta J = 2$, parity yes
log ft ≅ 8~9
(First forbidden)
and so on.

All these cases with only a very few exceptions agree surprising well the predictions based on simple shell structure model and Fermi theory of β-decay.

It should be noticed that there is another group of super allowed transitions which occur between mirror nuclei

		log ft
$_0n^1 \rightarrow {}_1H^1 + e^-$	$s_{1/2}$	3.2 ± 0.2
$_1H^3 \rightarrow {}_2He^3 + e^-$	$s_{1/2} \rightarrow s_{1/2}$	3.05
$_4Be^7 \rightarrow {}_3Li^7 + e^+$	$p_{3/2} \rightarrow p_{3/2}$	3.36

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M.N.16

Even A Nuclei

Now we go over to even A nuclei. It was already mentioned that there are only very few stable even nuclei with Z, N both odd. They are

$_1H^2$, $_3Li^6$, $_5B^{10}$, $_7N^{14}$

their spins being

1, 1, 3, 1.

Extra proton and neutron in these cases are supposed to be

$s_{1/2}, s_{1/2}$; $p_{3/2}, p_{3/2}$; $p_{3/2}, p_{3/2}$; $p_{1/2}, p_{3/2}$

In all these cases, one can only say that for some reason or other the extra neutron and proton tend to have the resultant spin which is different from zero.

However, there are great number of unstable nuclei with odd Z and odd N.

First group.

	$\Delta J = 1$	Parity No.	(allowed)
$_5B^{12} \to {}_6C^{12} + e^-$	$5 \sim 7$	$p_{3/2} \sim p_{1/2}$	4.2
$_7N^{12} \to {}_6C^{12} + e^+$	$7 \sim 5$	$p_{1/2} \sim p_{3/2}$	4.1

⋮

all of them have $\log ft \sim 4 \sim 5$.

Second group, $\Delta J = 0$ yes. (First Forbidden)

$_7N^{16} \to {}_8O^{16} + e^-$ $\quad 7 \sim 9$ $\quad p_{1/2} \sim s_{1/2}$ $\quad 6.9$

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all solid.

Brief Discussions of Two Exceptions

$_{11}N^{23}_{12}$: $Z=11$: $(s_{1/2})^2 (p_{3/2})^4 (p_{1/2})^2 (d_{5/2})^3$
spin $= \frac{3}{2}$

$_{25}Mn^{55}_{30}$: $Z=25$: $(s_{1/2})^2 (p_{3/2})^4 (p_{1/2})^2 (d_{5/2})^6$
$(s_{1/2})^2 (d_{3/2})^4 (f_{7/2})^5$
spin $= \frac{5}{2}$ ~~$(d_{5/2}\ ?\ ?)$~~

mag. moments of $_{11}N^{23}$ and $_{25}Mn^{55}$

	e.m.p	$l = j - \frac{1}{2}$	$l = j + \frac{1}{2}$	$J = 3/2\ (d_{5/2})^3$
$_{11}N^{23}$	2.21	3.78	0.12	2.8
$_{25}Mn^{55}$	3.	4.8	0.87	$J = 5/2\ (f_{7/2})^5$ 4

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M.N.17

Possible Theoretical Foundation
of shell model

First of all, we must admit again the crudeness of ~~an~~ independent particle model. There are indications that just the opposit model is more suited for dealing with nuclear reactions in general. As ~~point~~ emphasized by Bohr first, the sharpness and density of ~~levels~~ excited energy levels which were revealed by slow neutron experiments seem to show quite convincingly that whenever a ~~neutron or a nucleon~~ comes into the nucleus, it loses its energy and momentum very quick, ~~and~~ which are soon shared by a number of nucleons already present in the nucleus; so that a compound nucleus is formed. This idea sharply contradicts the notion of ~~orbit of~~ ~~than~~ a nucleon in the nucleus.

There is no satisfactory interpretation why these two mutually contradictory models are needed ~~in respective~~ for two cases. ~~Perhaps~~ We can find some vague reason in that indep. particle model is not so bad for the ground state and low excited states and in ~~the~~

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to Pauli principle, whereas, for
nuclear reactions, more highly
excited states are important, and
~~that~~ we have to take into account
a great number of different
configurations with nearly equal
energies, so that the notion of individual
orbit loses its validity almost
completely.
However, this is only a conjecture.
It may well be that ~~we do not foresee~~
~~yet something which~~ need some
fundamentally new idea in order to
~~understand~~ solve the present puzzle.

Now, even if we ~~understand~~ accept as such
the ~~same~~ indep. particle model, we
would like to have some justification
for more specific assumptions
made in this model. The fact
that ~~the~~ the appr. validity of
~~the square well~~ or the
oscillator potential could be
understood, if we assume that
the ~~particle density~~ particle density
is ~~uniform over~~ inside a sphere
of a definite radius. Then
the short range ~~force~~ attraction
between any pair of particles will

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M, N, 18

give rise to an effective potential similar to a square or oscillator well.

It is more difficult to understand, why spin-orbit coupling (and pairing effect) appears.

Mayer assumed that the attractive potential between the pairs of δ-force type.

$$V(1,2) = -g\,\delta(\varphi_1-\varphi_2)\,\delta(\cos\vartheta_1-\cos\vartheta_2)\cdot r_2^{-2}\cdot\delta(r_1-r_2)$$

$$\int V(1,2)\, r_2^2\, dr_2 \sin\vartheta_2\, d\vartheta_2\, d\varphi_2 = -g$$

so that there is no distinction between ordinary and Majorana forces. Thus, one can calculate the interaction energy between a pair of protons or neutrons 1 and 2 by using single particle eigenfunction with definite j's and l's. Then, one can calculate the total potential energy for the whole nucleus by assuming an anti-sym. combination of individual wave functions.

The results are; The lowest energy state for n-neutrons (or protons) with the same j is

$$J=0 \qquad \text{for } n: \text{even}$$

with the energy

$$E_0 \approx -\left(\frac{n}{2}\right)\frac{2j+1}{2}\cdot I$$

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and

$$J = j \qquad \text{for } n: \text{odd}$$

$$\bar{E}_j = \frac{n-1}{2}\,\frac{2j+1}{2}\,I$$

where I is defined as

$$I \equiv I^{nl} = \frac{\sigma}{4\pi}\int_0^\rho R_{nl}^4(r)\,r^2\,dr$$

which is roughly independent of n and l but depends on the size of the nucleus. Thus, a pair of ~~nucleons~~ (neutrons or protons) contributes to the binding energy with an amount proportional to $2j+1$, while single neutron or proton nucleon does not. Further, we have

$$\bar{E}_{even} - \bar{E}_{odd} = -\frac{2j+1}{2A}\,C$$

while empirically

$$\bar{E}_{even} - \bar{E}_{odd} = -\frac{36}{A^{3/4}} \times 10^{-3}\,Mc^2.$$

We can say roughly

$$C \sim 25\ \text{MeV}.$$

As for the ~~S.O.~~ spin-orbit coupling, a few MeV is sufficient for small A. Some of the meson theories can account for that, but unfortunately they are not the pseudoscalar theories.

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M.N.19

Nuclear Isomerism and Shell Structure

M. Goldhaber and R.D. Hill, R.M.P. 24 (1952), 179.

Another criterion of shell structure model is the interpretation of isomerism. It has been known for years that there are a number cases, in which a nucleus with a definite A and Z can be in two different states, both of which are relatively stable. If one is stable, then the other must be metastable.

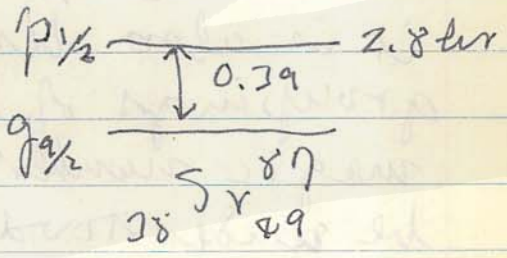
Platt and Weisskopf, p. 629

At present, a great number of long-lived isomers ($T_{1/2} \geq 1$ sec) is known and it is also known that there are abundant groupings of isomers just below the magic numbers 50, 82, 126. (for 2 or N) It could be understood on the basis of shell structure model. Namely, for instance, if 2 or N is a little less than 50, $g_{9/2}$ shell is not fulled up and we expect that there will be low energy states with large nuclear spin as well as those with small nuclear spin. The (the) radiative transition between two (probability of) states where ΔJ is large is strongly forbidden, so that the life-time of

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the higher energy state will be unusually long. (This was the interpretation originality proposed by Weizsäcker.) The situation will be similar just below Z or N = 82 or 126, where the shell $h_{11/2}$ or $i_{13/2}$ is not filled up.

For example, $_{38}Sr^{87}_{49}$ has, at least, two isomeric states, the higher state has the half-life 2.8 hr., the excitation energy being 0.39 MeV. The ground state is supposed to be $g_{9/2}$, while the excited state could be the one, in which one $p_{1/2}$ ~~state~~ orbit for the proton is vacant, so that



$$\Delta J = 4.$$

multi
The radiation is higher pole radiation and the probability of transition will be very small for small energy change.

In order to see the general situation, we have to ~~discuss~~ consider the theory of multipole radiation.

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M, N, 20

Theory of Multipole Radiation

In atomic physics, only the electric dipole radiation was of great importance. This was because

(i) magnetic interaction was, in general, much smaller than the electric interaction

(ii) Wave length of visible or infrared radiation was much larger than ~~that of~~ the atomic dimension.

This is not the case in nuclear physics. Magnetic interaction ~~due to the~~ is important and electric quadrupole radiation ~~was~~ is very important.

Thus, first of all, we have to ~~define~~ know how these multipole radiations are defined.

(Blatt and Weisskopf, p. 80 ~ 96)

Let us take an ~~plane~~ electromagnetic wave with the frequency ω in free space

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathcal{E}(\mathbf{r})\, e^{-i\omega t} + \mathcal{E}^*(\mathbf{r})\, e^{i\omega t} \\ \mathbf{H}(\mathbf{r},t) &= \mathcal{H}(\mathbf{r})\, e^{-i\omega t} + \mathcal{H}^*(\mathbf{r})\, e^{i\omega t} \end{aligned}\right\}$$

Maxwell field equations become

$$[\nabla \times \mathcal{H}] = -\frac{i\omega}{c}\mathcal{E} \qquad \text{etc}$$

$$[\nabla \times \mathcal{E}] = \frac{i\omega}{c}\mathcal{H}$$

$$(\nabla \mathcal{E} = \nabla \mathcal{H} = 0\ .)$$

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Usually, one can expand $\mathcal{E}, \mathcal{H}$ into plane transverse waves. However, we are for the moment more interested in the radiation emitted from a small source, so that the expansion into series of spherical waves ~~are~~ is more appropriate.

If we first take a scalar function $V(r,\theta,\varphi)$, one can expand, in general,

$$V(r,\theta,\varphi) = \sum_{l,m} f_{l,m}(r)\, Y_{lm}(\theta,\varphi).$$

(Under a certain rotation $(r,\theta,\varphi) \to (r,\theta',\varphi')$

$$V(r,\theta',\varphi') = \sum_{l,m} f_{l,m}(r)\, Y_{lm}(\theta',\varphi').$$

If further V satisfies a certain field or wave equation which is invariant ~~with respect~~ under rotation, $V(r,\theta',\varphi')$ must be also the solution of the original equation and so on.)

In other words, a scalar function can be decomposed into eigenfunctions of infinitesimal rotation operators

$$\mathfrak{M}_x^2 = \{[\mathbf{r} \times \mathbf{p}]\}^2$$

and $\mathfrak{M}_z = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$

Y_{lm} : $\mathfrak{M}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$ }

$\mathfrak{M}_z = \hbar m$

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M, N. 21

If we take a general vector function $\mathbf{A}$, one can perform the same expansion for each component of A_x, A_y, A_z.

$$A_k = \sum_{l,m} f_{k,l,m}(r)\, Y_{l,m}(\theta,\varphi)$$

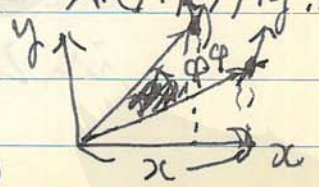
This again is the decomposition into eigenfunctions of the same infinitesimal rotation operators as above. However, the three components of a vector transform linearly under rotation, so that now the whole effect of rotation on a vector function is the same as on a wave function of a particle with the spin 1 which can be regarded as a vector function with three components. Namely,

$$\mathbf{A}(r,\theta,\varphi) = \sum_{k=1}^{3} A_k(r,\theta,\varphi)\,\mathbf{e}_k$$

$$= \sum_{\mu=-1,0,+1} A_\mu(r,\theta,\varphi)\,\mathbf{d}_\mu$$

$$\left.\begin{aligned} \mathbf{d}_{+1} &= \frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y) \\ \mathbf{d}_{-1} &= \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y) \\ \mathbf{d}_0 &= \mathbf{e}_z \end{aligned}\right\}$$

Under the infinitesimal rotation of an angle φ around z-axis will change a vector $\mathbf{A}(A_x, A_y, A_z)$ into



$$x' = r\cos(\theta+\varphi)$$
$$= r\cos\theta\cos\varphi - r\sin\theta\sin\varphi$$

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$$x' = x\cos\varphi - y\sin\varphi$$
$$y' = x\sin\varphi + y\cos\varphi \quad \Big\}$$

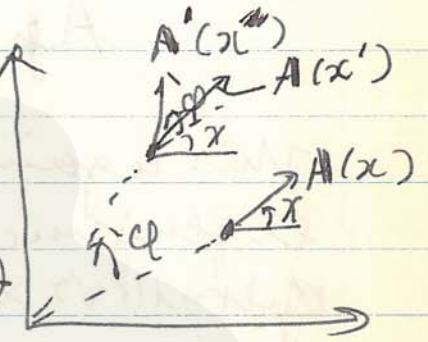
$A_x'(x') = |A|\cos(\chi+\varphi)$

$= A_x(x)\cos\varphi - A_y(x)\sin\varphi$

$A_y'(x') = A_x(x)\sin\varphi + A_y(x)\cos\varphi$

~~For an infinitesimal rotation (φ: very small)~~

~~$A_x'(x') = A_x(x)$~~



Thus, the change of the field δA at the point $x'(x', y', z')$ is

$$\Delta A_x = A_x'(x') - A_x(x') = A_x(x')\cos\varphi - A_y(x')\sin\varphi - A_x(x')$$

$$\Delta A_y = A_y'(x') - A_y(x') = A_x(x')\sin\varphi + A_y(x')\cos\varphi - A_y(x')$$

If the rotation is infinitesimal $\delta\varphi$.

$$x' = x - y\delta\varphi$$
$$y' = y + x\delta\varphi \quad \Big\}$$

$$\Delta A_x = A_x(x') - A_x(x') - A_y(x')\delta\varphi$$
$$= \left(y\frac{\partial A_x(x')}{\partial x} - x\frac{\partial A_x(x')}{\partial y} - A_y(x')\right)\delta\varphi$$

$$\Delta A_y = \left(y\frac{\partial A_y(x')}{\partial x} - x\frac{\partial A_y(x')}{\partial y} + A_x(x')\right)\delta\varphi$$

or

$$+i\hbar\Delta A_x = \left\{-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)A_x - i\hbar A_y\right\}\delta\varphi$$

$$i\hbar\Delta A_y = \left\{-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)A_y + i\hbar A_x\right\}\delta\varphi$$

$$i\hbar\Delta A_z = 0$$

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Thus, the total effect of infinitesimal rotation around z-axis on a vector field A is given by an operator

$$m_z + \text{[illegible}\ S_z$$

where

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, one can define

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

They satisfy the commutation relations

$$S_x S_y - S_y S_x = i S_z$$

etc.

$$\therefore S_x S_y - S_y S_x = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= i S_z$$

Thus, the vector operator

~~$M = m + iS$~~

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The eigenvalues of each of s_x, s_y, s_z are
$1, 0, -1$.

$$s_x^2 + s_y^2 + s_z^2 = 2$$

In particular, the vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} \qquad \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$

are (normalized) unit vectors belonging to the eigenvalues $1, 0, -1$ of s_z

$$\left(\because \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} = \begin{pmatrix}1\\ i\\ 0\end{pmatrix}\right.$$

$$\left.\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} = \begin{pmatrix}-1\\ i\\ 0\end{pmatrix} = -\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\right)$$

In this sense

$$Y_{l,m}(\theta,\varphi)\cdot d_\mu$$

can be regarded as the eigenvectors ~~for~~ belonging to the eigenvalues $\hbar l, \hbar m, \mu\ (=1, 0, -1)$ of the operators $\hbar^2 m^2, m_z, s_z$

Further, the vector operator

$$\mathbf{M} = \mathbf{m} + \hbar\mathbf{s}$$

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can be regarded as the total angular momentum operator satisfying the C. R.

$$M_x M_y - M_y M_x = i\hbar M_z$$

etc.

The eigenvector functions which are eigenvectors of the operators M^2, M_z, m^2 can be constructed from $Y_{l,m}(\theta,\varphi)\,d\mu$ in the following way:

$$Y^M_{J l \nu_\mu}(\theta,\varphi) = \sum_{m=-l}^{+l} \sum_{\mu=-1}^{+1} C_{l\nu_\mu}(J,M,m,\mu) Y_{l,m}(\theta,\varphi)\, d_\mu$$

where ~~J is~~ $l = J+1$ or J or $J-1$.

Thus, there are three kinds of vector spherical harmonics*.

Then, an arbitrary vector function A can be expended into series of vector eigenfunctions Y^M_{Jl},

$$A = \sum_{l,m} f_{l,m}(r)\, Y_{l,m}(\theta,\varphi) = \sum_{J} \sum_{l=J+1}^{J+1+J} \sum_{M=-J}^{M} f_{Jl}(r)\, Y^M_{Jl\nu_\mu}(\theta,\varphi),$$

because, ~~for any fixed value of r, $f_{lm} Y_{lm}$ can be written as a linear combination of mutually or thogonal vector~~ $Y^M_{Jl}(\theta,\varphi)$ constitute a complete ~~by~~

* Since m^2 commutes with M^2 and M_z, one can construct the simultaneous eigenfunctions with a definite set of values J, M, l.

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set of orthogonal vector function of θ, φ in the sense that

$$\int \mathbf{Y}^{M}_{Jl}(\theta,\varphi)\,\mathbf{Y}^{M'}_{J'l'}(\theta,\varphi)\sin\theta\,d\theta\,d\varphi$$

$$= \delta_{JJ'}\,\delta_{MM'}\,\delta_{ll'},$$

if we normalize their property. Among these vector eigenfunctions $\mathbf{Y}^{M}_{Jl}(\theta,\varphi)$, $\mathbf{Y}^{M}_{JJ}=\mathbf{Y}^{M}_{ll}$ has the parity $-(-1)^J=-(-1)^l$, while $\mathbf{Y}^{M}_{J,J-1}=\mathbf{Y}^{M}_{l+1,l}$ and $\mathbf{Y}^{M}_{J,J+1}=\mathbf{Y}^{M}_{l-1,l}$ have the parity $-(-1)^l=(-1)^J$. (Provided that $d\mu$ changes sign under reflection.*)

* However, it is more convenient to assume that, as Blatt and Weisskopf has done, $d\mu$ with the components $(d\mu_x, d\mu_y, d\mu_z)$ remains the same after reflection or $d\mu_x'=+d\mu_x$ etc.; this amounts to the same thing as to use the ~~fixed~~ unit vectors, which is defined, ~~first~~, with respect to a certain coordinate system. Then we have the result

* ~~Blatt and Weisskopf passed it as~~ assumed that the parities are $(-1)^l=(-1)^J$ and $(-1)^l=-(-1)^J$ respectively for $l=J$ and $l=J\pm1$. This is true only if we assume $d\mu$ does not change under reflection.

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M, N, 2

Now, let us go back to the expansion of electric and magnetic vectors $\mathbf{E}$ and $\mathbf{H}$ or $\mathcal{E}$ and $\mathcal{H}$.

$$\mathcal{E} = \sum\sum\sum f^{M}_{Je}(r)\, Y^{M}_{Je}(\theta,\varphi)$$

$$\mathcal{H} = \sum\sum\sum g^{M}_{Je}(r)\, Y^{M}_{Je}(\theta,\varphi)$$

There are relations between the expansion coeff. due to Maxwell field equations. Namely, if a vector function

$$[\nabla \times Y^{M}_{Je}(\theta,\varphi)]$$

is expanded into series of $Y^{M'}_{J'e'}$, the coefficient will be zero unless J' is ~~cont~~ appears in

$$\mathbf{D}_1 \times \mathbf{D}_J = \sum_{J'} \mathbf{D}_{J'}$$

or $J' = J+1,\ J,\ J-1$. Furthermore, ~~under the reflection~~ ∇ changes sign, ~~so that~~ $[\nabla \times Y^{M}_{J,J}(\theta,\varphi)]$ contains ~~$Y^{M'}_{J+1,J}(\theta,\varphi)$~~ (Further, only $l' = l+1$ and $l-1$ is possible)

Thus, for instance, $[\nabla \times Y^{M}_{J,J}(\theta,\varphi)]$ contains

$Y^{M'}_{J+1,J+1}$, $Y^{M'}_{J+1,J}$, $Y^{M'}_{J,J+1}$, $Y^{M'}_{J,J-1}$, $Y^{M'}_{J-1,J-1}$ ($l' = J+2$ is excluded)

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~~Therefore~~ (Which respect to the reflection, $Y^{M'}_{J,J\pm1}$ $Y^{M'}_{J+1,J+1}$, $Y^{M'}_{J-1,J-1}$ have all the parity $-(-1)^J$ which is opposite to $+(-1)^J$ of $Y^{M'}_{J,J}$.) Thus, we can conclude that

$$[\nabla \times Y^{M}_{J,J}(\theta,\varphi)] = \sum_{M'} a^{M'} Y^{M'}_{J,J+1}(\theta,\varphi) + \sum_{M'} b^{M'} Y^{M'}_{J,J-1}(\theta,\varphi) + \sum_{M'} c^{M'} Y^{M'}_{J+1,J+1} + \sum_{M'} d^{M'} Y^{M'}_{J-1,J-1}$$

(One can easily see that $M' = M-1, M, M+1$)

(On the contrary, $[\nabla \times Y^{M}_{J,J+1}(\theta,\varphi)]$ contains

$Y_{J+1,J+2}$ $Y_{J+1,J}$
$Y_{J,J}$, $Y_{J,J+1}$,
$Y_{J-1,J}$)

~~Among them, only~~ $Y_{J,J}$;

In this way, an electric field of the type $Y^{(M)}_{J,J}(\theta,\varphi)$ is coupled with the magnetic fields of the types

$Y_{J+1,J+1}$; $Y_{J,J+1}$; $Y_{J,J-1}$, $Y_{J-1,J-1}$.

while $Y^{M}_{J,J\pm1}$ is coupled with those Y_{J}. with the opposite parity.

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(M, N. 25)

There is one more relation, which is to be taken into account. Namely

$$M_j \cdot [\nabla \times \mathbf{B}] = [\nabla \times M_j \mathbf{B}] \qquad j = x, y, z$$

where $\mathbf{B}(r, \theta, \varphi)$ is any vector field.

Proof:

$$\left[-ih\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \begin{pmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{pmatrix} \right]$$

$$= \begin{pmatrix} -ih\frac{\partial B_z}{\partial x} \\ -ih\frac{\partial B_z}{\partial y} \\ ih\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x}\right) \end{pmatrix}$$

$$h\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{pmatrix} = -ih\begin{pmatrix} \frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \\ -\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) \\ 0 \end{pmatrix}$$

$$-ih\left\{\nabla \times \begin{pmatrix} B_y \\ -B_x \\ 0 \end{pmatrix}\right\} = -ih\begin{pmatrix} \frac{\partial B_y}{\partial z} \\ \frac{\partial B_y}{\partial z} \\ -\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \end{pmatrix}$$

$$-ih\begin{pmatrix} -\frac{\partial B_z}{\partial x} \\ -\frac{\partial B_z}{\partial y} \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \end{pmatrix}$$

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Thus, if $\mathbf{B}$ is an eigenvector function of $\mathcal{M}^2$, M_z belonging to the eigenvalues J, M, then $[\nabla \times \mathbf{B}_J]$ is also an eigenvector function of belonging to the same values J, M. However, the parity is opposite. Thus, we can have

$$\left.\begin{aligned} [\nabla \times Y^M_{J,J}] &= a_J Y^{M}_{J,J+1} + b_J Y^M_{J,J-1} \\ [\nabla \times Y^M_{J,J+1}] &= c_J Y^M_{J,J} \\ [\nabla \times Y^M_{J,J-1}] &= d_J Y^M_{J,J} \end{aligned}\right\}$$

Thus, we are left with two cases:

$$\mathcal{E} \sim Y^M_{J,J} \qquad \mathcal{H} \sim Y^M_{J,J+1}, Y^M_{J,J-1}$$
$$\mathcal{E} \sim Y^M_{J,J+1}, Y^M_{J,J-1} \qquad \mathcal{H} \sim Y^M_{JJ}$$

Now, it is clear that the electromagnetic as described by $\mathcal{H} \sim Y_{J,J}$ ($\mathcal{H} \sim Y^M_{J,J+1}, Y^M_{J,J-1}$) or $\mathcal{E} \sim Y^M_{J,J+1}, Y^M_{J,J-1}$ ($\mathcal{H} \sim Y_{J,J}$) describes the photons with angular momentum $J\hbar$ (or the absolute square of ang. momentum is $\hbar^2 J(J+1)$). This, of course, consists of intrinsic and orbital a.m. Actually, one

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can prove that by constructing the total energy (time average)

$$\frac{1}{8\pi}\int (E^2 + H^2)\, dv \to \frac{1}{4\pi}\int (\mathcal{E}^*\mathcal{E} + \mathcal{H}^*\mathcal{H})\, dv$$

and the total angular momentum (time average)

$$\mathbf{M}_0 = \frac{1}{4\pi c}\int \{\mathbf{r} \times (\mathbf{E} \times \mathbf{H})\}\, dv$$

$$= \frac{1}{4\pi c}\int \mathbf{r}\,(\mathcal{E}^* \times \mathcal{H} + \mathcal{E} \times \mathcal{H}^*)\, dv$$

$$\mathbf{M}_z = \frac{1}{4\pi c}\frac{1}{2\pi}\int \{\mathcal{E}^*(m_z\mathcal{E}) + \mathcal{H}^*(m_z\mathcal{H})\}\, dv$$

$$\to \frac{\hbar\omega}{\hbar\omega}(M\hbar)\cdot\frac{1}{4\pi}\int \mathcal{E}^*\mathcal{E} + \mathcal{H}^*\mathcal{H}\, dv$$

$$= M\hbar \cdot (\text{number of photons})$$

Now, the electric multipole of the order J is ~~given~~ defined by

		parity
$\mathcal{H}^{(e)}_{J,M} \sim$	$Y^M_{J,J}(\theta,\varphi)$	$(-1)^J$
$\mathcal{E}^{(e)}_{J,M} \sim$	$Y^M_{J,J\pm1}(\theta,\varphi)$	$-(-1)^J$

Magnetic multipole radiation of the order J is defined by

		parity
$\mathcal{E}^{(m)}_{J,M} \sim$	$Y^M_{J,J}(\theta,\varphi)$	$(-1)^J$
$\mathcal{H}^{(e)}_{J,M} \sim$	$Y^M_{J,J\pm1}(\theta,\varphi)$	$-(-1)^J$

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the transition probability is proportional to the absolute square of

$$\int \Psi_f^* (jA) \Psi_i \, dv$$

A has the same parity E or $\mathfrak{E}$ and j has the parity $(-1)^J$, so that jA has the same parity as ~~$\mathfrak{E}$~~ ~~also~~ both for el. and mag. multipole radiation.*

Thus we have the selection rule for the parity change of initial and final states

Electric multipole: parity change $(-1)^J$

or yes for ~~J~~: odd multipole (dipole)
no for J: even (quadrup)

Magnetic multipole: yes for J: even multip. (quad)
no for J: odd multip. (dipole)

For instance, for an electric dipole radiation, $\Delta J = \pm 1, 0$ $(l = 0, 1)$ parity yes

magnetic dipole radiation

$\Delta J = \pm 1, 0$ $(l = 0, 1)$ parity no

e.d.: $P \rightleftarrows S$ etc,

m.d.: $S \not\rightleftarrows S$ etc.

---

* In non-relativistic approximation, one may write

$$\int \tilde{\Psi}_f \left(\frac{-e}{Mc} Ap + e A_0 - \mu H \right) \Psi_i$$

$$= \int \tilde{\Psi}_f \left(\frac{-e}{Mc} \mathbf{A}p + e A_0 - \frac{e\hbar}{2Mc} g \boldsymbol{\sigma} \mathbf{H} \right) \Psi_i$$

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M. N. 2)

Hence $\operatorname{div} \mathbf{Y}^m_{l,l} = 0$

Problem: Show that

$$\mathbf{Y}^m_{l,l}(\theta,\varphi) = \frac{\mathbf{m}\, Y_{lm}(\theta,\varphi)}{\hbar\sqrt{l(l+1)}} = \frac{\mathbf{L}\, Y_{lm}(\theta,\varphi)}{\sqrt{l(l+1)}}$$

where $\mathbf{m} = [\mathbf{r} \times \mathbf{p}]$, $\hbar\mathbf{L} = \mathbf{m}$

$m_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$\mathbf{L} = -i(\mathbf{r}\times\nabla)$

Proof: $m_z \begin{pmatrix} m_x Y_{lm}(\theta,\varphi) \\ m_y Y_{lm}(\theta,\varphi) \\ m_z Y_{lm}(\theta,\varphi) \end{pmatrix} = \begin{pmatrix} m_x(m_z Y_{lm}) \\ m_y(m_z Y_{lm}) \\ m_z(m_z Y_{lm}) \end{pmatrix}$

$$= \begin{pmatrix} i\hbar\, m_y Y_{lm} \\ -i\hbar\, m_x Y_{lm} \\ 0 \end{pmatrix}$$

$$\hbar S_z \cdot \mathbf{m} Y_{lm} = \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_x Y_{lm} \\ m_y Y_{lm} \\ m_z Y_{lm} \end{pmatrix}$$

$$= \begin{pmatrix} -i\hbar\, m_y Y_{lm} \\ i\hbar\, m_x Y_{lm} \\ 0 \end{pmatrix}$$

$\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$
$+ \frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$
$+ \frac{\partial}{\partial z}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = 0$

$M_z \cdot \mathbf{m} Y_{lm} = \mathbf{m}(m_z Y_{lm})$

$M_z^2 \cdot \mathbf{m} Y_{lm} = M_z \cdot \mathbf{m}(m_z Y_{lm})$

$= \mathbf{m}(m_z^2 Y_{lm})$

$M^2 \cdot \mathbf{m} Y_{lm} = \mathbf{m}(M^2 Y_{lm})$

$M_z(\mathbf{m} Y_{lm}) = \hbar\, \mathbf{m}(\mathbf{m} Y_{lm})$

$M^2(\mathbf{m} Y_{lm}) = \hbar^2 l(l+1)(\mathbf{m} Y_{lm})$

Thus, $J = l$, $M = m$.

$$\mathbf{Y}^m_{l,l}(\theta,\varphi) = \frac{\mathbf{m}\, Y_{lm}(\theta,\varphi)}{\hbar\sqrt{l(l+1)}}$$

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- Radial parts of the multipole eigenfunctions in free space, $\mathcal{E}, \mathcal{H}$ have to satisfy

$$\nabla \times \mathcal{H} = -\frac{i\omega}{c}\mathcal{E}$$

$$\nabla \times \mathcal{E} = \frac{i\omega}{c}\mathcal{H}$$

$$(\nabla \mathcal{E} = \nabla \mathcal{H} = 0)$$

$$(\nabla^2 + k^2)\mathcal{E} = 0 \qquad k = \frac{\omega}{c}$$

$$(\nabla^2 + k^2)\mathcal{H} = 0$$

Thus, if we put $J = l$, $M = m$ in $\mathcal{H}^{(e)}_{J,M}$:

$$\mathcal{H}^{(e)}_{l,m} = \frac{1}{kr} f_l(r) Y^m_{ll}(\theta,\varphi)$$

$$= \frac{1}{kr} f_l(r) \frac{m Y_{lm}(\theta,\varphi)}{k\sqrt{l(l+1)}}$$

$$(\nabla^2 - k^2)\mathcal{H}^{(e)}_{l,m} \propto \left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) - \frac{l(l+1)}{r^2} + k^2\right\} \times f_l(r) \cdot \frac{m Y_{lm}(\theta,\varphi)}{k\sqrt{l(l+1)}} = 0$$

$$f_l(r) = \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+\frac{1}{2}}(kr)$$

~~For $kr \ll l$: $f_l(r) \sim \left(\frac{\pi kr}{2}\right)^{1/2} \frac{1}{\Gamma(l+\frac{3}{2})}\left(\frac{kr}{2}\right)^l$~~

$$\mathcal{E}^{(e)}_{lm} = \frac{i}{k}\left(\nabla \times \left(\frac{1}{kr} f_l(r) Y^m_{ll}(\theta,\varphi)\right)\right)$$

(2^l-pole electric)

~~$\mathcal{H}^{(e)}_{lm}$~~

$$\mathcal{E}^{(m)}_{lm} = \frac{1}{kr} f_l(r) Y^m_{ll}(\theta,\varphi)$$

$$\mathcal{H}^{(m)}_{lm} = -\frac{i}{k}\left(\nabla \times \frac{1}{kr} f_l(r) Y^m_{ll}(\theta,\varphi)\right)$$

(magnetic 2^l-pole)

$$\mathcal{E} = \sum_{l=l_{min}}^{\infty}\sum_{m}\left(a^{(e)}_{lm}\mathcal{E}^{(e)}_{lm} + a^{(m)}_{lm}\mathcal{E}^{(m)}_{lm}\right)$$

$$\mathcal{H} = \sum_{l=1}^{\infty}\sum_{m}\left(a^{(e)}_{lm}\mathcal{H}^{(e)}_{lm} + a^{(m)}_{lm}\mathcal{H}^{(m)}_{lm}\right)$$

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Remark: For $l=0$

$$m Y_{l,m}(\theta,\varphi) = 0$$

In fact, $Y^{o}_{0,0} = 0$, because it is impossible to construct $J=0$ state by combining $Y_{0,0}$ and $d\mu$. ~~Similarly~~, Hence, of course

$$[\nabla \times Y^{o}_{0,0}] = 0 = a_J Y_{0,1}$$

or $a_J = 0$

Thus, both $\mathcal{M}^{(e)}_{l,m}$, $\mathcal{E}^{(e)}_{l,m}$; ~~$\mathcal{E}^{(m)}_{l,m}$~~, $\mathcal{H}^{(m)}_{l,m}$ start from $l=1$.

Selection rules

The transition probability is proportional to the absolute square of

$$\oint \tilde{\Psi}_f \left(eA_0 - \frac{e}{Mc} A p - \mu H\right) \Psi_i \, dv,$$

if we consider one nucleon-transition in non-relativistic approximation.

Since, A_0 A are related to E

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla A_0$$

$A_0 = a_0 e^{-i\omega t} + a_0^* e^{i\omega t}$ $\quad A = a e^{-i\omega t} + a^* e^{i\omega t}$

or $\quad \mathcal{E} = i\kappa a - \nabla a_0$

In particular, one can choose ~~a~~ $A_0 = 0$, and $\mathcal{E} = i\kappa a$, so that the expression

$$\oint \tilde{\Psi}_f \left(\frac{e}{Mc}\frac{i}{\kappa}\mathcal{E}p - \mu \mathcal{H}\right) \Psi_i \, dv$$

is relevant to the selection rules.

For electric 2^l-pole radiation, $\mathcal{E}^{(e)}_{l,m}$ contains $Y_{l+1,m'}$, $Y_{l-1,m'}$ corresponding

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to the total angular momentum $J = l$.
Hence, the transition probability is different from zero, only if for 2^l-pol rad.

$$\left.\begin{aligned} &|J_i - J_f| \le l \le J_i + J_f \\ \text{and}\quad &M_i - M_f = m \end{aligned}\right\}$$

The same is true for the magnetic 2^l-pole radiation.

In particular, since there is no $l=0$ radiation, the ~~Trans~~ radiative transition between $J_i = J_f = 0$ is strictly forbidden.

Further,

$\pi_i = \pi_f$ for even parity radiation
(electric 2^l-radiation with even l)
(magnetic 2^l-radiation with odd l)

$\pi_i = -\pi_f$ for odd parity radiation
(electric 2^l-radiation with odd l)
(magnetic 2^l-radiation with even l)

In the atomic system, only electric dipole ~~lowest~~ radiation were of importance, and the selection rule of

$$\Delta J = 0, \pm 1, \qquad \Delta M = 0, \pm 1$$

parity change : yes.

which is a particular case of above general selection rules.

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M.IV.29

Since as we shall see later, the transition probability decreases very rapidly with increasing l for $kr_0 \ll 1$, it is useful to know the lowest $l = l_{min}$ for a given pair of states with J_i, π_i; J_f, π_f.

a. $J_i \neq J_f$:

	electric radiation	magnetic radiation
$\pi_i \pi_f = (-1)^{J_i - J_f}$ (parity favored)	$l_{min} = \lvert J_i - J_f \rvert$	$l_{min} = \lvert J_i - J_f \rvert + 1$ (except J_i or $J_f = 0$)
$\pi_i \pi_f = (-1)^{J_i - J_f + 1}$ (parity unfavored)	$l_{min} = \lvert J_i - J_f \rvert + 1$ (except J_i or $J_f = 0$)	$l_{min} = \lvert J_i - J_f \rvert$

b. $J_i = J_f \neq 0$.

$\pi_i = \pi_f$	$l_{min} = 2$ (except $J_i = J_f = \frac{1}{2}$)	$l_{min} = 1$
$\pi_i = -\pi_f$	$l_{min} = 1$	$l_{min} = 2$ (except $J_i = J_f = \frac{1}{2}$)

As we shall see that, for the same value of l, emission of electric radiation has larger probability than magnetic radiation, parity favored case is almost purely electric 2^l-pole radiation of $l = \lvert J_i - J_f \rvert$. In parity unfavored case, both magnetic 2^l-pole radiation and electric 2^{l+1}-pole radiation may ~~be~~ be there. (In most cases, magnetic radiation is predominant)

Reference: Blatt and Weisskopf, XII.

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Estimation of Transition Probabilities

Above arguments were not strictly quantum theoretical in that the operator character of the electromagnetic fields was not taken into account. In order to make the argument completely quantum mechanical, we have to regard $\mathfrak{E}, \mathfrak{H}$ as operators operating on Ψ which describes the state of the whole system, including photons as well as nucleons. Certainly, we could be done by normalizing the eigenfunctions for the spherical wave suitable and and replacing the coefficients of expansions by annihilation and creation operators of photons of definite angular momentum.

Namely, the normalization factor would be determined by

$$\frac{1}{4\pi}\int\{\mathcal{E}^{*(e)}_{lm}(k,r,\theta,\varphi)\mathcal{E}^{(e)}_{l'm'}(k',r,\theta,\varphi) + \mathcal{H}^{*(e)}_{lm}(k,r,\theta,\varphi)\mathcal{H}^{(e)}_{l'm'}(k',r,\theta,\varphi)\}$$
$$r^2 dr\cdot\sin\theta d\theta d\varphi \approx \hbar c\,\frac{\delta(k-k')}{k^2}\times\delta_{ll'}\delta_{mm'}$$

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M.N.30

However, before going over to completely q.m. treatment of the problem, we had better take into account the following situation. Namely, if one expand the scalar or vector function into Fourier series or Fourier integral, ~~all plane~~ the eigenfunctions, which are plane waves, are all free from singularities and constitute a complete set of orthogonal functions. However, if one expand a scalar or vector function into series of spherical waves, one encounters a strange situation: If one excludes a small region around the origin, $r=0$, one needs both types of solutions of

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right) \psi_l(r) = 0$$

$$\begin{cases} f_l(kr) = \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+\frac{1}{2}}(kr) \\ g_l(kr) = \left(\frac{\pi kr}{2}\right)^{1/2} J_{-l-\frac{1}{2}}(kr) \end{cases}$$

For large values of $kr >> 1$, asymptotically *

$$f_l(kr) \simeq \sin\left(kr - \tfrac{1}{2}l\pi\right)$$

$$g_l(kr) \simeq (-1)^l \cos\left(kr - \tfrac{1}{2}l\pi\right)$$

---

* $J_n(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}n\pi - \frac{1}{4}\pi\right)$

$\cos\left(x + \frac{l}{2}\pi\right) \sim \cos(l\pi)\cdot\cos\left(x - \frac{1}{2}l\pi\right)$

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For $kr = x \ll 1$:

$$J_{l+\frac{1}{2}}(x) = \left(\frac{2x}{\pi}\right)^{1/2} \frac{(2x)^l\, l!}{(2l+1)!}$$

$$J_{-l-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \frac{(2x)^{-l-1}\, 2\,(2l)!\,(-1)^l}{l!}$$

$$\times \left(1 + \frac{x^2}{4l-2}\right)$$

(Mott, Theory of Atomic Collisions, p. 35)

or for $kr \ll 1$

$$f_l(kr) = (kr) \cdot \frac{(2kr)^l\, l!}{(2l+1)!}$$

$$\propto (kr)^{l+1}$$

$$g_l(kr) = (kr) \frac{(2kr)^{-l-1} \cdot 2 \cdot (2l)!}{l!} (-1)^l$$

$$\propto (kr)^{-l}$$

or $$\frac{f_l(kr)}{kr} \propto (kr)^l$$

$$\frac{g_l(kr)}{kr} \propto (kr)^{-l-1}$$

Thus, the second solution $g_l(kr)$ is not regular at $r = 0$, but still necessary for the expansion for $r \geq r_0$.

Now suppose that there is a source of the electromagnetic field inside the

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(M, N, 31)

sphere $r = r_0$. Then, the field equations are modified for $r \le r_0$:

$$\nabla \times \mathcal{H} = -\frac{i\omega}{c}\mathcal{E} + \frac{4\pi}{c}\mathcal{J}$$

$$\nabla \times \mathcal{E} = \frac{i\omega}{c}(\mathcal{H} + 4\pi \mathcal{M})$$

$$\mathrm{div}\, \mathcal{J} = i\omega \rho$$

$$(\mathrm{div}\, \mathcal{E} = 4\pi\rho, \quad \mathrm{div}\,(\mathcal{H} + 4\pi\mathcal{M}) = 0)$$

$$\nabla \times \nabla \times \mathcal{H} - k^2 \mathcal{H} = \frac{4\pi}{c}(\nabla \times \mathcal{J} + c k^2 \mathcal{M})$$

$$\nabla \times \nabla \times \mathcal{E} - k^2 \mathcal{E} = \frac{4\pi i k}{c}(\mathcal{J} + c\, \nabla \times \mathcal{M})$$

with $k = \omega / c$

~~The general~~ One General solution of these equations is the sum of a particular solution and the general solution of field equations in free space.

Now, let us again expand

$$\mathcal{E} = \sum (b_{lm}^{(e)} \mathcal{E}_{lm}^{(e)} + b_{lm}^{(m)} \mathcal{E}_{lm}^{(m)})$$

$$\mathcal{H} = \sum (b_{lm}^{(e)} \mathcal{E}_{lm}^{(e)} + b_{lm} \mathcal{H}_{lm}^{(m)})$$

However,

$$\mathcal{E} = \sum (a_{lm}^{(e)} \mathcal{E}_{lm}^{(e)} + a_{lm}^{(m)} \mathcal{E}_{lm}^{(m)}) + \sum (b_{lm}^{(m)} \bar{\mathcal{E}}_{lm}^{(e)*} + b_{lm}^{(m)} \bar{\mathcal{E}}_{lm}^{(m)})$$

where $\bar{\mathcal{E}}_{lm}^{(e)}$ is different from $\mathcal{E}_{lm}^{(e)}$ in that $f_l(kr)$ is replaced by $g_l(kr)$.

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One can similarly expand $\mathcal{H}$:

$$\mathcal{H} = \sum \left(a^{(e)}_{lm} \mathcal{H}^{(e)}_{lm} + a^{(m)}_{lm} \mathcal{H}^{(m)}_{lm} \right) + \sum \left(b^{(e)}_{lm} \overline{\mathcal{H}}^{(e)}_{lm} + b^{(m)}_{lm} \overline{\mathcal{H}}^{(m)}_{lm} \right)$$

These expansions ~~are certain~~ with the same ~~constant~~ coefficients, a's and b's for $\mathcal{E}, \mathcal{H}$ are correct for $r > r_0$, where the free field eq. must be satisfied.

However, for $r \leq r_0$, this is not so.

~~At we have taken the part~~ assume a solution

~~$$\mathcal{H}'^{(e)}_{lm} = \frac{1}{kr} u_l(r) Y^m_{ll}(\theta, \varphi)$$~~

~~$$\mathcal{E}'^{(e)}_{lm} = \frac{i}{k} (\nabla \times (\frac{1}{kr} u_l(r) Y^m_{ll}(\theta, \varphi))$$~~

~~and insert them in the field equations, we obtain~~

~~$$\nabla \times \mathcal{H}'^{(e)}_{lm} = \nabla \times \frac{1}{kr} u_l(r) Y^m_{ll}(\theta, \varphi)$$~~

~~$$= - ik \mathcal{E}'^{(e)}_{lm} + \frac{4\pi}{c} \ldots$$~~

let us start from the $\mathcal{E}$ equation:

$$\nabla \times \nabla \times \mathcal{H}'^{(e)} - k^2 \mathcal{H}'^{(e)} = \frac{4\pi}{c} (\nabla \times \mathcal{J} + c k^2 \mathcal{M})$$

both $\mathcal{H}$, and the right hand side can be expanded into series of $Y^m_{ll}(\theta, \varphi)$ and $Y^m_{l+1,l}$, $Y^m_{l-1,l}$

$$\nabla \times \nabla \times \mathbf{A} = \operatorname{grad}(\operatorname{div} \mathbf{A}) - \Delta \mathbf{A}$$

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(M, N. 32)

by comparing the coefficients of $Y_{\ell}^{m}(\theta,\varphi)$ on both hand side, will be we have

$$\left(-\frac{d}{dr^2}+\frac{l(l+1)}{r^2}-k^2\right)u_\ell(r) = K^{(e)}_{\ell,m}(r)$$

A particular solution of this equation is

$$u_\ell(r) = \int_0^{r_0} f_\ell(r')\, h_\ell(r')\, K^{(e)}_{\ell m}(r')\, dr' + \int_{r_0}^{\infty} h_\ell(r)\, K^{(e)}_{\ell m}(r')\, dr'$$

where $h_\ell(r)$ is a solution of

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-k^2\right) h_\ell(r) = 0.$$

$$u_\ell(r) = \int_0^{\infty} G(r,r')\, K^{(e)}_{\ell m}(r')\, dr'$$

where

$$G(r,r') = +\frac{(-1)^\ell}{k_\ell} f_\ell(r)\, h_\ell(r') \qquad r'>r$$
$$= +\frac{(-1)^\ell}{k} h_\ell(r')\, f_\ell(r') \qquad r'<r$$

$h_\ell(r)$ being a solution of homogeneous equation of the form $h_\ell(r) = g_\ell(r) + c\, f_\ell(r)$.

$$\left(-\frac{d}{dr^2}+\frac{l(l+1)}{r^2}-k^2\right)\left[\int_0^{\infty}\right.$$

$$= \left(-\frac{d}{dr}+\frac{l(l+1)}{r^2}-k^2\right)\left[\int_0^{r} h_\ell(r')\, K^{(e)}_{\ell m}(r')\, f_\ell(r')\, dr'\right.$$

$$\left. + \int_r^{\infty} f_\ell(r')\, K^{(e)}_{\ell m}(r')\, h_\ell(r')\, dr'\right]$$

$$= +\left[\frac{df(r)}{dr} h(r) - \frac{dh(r)}{dr} f(r)\right] K(r)$$

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Since

$$\frac{d}{dr}\left(\frac{df}{dr}h - \frac{dh}{dr}f\right) = 0,$$

$\frac{df}{dr}h - \frac{dh}{dr}f$ is constant, so what one can evaluate it for $kr \gg 1$:

$$f_\ell(r) \sim \sin\left(kr - \frac{\ell}{2}\pi\right)$$

$$h_\ell(r) \sim (-1)^\ell \cos\left(kr - \frac{\ell}{2}\pi\right) + C \sin\left(kr - \frac{\ell}{2}\pi\right)$$

$$\frac{df_\ell}{dr}h_\ell - \frac{dh_\ell}{dr}f_\ell = (-1)^\ell k \cos^2\left(kr - \frac{\ell}{2}\pi\right) + Ck\cos\left(kr - \frac{\ell}{2}\pi\right)\sin\left(kr - \frac{\ell}{2}\pi\right) + (-1)^\ell k \sin^2\left(kr - \frac{\ell}{2}\pi\right) - Ck\cos\left(kr - \frac{\ell}{2}\pi\right)\sin\left(kr - \frac{\ell}{2}\pi\right) = (-1)^\ell k$$

Now, for $r > r_0$, we have

$$u_\ell(r) = \frac{(-1)^\ell}{k}\int_0^{r_0} f_\ell(r') K_\ell(r')\, dr'\, h_\ell(r)$$

Thus, one may choose $h_\ell(r)$ such that it corresponds to an outgoing wave, namely

$$C = (-1)^\ell i$$

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$$h_l(r) \cong (-1)^l \exp i(kr - \frac{l}{2}\pi)$$

$$= (-1)^{\frac{l}{2}} \exp i(kr)$$

$$\sum_{l,m} K_{lm}^{(e)}(r)\, Y_{ll}^{m}(\theta,\varphi) + \cdots$$

$$= \frac{4\pi}{c}\left(\nabla \times \mathcal{J} + ck^2 \mathcal{M}\right)$$

or

$$K_{lm}^{(e)}(r) = \frac{4\pi}{c}\int \hat{Y}_{ll}^{m*}(\theta,\varphi)\left(\nabla \times \mathcal{J} + ck^2 \mathcal{M}\right) \times d\omega$$

$$u_{lm}^{(e)}(r) = \frac{(-1)^l}{k}\int_0^\infty f_l(r)\, K_{lm}^{(e)}(r)\, dr \cdot h_l(r)$$

$$= \frac{(-1)^l}{k}\,\frac{4\pi}{c}\, h_l(r)\int_0^\infty \frac{1}{r^2} f_l(r)\, Y_{ll}^{m*}(\theta,\varphi) \times \left(\nabla \times \mathcal{J} + ck^2 \mathcal{M}\right) dv$$

$$= (-1)^l \cdot \frac{4\pi}{c}\, h_l(r)\int_0^\infty \frac{f_l(r)}{kr}\,\frac{1}{r}\, Y_{ll}^{m*}(\theta,\varphi)\left(\nabla \times \mathcal{J} + ck^2 \mathcal{M}\right) dv$$

Similarly, one obtains the solutions for Y_{ll}^{m} part of $\mathcal{E}$ by making use of the other equation.
If we add these particular solutions to $\mathcal{E}$ and $\mathcal{H}$ resp. the free space solutions of $\mathcal{E}, \mathcal{H}$ respectively (by taking into account the relations

$$\nabla \times \mathcal{H} = -\frac{i\omega}{c}\mathcal{E}$$

$$\nabla \times \mathcal{E} = \frac{i\omega}{c}\mathcal{H}$$
)

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we leave the general solution of the field equations.

~~Thus, we~~ Summarizing ~~the~~ above results, one can say that the particular solution of Maxwell field equations corresponding to the presence of outgoing wave alone is given by

$$\mathcal{E} = \sum \left\{ a^{(e)}_{lm} \left(\frac{i}{k} \nabla \times \frac{h^{(+)}_l(r)}{kr} Y^m_{ll}(\theta,\varphi) \right) + a^{(m)}_{lm} \left(\frac{h^{(+)}_l(r)}{kr} Y^m_{ll}(\theta,\varphi) \right) \right\}$$

$$\mathcal{H} = \sum \left\{ a^{(e)}_{lm} \left(\frac{h^{(+)}_l(r)}{kr} Y^m_{ll}(\theta,\varphi) \right) + a^{(m)}_{lm} \left(-\frac{i}{k} \nabla \times \frac{h^{(+)}_l(r)}{kr} Y^m_{ll}(\theta,\varphi) \right) \right\}$$

for $r \geq r_0$,

where

$$h_l(r) = g_l(r) + i(-1)^l f_l(r)$$

$$\sim (-1)^l \exp i(kr - \tfrac{1}{2} l\pi)$$

$$u^{(+)}_l(r) = (-1)^l h_l(r)$$

$$\sim \exp i(kr - \tfrac{1}{2} l\pi)$$

$$f_l(r) = \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+\frac{1}{2}}(kr)$$

$$g_l(r) = \left(\frac{\pi kr}{2}\right)^{1/2} J_{-l-\frac{1}{2}}(kr)$$

$$a^{(e)}_{lm} = \frac{4\pi k}{c} \iiint_{\sigma} \frac{f_l(r)}{kr} Y^{m*}_{ll}(\theta,\varphi) (\nabla \times \mathbf{J} + c k^2 \mathbf{M}) \, dv$$

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$$a_{lm}^{(m)} = \frac{4\pi i k^2}{c} \iiint \frac{f_l(r)}{kr} Y_{ll}^{m*}(\theta,\varphi) \left(\mathbf{J} + c\nabla \times \mathbf{M}\right) dv$$

(See Remark **)

By making use of the relation

$$Y_{l,l}^{m}(\theta,\varphi) = \frac{m\, Y_{lm}(\theta,\varphi)}{\pm\sqrt{l(l+1)}},$$

$$= \frac{\mathbf{L}\, Y_{lm}(\theta,\varphi)}{\sqrt{l(l+1)}}$$

$$\mathbf{L} = -i(\mathbf{r} \times \nabla),$$

and, further, assuming $kr_1 \ll 1$, so that

$$\frac{f_l(r)}{kr} \simeq \frac{\pi^{1/2}}{(2kr)} \frac{(2kr)^l l!}{(2l+1)!},$$

$$a_{lm}^{(e)} = \frac{4\pi}{c} \frac{k(2k)^l l! \cdot (i)}{(2l+1)!\sqrt{l(l+1)}} \iiint (\mathbf{r} \times \nabla) \tilde{Y}_{lm}(\theta,\varphi)$$

$$\times r^l \left(\mathbf{J} + c\nabla \times \mathbf{M}\right) dv$$

or See Blatt and Weisskopf, p. 806

$$= \times \nabla \tilde{Y}_{lm}(\theta,\varphi)$$

$$a_{lm}^{(e)} = -\frac{4\pi k^{l+2} 2^l l!}{(2l+1)!} \left(\frac{l+1}{l}\right)^{1/2} (Q_{lm} + Q'_{lm})$$

where

$$Q_{lm} = \int r^l Y_{lm}^*(\theta,\varphi)\, \rho(r)\, dv$$

$$Q'_{lm} = -\frac{ik}{l+1} \int r^l Y_{lm}^*(\theta,\varphi)\, \mathrm{div}(\mathbf{r} \times \mathbf{M})\, dv$$

$$a_{lm}^{(m)} = \frac{4\pi k^{l+2} 2^l \cdot l!}{(2l+1)!} \left(\frac{l+1}{l}\right)^{1/2} (M_{lm} + M'_{lm})$$

$$M_{lm} = -\frac{1}{c(l+1)} \int r^l Y_{lm}^*(\theta,\varphi)\, \mathrm{div}(\mathbf{r} \times \mathbf{J})\, dv$$

$$M'_{lm} = -\int r^l Y_{lm}^*(\theta,\varphi)\, dv\, \mathbf{M} \cdot dv$$

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From this, one can derive readily the intensity of electric and magnetic multipole radiation due to the classical charge-current and magnetic polarization.

In order to go over to the complete quantum mechanical treatment of the electromagnetic field as well as the mechanical system which interacts with the el.mag. field, one may adopt either of two methods.

(i) Quantize directly the free electromagnetic field by regarding $\mathcal{E}$, $\mathcal{H}$ as operators, which satisfy the familiar commutation relations and then re-expand them in terms of vector-spherical harmonics. The interaction between the q.el.mg. field and the q.m. system can be dealed with according to the standard perturbation theory. Although this method is satisfactory from theoretical standpoint, the actual calculations are rather complicated.

(ii) One can make full use of the above classical calculations. Namely,

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~~If the above solution is suitably~~ ~~one can~~ multiplied
~~the~~ the intensity of the radiation ~~divided~~
~~by the corresponding to~~ as calculated
from the integration ~~integrate~~ Poynting vector
over a ~~large~~ sphere of large radius
divided by $\frac{\omega}{c}\hbar$ will give the
number of photons emitted per unit
time. (or the spontaneous transition probability per
unit time.) The calculation of
Q_{lm} etc. can be done quantum
mechanically.

In order to have a more general
transition probability including the
induced as well as the spontaneous
transition, one must again go back
to the comparison with the standard
q.e.d. in which plane waves
instead of the spherical waves are
used for expansion.

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$$\nabla \times \mathcal{H} = -\frac{i\omega}{c}\mathcal{E} + \frac{4\pi}{c}\mathcal{J}$$

$$\nabla \times \mathcal{E} = \frac{i\omega}{c}(\mathcal{H} + 4\pi\mathcal{M})$$

$$\operatorname{div}\mathcal{J} = i\omega\rho$$

$$\nabla\times\nabla\times\mathcal{H} - k^2\mathcal{H} = \frac{4\pi}{c}(\nabla\times\mathcal{J} + c k^2 \mathcal{M})$$

$$\nabla\times\nabla\times\mathcal{E} - k^2\mathcal{E} = \frac{4\pi i k}{c}\left(\mathcal{J} + \frac{c}{ik}\nabla\times\mathcal{M}\right)$$

$$\nabla\times\nabla\times\mathcal{H} = \operatorname{grad}(\operatorname{div}\mathcal{H}) - \Delta\mathcal{H}$$

$$\mathcal{H}^{l(e)}_{lm} = \frac{u_{El}(r)}{kr} Y_{ll}^{m}(\theta,\varphi)$$

$$u_l^{(+)}(r) = (-1)^l g_l(r) + i f_l(r)$$

$$\cong \exp i\left(kr - \tfrac{1}{2}l\pi\right)$$

The vector $Y_{ll}^{m}(\theta,\varphi)$ is transverse ~~i.e.~~ and satisfies

$$\operatorname{div} Y_{ll}^{m}(\theta,\varphi) = 0$$

Thus

$$\operatorname{div}\{i f_{lm}(r) Y_{ll}^{m}(\theta,\varphi)\} = \frac{\partial f(r)}{\partial x} Y$$

$$= (\operatorname{grad} f(r)) Y_{ll}^{m}(\theta,\varphi) + f_{lm}(r) \operatorname{div} Y_{ll}^{m}(\theta,\varphi)$$

$$= \frac{r}{r} f'(r)\, Y_{ll}^{m}(\theta,\varphi)$$

$$= 0$$

because

$$r Y_{ll}^{m}(\theta,\varphi) = -i\, r\,(r\times\nabla)\, Y_{lm}$$

$$= 0$$

or Y_{ll}^{m} is a transverse vector.

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$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-k^2\right)u_{lm}^{(e)}(r)=K_{lm}^{(e)}(r)$$

$$u_{lm}^{(e)}(r)=\frac{1}{ik}\int_0^\infty f_l(r')K_{lm}^{(e)}(r')dr'\cdot u_l^{(+)}(r)$$

for $r>r_0$.

$$\mathcal{E}_{lm}^{\prime(e)}=\frac{i}{k}\left\{\nabla\times\left(\frac{u_{lm}(r)}{kr}Y_{ll}^{m}(\theta,\varphi)\right)\right\}$$

$$\left\{-\frac{4\pi}{c}\mathcal{J}_{lm}^{(e)}(\theta,\varphi)\right\}$$

Here, $\mathcal{H}_{lm}^{\prime(e)}$, $\mathcal{E}_{lm}^{\prime(e)}$ satisfy the relations

$$\nabla\times\mathcal{H}_{lm}^{\prime(e)}=-ik\mathcal{E}_{lm}^{\prime(e)}$$

$$\nabla\times\mathcal{E}_{lm}^{\prime(e)}=\frac{i}{k}\left(k^2\mathcal{H}_{lm}^{\prime e}+\frac{4\pi}{c}\nabla\times\mathcal{J}_{lm}^{(e)}+4\pi k^2\mathcal{M}_{lm}^{(e)}\right)$$

$$\mathcal{E}_{lm}^{(m)}=\frac{v_{lm}}{ikr}Y_{ll}^{(m)}(\theta,\varphi)$$

$r>r_0$

$$\mathcal{H}_{lm}(r)=\frac{1}{ik}\int_0^\infty f_l(r')K_{lm}^{(m)}(r')dr'\times u_l^{(+)}(r)$$

where

$$\frac{4\pi}{c}\left(\nabla\times\mathcal{J}+ck^2\mathcal{M}\right)=\sum_{l,m}K_{lm}^{(e)}(r)Y_{ll}^{m}(\theta\varphi)$$

$+\cdots$

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Remark **

$$\begin{cases} \nabla\times\nabla\times\mathcal{H}^{(e)} - k^2\mathcal{H}^{(e)} = \frac{4\pi}{c}\left(\nabla\times\mathcal{J}^{(e)} + ck^2\mathcal{M}^{(e)}\right) \\ \nabla\times\mathcal{H}^{(e)} = -\frac{i\omega}{c}\mathcal{E}^{(e)} + \frac{4\pi}{c}\mathcal{J}^{(e)} \end{cases}$$

$$\begin{cases} \nabla\times\nabla\times\mathcal{E}^{(m)} - k^2\mathcal{E}^{(m)} = \frac{4\pi ik}{c}\left(\mathcal{J}^{(m)} + c\nabla\times\mathcal{M}^{(m)}\right) \\ \nabla\times\mathcal{E}^{(m)} = \frac{i\omega}{c}\left(\mathcal{H}^{(m)} + 4\pi\mathcal{M}^{(m)}\right) \end{cases}$$

$$\left.\begin{aligned} \nabla\mathcal{H}^{(e)} &= \nabla\mathcal{H}^{(e)} = 0 \\ \nabla\mathcal{E}^{(m)} &= \nabla\mathcal{E}^{(m)} = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \nabla\mathcal{M}^{(e)} &= 0 \\ \nabla\mathcal{J}^{(m)} &= 0 \end{aligned}\right\}$$

$$\mathcal{J} = \mathcal{J}^{(e)} + \mathcal{J}^{(m)}$$

$$\mathcal{J}^{(m)} \sim Y^{m}_{l\,l}$$

$$\mathcal{J}^{(e)} \sim Y^{m}_{l,\,l\pm1}$$

$$\nabla\times\mathcal{J} = \nabla\times\mathcal{J}^{(e)} \sim Y^{m}_{l\,l}$$

$$\mathcal{M} = \mathcal{M}^{(e)} + \mathcal{M}^{(m)}$$

$$\mathcal{M}^{(e)} \sim Y^{m}_{l\,l}$$

$$\mathcal{M}^{(m)} \sim Y^{m}_{l,\,l\pm1}$$

$$\nabla\times\mathcal{M} = \nabla\times\mathcal{M}^{(m)} \sim Y^{m}_{l\,l}$$

$$\frac{4\pi}{c}\left(\nabla\times\mathcal{J}^{(e)} + ck^2\mathcal{M}^{(e)}\right) = \frac{4\pi}{c}\left(\nabla\times\mathcal{J} + ck^2\mathcal{M}^{(e)}\right)$$

$$= \sum_{l,m} K^{(e)}_{lm}(r)\, Y^{m}_{l\,l}(\theta,\varphi)$$

$$K^{(e)}_{lm}(r) = \frac{4\pi}{c}\int_{d\Omega} Y^{m\,*}_{l\,l}(\theta,\varphi)\left(\nabla\times\mathcal{J} + ck^2\mathcal{M}^{(e)}\right)$$

$$= \frac{4\pi}{c}\int_{d\Omega} Y^{m\,*}_{l\,l}(\theta,\varphi)\left(\nabla\times\mathcal{J} + ck^2\mathcal{M}\right)$$

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$$\frac{4\pi i k}{c}\left(\mathcal{J} + c\nabla \times \mathfrak{M}\right)$$

$$= \sum K_{lm}^{(m)}(r)\, Y_{ll}^{m}(\theta,\varphi) + \cdots$$

$$\mathfrak{M}_{lm}^{(m)} = -\frac{i}{k}\left(\nabla \times \mathcal{E}_{lm}^{(m)}\right) - 4\pi \mathfrak{M}_{lm}^{(m)}$$

$$\mathcal{J} = \sum_{lm} \mathcal{J}_{lm}^{(m)}(\theta,\varphi) + \sum_{lm} \mathcal{J}_{lm}^{(e)}(\theta,\varphi)$$

$$\mathcal{J}_{lm}^{(e)} \propto Y_{l,l+1}^{m}(\theta,\varphi)$$

$$\mathfrak{M} = \sum_{lm} \mathfrak{M}_{lm}^{(m)}(\theta,\varphi) + \cdots$$

$$\mathfrak{M}_{lm}^{(m)}$$

$$\nabla \times f(r) Y_{ll}^{m} = \nabla \times \qquad Y = (r \times \nabla) Y_{lm}$$

$$\frac{\partial}{\partial y}\left(f(r)(Y_{ll}^{m})_z\right) - \frac{\partial}{\partial z}\left(f(r)(Y_{ll}^{m})_y\right)$$

$$= f(r)(\nabla \times Y_{ll}^{m})_x + f'(r)\left(\frac{r}{r} \times Y\right)_x$$

$$r \times (r \times \nabla) = r(r\nabla) - r^2 \cdot \nabla$$

$$y\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - z\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial y}\right) - \left(z^2 + y^2\right)\frac{\partial}{\partial x}$$

$$= x(r\nabla) - r^2 \cdot \nabla$$

$$f(r)(\nabla \times Y_{ll}^{m}) + f'(r)\{x(r\nabla) - r^2\nabla\} Y_{lm}^{m}$$

$$(r \times \nabla) \times Y_{lm}$$

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$$K^{(m)}_{lm}(r) = \frac{4\pi i k}{c}\int Y^{m*}_{ll}(\theta,\varphi)\left(j^{(m)} + c\,\nabla\times M^{(m)}\right)d\Omega$$

$$= \frac{4\pi i k}{c}\int Y^{m*}_{ll}(\theta,\varphi)\left(j + c\,\nabla\times M\right)d\Omega$$

$$e^{i\varphi} i kr = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kr)\, Y^*_{lm}(\Theta,\Phi)$$

$$Y_{lm}(\theta,\varphi)$$

$$j_l(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr)$$

$$\simeq \frac{\sin(kr - \frac{1}{2}l\pi)}{kr}$$

$$\int_0^R j_l^2(kr)\, r^2 dr$$

$$\simeq \frac{1}{k^2}\int_0^R \sin^2(kr - \tfrac{1}{2}l\pi)\, dr$$

$$\simeq \frac{1}{k^3}\int_0^{kR} \sin^2 x\, dx$$

$$\simeq \frac{R}{2k^2}$$

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$$V(r_1,r_2) = \sum C^{(r_1,r_2)}_{l_1 m_1, l_2 m_2} Y_{l_1 m_1}(\theta_1,\varphi_1)\, Y_{l_2 m_2}(\theta_2,\varphi_2)$$

$$r_{12} = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_{12}$$

$$\cos\theta_{12} = \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2) + \cos\theta_1 \cos\theta_2$$

$$V(r_{12}) = \sum_l V_l(r_1,r_2) P_l(\cos\theta_{12})$$

$$= \sum_l V_l(r_1,r_2) \sum_{m=-l}^{l} Y^*_{lm}(\theta_1,\varphi_1)\, Y_{lm}(\theta_2,\varphi_2)$$

$$V(r_{12})\cdot S_{12} = \sum V_l(r_1,r_2)\, Y^*_{lm}(\theta_1,\varphi_1)\, Y_{l\pm 2, m\pm 2}(\theta_2,\varphi_2)$$

$$S_{12} = \frac{3(\sigma_1(r_1-r_2))(\sigma_2(r_1-r_2))}{|r_1-r_2|^2} - \sigma_1\sigma_2$$

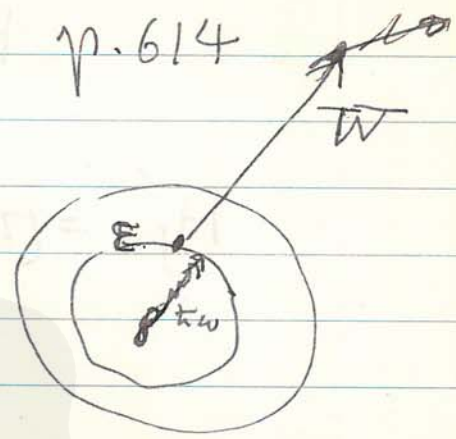
$$\sigma_1(r_1-r_2)\cdot\sigma_2(r_1-r_2)$$
$$= (\sigma_1 r_1)(\sigma_2 r_1) - (\sigma_1 r_1)(\sigma_2 r_2) - (\sigma_1 r_2)(\sigma_2 r_1) + (\sigma_1 r_2)(\sigma_2 r_2)$$
$$= \sigma_{1x}\sigma_{2x}\sin^2\theta_1\sin^2\varphi_1 + \sigma_{1y}\sigma_{2x}\sin^2\theta_1\sin^2\varphi_1 + \sigma_{1z}\sigma_{2x}\cos^2\theta_1 + (\sigma_{1x}\sigma_{2y}+\sigma_{1y}\sigma_{2x})\sin^2\theta_1\sin\varphi_1\cos\varphi_1 + \cdots$$
$$+ \cdots$$
$$= Y_{2m}(\theta_1,\varphi_1),\ Y_{2m}(\theta_2,\varphi_2)$$
$$(Y_{1m}(\theta_1,\varphi_1), Y_{1m}(\theta_2,\varphi_2))$$

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M. N. 39

Internal Conversion

Hatt and Weisskopf, XII. p. 614



$W = \hbar\omega - |\varepsilon|$

$P_{total} = P_{\gamma} + P_{I.C.}$

$= (1+\alpha) P_\gamma$

$P_{I.C.} = \alpha P_\gamma$

α; internal conversion coefficient

$\alpha = \alpha_K + \alpha_L + \cdots$

α: larger for smaller $\hbar\omega$ and larger Z.

Let us consider the following simplified case of K-conversion. We restrict ourselves to the electrostatic interaction and use the non-relat. wave eq. for the electron. Further we use the wave approximation for the ejected electron.

$$v_i = (\pi a^3)^{-1/2} \exp\left(-\frac{r}{a}\right)$$

$$a = \frac{a_0}{Z} \qquad a_0 = \frac{\hbar^2}{me^2}.$$

$$v_f = V^{-1/2} \exp i\mathbf{k}\mathbf{r}$$

$$\text{Transition probability} = \frac{2\pi}{\hbar} |H'_{fi}|^2 \rho(k) d\Omega$$

$$\rho(k) = V \cdot \frac{m\hbar k}{(2\pi\hbar)^3}$$

$$T_{fi}^{(1)} = 2 \cdot \frac{2\pi}{\hbar} \rho(k) \int |H'_{fi}|^2 d\Omega$$

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$$H' = \sum_{i=1}^{Z} \frac{e^2}{|r - R_i|}$$

$$H_{fi}' = (\pi a^3 V)^{-1/2} \sum_{i=1}^{Z} \int (dr) \int (dR_i)^{3Z} \exp(-iPr)$$
$$\times \psi_f^* \frac{e^2}{|r - R_i|} \exp\left(-\frac{r}{a}\right) \psi_i$$

Main contribution to this integral comes from $r > r_i \quad i = 1, 2, \ldots Z$,
so we put

$$\frac{1}{|r - R_i|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{R_i^l}{r^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta_i, \Phi_i)$$

$$H_{fi}' = \frac{e}{(\pi a^3 V)^{1/2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} (Q_{lm})_{fi} J_{lm}$$

$$J_{lm} = \int e^{-\frac{r}{a}} e^{-ikr} r^{-(l+1)} Y_{lm}(\theta, \varphi) (dr)^3$$

We are interested in the case

$$ka >> 1$$

so that $\exp(-r/a) \cong 1$.

$$J_{lm} = 4\pi (i)^{-l} \frac{k^{l-2} \cdot 2^l (l-1)!}{(2l-1)!} Y_{lm}(\Theta, \Phi)$$

where $P = (k, \Theta, \Phi)$

$$H_{fi}' = \frac{(4\pi) e}{(\pi a^3 V)^{1/2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{i^l k^{l-2} \cdot 2^l l!}{(2l+1)!} (Q_{lm})_{fi}$$
$$\times Y_{lm}(\Theta, \Phi)$$

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$$T_{fi}^{(1)} = 128\pi \frac{me^2}{\hbar^3 a^3} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{k^{2l-3}}{[(2l+1)!!]^2} \left|(Q_{lm})_{fi}\right|^2$$

$$\alpha \cong 16 \frac{l}{l+1} \frac{Z^3}{a_0^4} \frac{k^{2l-3}}{\kappa^{2l+1}}$$

$$\kappa = \frac{\omega}{c}$$

$$\frac{\hbar^2 k^2}{2m} \cong \hbar\omega - B_K \cong \hbar\omega$$

$$\alpha_K \cong Z^3 \left(\frac{e^2}{\hbar c}\right)^4 \frac{l}{l+1} \left(\frac{2mc^2}{\hbar\omega}\right)^{l+5/2}$$

0→0 transition is exceptional in that the contribution to the matrix element comes from $r < R_i$ $(i = 1, 2, \cdots 2)$

$$T_{fi}^{0\to 0} = \frac{32}{9} \frac{e^2}{\hbar c} Z^3 \left(\frac{r_0^2}{a_0^2}\right)^2 2\left(\frac{\Delta E}{mc^2}\right)^{1/2} \frac{mc^2}{\hbar}$$

$$\Delta E = E_f - E_i - B_K = \frac{\hbar^2 k^2}{2m}$$

$$T_{fi}^{0\to 0} = 0.5\, Z^3 \left(\frac{\Delta E}{mc^2}\right)^{1/2} 10^4 \text{ sec}^{-1}$$

for $r_0^2 \sim (6 \times 10^{-13} \text{ cm})^2$

ψ_i, ψ_f must have the same parity.

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$$Y_{ll}^{m}(\theta,\varphi) = -i\,(r\times\nabla)Y_{lm}/\sqrt{l(l+1)}$$

$$\nabla\times f(r)Y_{ll}^{m} = f(r)(\nabla\times Y_{ll}^{m}) + \frac{df}{dr}\left(\frac{r}{r}\times Y_{ll}^{m}\right)$$

$$\nabla\times(r\times\nabla) = r\cdot\nabla^2 - (r\nabla)\cdot\nabla - 2\nabla$$

$$r\times(r\times\nabla) = r(r\nabla) - r^2\cdot\nabla$$

$$\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= x\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} - 2\frac{\partial}{\partial x}$$

$$= x\cdot\nabla^2 - (r\nabla)\frac{\partial}{\partial x} - 2\frac{\partial}{\partial x}$$

$$\nabla\times f(r)Y_{ll}^{m} = -i\,f(r)(\nabla^2 - (r\nabla)\nabla - 2\nabla)Y_{lm}/\sqrt{l(l+1)}$$

$$-i\frac{df}{dr}(r(r\nabla) - r^2\nabla)Y_{lm}/\sqrt{l(l+1)}$$

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* Remark! Spin and Parity of the π-mesons:

$$p + p \rightleftarrows d + \pi^+$$

$$(\sigma_{\rightarrow})_{av.} = \frac{2\pi}{\hbar}(|H_{fi}|^2)_{av.}\rho(E_f)/v_i$$

$$(\sigma_{\leftarrow})_{av} = \frac{2\pi}{\hbar}(|H_{if}|^2)_{av.}\rho(E_i)/v_f$$

$$\frac{(\sigma_{\rightarrow})_{av}}{(\sigma_{\leftarrow})_{av}} = \frac{v_f\rho(E_f)}{v_i\rho(E_i)}$$

provided that $(|H_{fi}|^2)_{av}$ is the same for → and ←.

→ : $p + p \rightarrow d + \pi^+$

spin 0 mesons: $^1S \rightarrow {}^3S + S$; $^3P \rightarrow {}^3S + P$

$$p_\mu = \frac{}{\sqrt{1-(\frac{v_f}{c})^2}}$$

$$\frac{\rho(E_f)}{\rho(E_i)} = \frac{p_\mu c}{\sqrt{1-(\frac{v_\mu}{c})^2}} \cdot / p_p \cdot Mc$$

$$E_\mu^2 \cdot \frac{v_\mu}{c} / Mv_p \cdot Mc^2$$

$$\frac{v_\mu^2}{v_p^2} \cdot \frac{E_\mu^2}{(Mc^2)^2}$$

If spin of π-meson is larger than 0, we have

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May, 1953

E. Wichmann

(1)

Study of the quantization of wave fields by using spherical coordinates

I. Scalar Field

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x)$$

Fourier-Bessel transform for $l \geq 0$

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty p^2 f(p)\, j_l(px)\, dp$$

$$f(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 F(x)\, j_l(px)\, dx$$

$$\frac{2}{\pi} \int_0^\infty p^2 j_l(px')\, j_l(px)\, dp = \frac{1}{x^2}\delta(x-x')$$

$$\frac{d^2 j_l(kx)}{dx^2} + \frac{2}{x}\frac{d j_l(kx)}{dx} + \left(k^2 - \frac{l(l+1)}{x^2}\right) j_l(kx) = 0$$

$$\frac{d}{dx}\left(x^2 \frac{d j_l}{dx}\right)$$

$$\frac{d^2 j_l(k'x)}{dx^2} + \frac{2}{x}\frac{d j_l(k'x)}{dx} + \left(k'^2 - \frac{l(l+1)}{x^2}\right) j_l(k'x) = 0$$

$$\int_0^\infty x^2 (k^2 - k'^2)\, j_l(kx)\, j_l(k'x)\, dx$$

$$= -\int_0^\infty \left[j_l(k'x) \frac{d}{dx}\left(x^2 \frac{d j_l(kx)}{dx}\right) - j_l(kx) \frac{d}{dx}\left(x^2 \frac{d j_l(k'x)}{dx}\right)\right] dx$$

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$$= -\left[j_\ell(k'x)\, x^2 \frac{d j_\ell(kx)}{dx} - j_\ell(kx)\, x^2 \frac{d j_\ell(k'x)}{dx} \right]_0^\infty$$

$$\mathcal{F}_{\ell+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{\ell+\frac{1}{2}} \left(-\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$j_{\ell+\frac{1}{2}}(x) = x^\ell \left(-\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$j_\ell(kx) = (kx)^\ell \left(-\frac{1}{kx}\frac{d}{d(kx)}\right)^\ell \frac{\sin(kx)}{kx}$$

$$\frac{d j_\ell(kx)}{dx} = k(kx)^{\ell-1} \left(-\frac{1}{kx}\frac{d}{d(kx)}\right)^\ell \frac{\sin(kx)}{kx}$$

$$-k(kx)^{\ell+1} \left(-\frac{1}{kx}\frac{d}{d(kx)}\right)^{\ell+1} \frac{\sin(kx)}{kx}$$

$$= \frac{1}{x} j_\ell(kx) - k\, j_{\ell+1}(kx)$$

$$= \left[kx^2 j_\ell(k'x) j_{\ell+1}(kx) - k'x^2 j_\ell(kx) j_{\ell+1}(k'x) \right]_0^\infty$$

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$$\{\psi(\vec{x}), \dot{\psi}(\vec{x}')\} = i\hbar\,\delta(\vec{x}-\vec{x}')$$

$$\psi(\vec{r}) = \sqrt{\frac{2}{\pi}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty x^2 q_{lm}(x) j_l(xr) \times dx\, Y_{lm}(\theta,\varphi)$$

$$\dot{\psi}(\vec{r}) = \sqrt{\frac{2}{\pi}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty x^2 p^*_{lm}(x) j_l(xr) \times dx\, Y_{lm}(\theta,\varphi)$$

$$\left.\begin{aligned} p_{l,-m}(x) &= p^*_{l,m}(x) \\ q_{l,-m}(x) &= q^*_{l,m}(x) \end{aligned}\right\}$$

$$\{q_{l_1 m_1}(x_1),\ p_{l_2 m_2}(x_2)\}$$

$$= i\hbar\,\delta_{l_1 l_2}\cdot\delta_{m_1 m_2}\cdot\frac{1}{x_1^2}\,\delta(x_1-x_2)$$

$$\mathcal{H} = \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int x^2 dx \left[p^*_{lm}(x)\,p_{lm}(x) + \omega_x^2\, q^*_{lm}(x)\, q_{lm}(x)\right]$$

$$\omega_x = c\sqrt{\mu^2 + x^2}$$

$$q_{lm}(x) = \frac{1}{x}\sqrt{\frac{\hbar}{2\omega_x}}\left[a_{lm}(x) + a^*_{l,-m}(x)\right]$$

$$p_{lm}(x) = \frac{i}{x}\sqrt{\frac{\hbar\omega_x}{2}}\left[a^*_{lm}(x) - a_{l,-m}(x)\right]$$

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$$[a_{l_1 m_1}(x_1),\ a^{*}_{l_2 m_2}(x_2)] = \delta_{l_1 l_2}\,\delta_{m_1 m_2}\,\delta(x_1 - x_2)$$

$$\mathcal{H} = \frac{\hbar}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty}\Big[a^{*}_{lm}(x)\,a_{lm}(x) + a_{lm}(x)\,a^{*}_{lm}(x)\Big]_0^{\omega x} dx$$

$$= \frac{\hbar}{2}\sum\sum\int \omega(x)\,dx\,\big[2N_{lm}(x) + 1\big]$$

$$\dot{a}_{lm}(x) = -i\omega(x)\,a_{lm}(x)$$

$$\psi(\vec{r},t) = \sqrt{\frac{\hbar}{\pi}}\sum\sum\int \frac{x\,dx}{\sqrt{\omega(x)}}\,j_l(xr)$$

$$\Big[a_{lm}(x)\,e^{-i\omega(x)t}\,Y_{lm}(\theta,\varphi) + a^{*}_{lm}(x)\,e^{i\omega(x)t}\,Y^{*}_{lm}(\theta,\varphi)\Big]$$

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Angular momentum

$$\vec{j} = -\frac{i}{2}\int\int\int_{\hat{r}} d\Omega_r \left(\dot{\psi}(F)(\vec{\gamma}\psi) + (\vec{\gamma}\psi)\dot{\psi}\right)$$

$$\vec{\gamma} = -i\vec{r}\times\nabla \qquad \vec{\gamma}\, j_\ell(\kappa r) = 0$$

$$\vec{j} = -\frac{i}{2}\sum\sum\sum\int \kappa^2 d\kappa \left[\psi_{\ell_1 m_1}(\kappa) g_{\ell_2 m_2}(\kappa) + g_{\ell_2 m_2}(\kappa)\psi_{\ell_1 m_1}(\kappa)\right](\ell_1 m_1|\vec{\gamma}|\ell_2 m_2)$$

$j_z = \hbar m$ $\quad m$: 0 or positive integer

$j_x + i j_y = \cdots$

$j_x - i j_y = \cdots$

$\vec{j}\times\vec{j} = i\hbar\vec{j}$

$(\vec{j})^2 = \hbar^2 n(n+1)$

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II. Photon Field

$$\vec{\gamma} = -i\vec{r}\times\nabla$$

$$\vec{u}_{lm} = \frac{\vec{\gamma}}{\sqrt{l(l+1)}} j_l(xr) Y_{lm}(\theta,\varphi)$$

$$\vec{v}_{lm}(\vec{r},x) = \frac{1}{x}\nabla\times\vec{u}_{lm}(\vec{r},x)$$

$$\nabla\cdot\vec{v}_{lm} = 0$$

$$\nabla\vec{u}_{lm} = 0$$

$$\left.\begin{aligned}\Delta\vec{u}_{lm} &= -x^2\vec{u}_{lm}\\ \Delta\vec{v}_{lm} &= -x^2\vec{v}_{lm}\end{aligned}\right\}$$

$$([\Delta,\vec{\gamma}]=0,\quad [\Delta,\nabla]=0)$$

$$\iiint d\tau_{\vec{r}}\ \vec{u}^*_{l_1m_1}(\vec{r},x_1)\vec{u}_{l_2m_2}(\vec{r},x_2)$$

$$= \frac{-1}{\sqrt{l_1l_2(l_1+1)(l_2+1)}}\iiint d\tau_{\vec{r}}\ \vec{u}^*_{l_1m_1}\gamma^2 u_{l_2m_2}$$

$$= \delta_{l_1l_2}\delta_{m_1m_2}\frac{1}{x_1^2}\delta(x_1-x_2)\frac{\pi}{2}$$

$$u_{lm} = j_l(xr)Y_{lm}(\theta,\varphi)$$

$$\gamma^2 u_{lm} = -l(l+1)u_{lm}$$

$$\iiint d\tau_{\vec{r}}\ \vec{v}^*_{l_1m_1}(\vec{r},x_1)\vec{v}_{l_2m_2}(\vec{r},x_2)$$

$$= \frac{1}{x_1^2}\delta(x_1-x_2)\frac{\pi}{2}\delta_{l_1l_2}\delta_{m_1m_2}$$

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$$\iiint d\tau_{\vec{r}}\ \vec{u}^{*}_{l_1 m_1}(\vec{r}, x_1)\,\vec{v}_{l_2 m_2}(\vec{r}, x_2)$$

$$= 0$$

$$\vec{A}(\vec{r}) = \frac{\sqrt{2}}{\sqrt{\pi \varepsilon_0}} \sum \sum \int dx \Big[a_{lm}(x)\,\vec{u}_{lm}(\vec{r}, x) + b_{lm}(x)\,\vec{v}_{lm}(\vec{r}, x) \Big]$$

$$\vec{A}^{*} = \vec{A}$$

$$\mathcal{L} = \frac{1}{2} \int d\tau_{\vec{r}} \left[\varepsilon_0 \vec{E}^2 - \mu_0 \vec{H}^2 \right]$$

$$\vec{E} = -\dot{\vec{A}}, \qquad \vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$$

$$a_{l,m}(x) = \sqrt{\frac{\hbar}{2\omega}} \left[Q^{E}_{lm}(x) + Q^{*E*}_{l,-m}(x) \right]$$

$$b_{lm}(x) = -i \sqrt{\frac{\hbar}{2\omega}} \left\{ Q^{M}_{lm}(x) + Q^{M*}_{l,-m}(x) \right\}$$

$$\left[Q^{(s_1)}_{l_1 m_1}(x_1),\ Q^{(s_2)*}_{l_2 m_2}(x_2) \right] = \delta_{s_1 s_2} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta(x_1 - x_2)$$

$$\mathcal{H} = \frac{\hbar}{2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} d\omega\, \omega(x) \Big\{ \Big[Q^{E*}_{lm}(x) Q^{E}_{lm}(x) + Q^{E}_{lm}(x) Q^{E*}_{lm}(x) \Big] + \Big[Q^{M*}_{lm}(x) Q^{M}_{lm}(x) + Q^{M}_{lm}(x) Q^{M*}_{lm}(x) \Big] \Big\}$$

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$$\vec{A}(\vec{r}) = \sqrt{\frac{\hbar}{\pi c}} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} \frac{c\,dk}{\sqrt{\omega(k)}}$$

$$\{[Q^{E}_{lm}(k)\,\vec{u}_{lm}(\vec{r},k) + Q^{E*}_{lm}(k)\,\vec{u}^{*}_{lm}(\vec{r},k)]$$

$$* -i\left[Q^{M}_{lm}(k)\,v_{lm}(\vec{r},k) - Q^{M*}_{lm}(k)\,v_{lm}(\vec{r},k)\right]\}$$

$$Q^{E}_{lm}(k) = Q^{E}_{lm}(k)\,e^{-i\omega(k)t}$$

etc.

$$\vec{J} = \frac{1}{2c^2} \iiint_{\tau_{\vec{r}}} d\tau_{\vec{r}}\ \vec{r} \times [\vec{E} \times \vec{H} - \vec{H} \times \vec{E}]$$

$$J_z = \hbar \sum\sum m \int_0^{\infty} dk \left[Q^{E*}_{lm}(k)\,Q^{E}_{lm}(k) + Q^{M*}_{lm}(k)\,Q^{M}_{lm}(k) \right]$$

$$[\vec{J} \times \vec{J}] = i\hbar \vec{J}$$

$$J^2 = \hbar^2 l(l+1)$$