NOTE-BOOK

Differentiale Rechnung
Band II.

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N206

(4) $y=\varphi(x)$ ヲ interval I ニ於ケル x ノ fn トス.

而シテ x ノ一ツノ極限値ヲ a.

$z=F(y)$ ヲ interval I' ニ於ケル conti fn トス.

而シテ $\lim\limits_{x=a} \varphi(x) = b$ トス.

or $|\varphi(a+\sigma)-b|<\varepsilon \qquad |\sigma|\leq\delta$.

or $b-\varepsilon < \varphi(a+\sigma) < b+\varepsilon, \qquad |\sigma|\leq\delta$

又 $\lim\limits_{y=b\pm 0} F(y)=F(b)$

即チ. $|F(b+\sigma')-F(b)|<\varepsilon' \qquad |\sigma'|\leq\delta'$

今 $0\leq\varepsilon\leq\delta'$ トスレバ $|\sigma|$ ト ε トハ同ジ條件ニ従フカラ,

$|F\{\varphi(a+\sigma)\}-F(b)|<\varepsilon' \qquad |\sigma|\leq\delta$.

$$\therefore \lim_{x=a} F\{\varphi(x)\} = F(b) = F\{\lim_{x=a}\varphi(x)\}$$

$y=\varphi(x)$ ヲ x ノ cont. fn トス. $A\leq x\leq B \to C\leq y\leq D$

次ニ $z=F(y)$ ヲ y ノ cont fn トス. $C\leq y\leq D$.

然ルトキハ

$$\left.\begin{array}{l} \lim\limits_{x=a\pm 0}\varphi(x)=\varphi(a) \\ \\ \lim\limits_{y=b\pm 0}F(y)=F(b) \end{array}\right\} \text{by def}$$

今 $\varphi(a)=b$ トオケバ 前ノ証明ニヨリ.

$$\lim_{x=a\pm 0} F\{\varphi(x)\} = F\{\lim_{x=a\pm 0}\varphi(x)\} = F\{\varphi(a)\}$$

∴ x ノ cont. fn y ノ cont. fn z ハ x ノ cont. fn デアル.

(一般ニ　$y=\varphi(x)$　$z=F(y)$　$w=G(z)$ . . . .)

即チ一ツノ cont fn ノ cont fn ハ矢張リ cont fn ナリ.

Example 1.　$y=f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots\cdots$
$+a_{n-1}x+a_n$　n; pos. int

§2, ノ Ex1 ニヨリ　Ax ハ x ノスベテノ値ニ於テ cont ナリ.

∴ A_kx^k (k; pos. int.) ハスベテノ有限ノ値ニ於テ cont.

従ッテ之等ノ和ナル polynomial ハ x ノ凡テノ有限ノ値ニ於テ cont ナリ.

2.　$y=f(x)=\dfrac{a_0x^n+a_1x^{n-1}+\cdots\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots\cdots+b_{m-1}x+b_m}$　m, n; pos. int

ソノ分母子ハ Ex1. ニヨリ x ノ凡テノ有限ノ値ニ於テ cont ナリ.

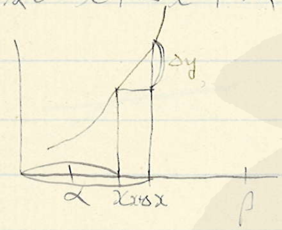
∴ 其ノ quotient タル rational fraction ハ x ノ凡テノ有限ノ値ニ於テ cont ナリ. 但シ分母ヲ0ナラシムル x ノ値ハハブク.

3.　$y=f(x)=\tan x=\dfrac{\sin x}{\cos x}$

§2 ノ例2ニヨリ $\sin x$ ハ x ノスベテノ有限ノ値ニ於テ cont ナリ. 同様ニ $\cos x$ モ cont. ∴ 其ノ商ナル $\tan x$ モ亦 x ノスベテノ値ニ於テ cont. ($\cos x=0$ ノ点ハハブク)

Differentiation of Functions 微分法.

§1. Differential quotient 微(分)商　$y=f(x)$ ヲ interval (α,β) ニ於ケル x ノ cont fn トス。(α,β) 内ニ於ケル x ノ二ツノ値ヲ x 及ビ $x+\Delta x$ トス。コノ二ツノ値ニ対応スル fn ノ値ヲ y 及ビ $y+\Delta y$ トスレバ、fn ノ受クル変化ハ



$$\Delta y = f(x+\Delta x) - f(x)$$

ニテアラハサル。

今 x ヲ fix ニシテオケバ、

$$\frac{\Delta y}{\Delta x}$$ ハ Δx ノ fn デアル。

コレニ於テ Δx ヲ 0 ニ converge セシムル時ハ次ノ三ノ場合ヲ考フルコトヲ得.

〔1〕 Limiting value ガ finite determinate ナル場合

〔2〕 $+\infty$ or $-\infty$ ナル場合

〔3〕 不定ノ場合.

其ノ内最初ノ二ツノ場合ニ於テハ x ナル点ニ於テ differential quotient ガ存在スルト称シ 第三ノ場合ニ於テハ存在セズトイフ.

Δx ガ 0 ニ converge スル時. positive side カラ diminish スル時、negative side カラ ~~increase~~ diminish スル時トアリ.

$$\lim_{\Delta x=+0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = X_1 \text{ : Progressive diff. quot.}$$

$$\lim_{\Delta x=-0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = X_2 \text{ : Regressive dif. quot.}$$

x_1 ト x_2 トノ異ナル時トモ一般ニハ $x_1 = x_2$.
等シキ場合ニ於テハ其ノ点ニ於テ complete diff. quotient 或ハ
略シテ dif. quotient ガ存在スルト称ス.

dif. quo ヲ表ハスニ次ノ如キ記号ヲ用フ.

$y = f(x)$ トスレバ

$$\frac{dy}{dx},\quad y',\quad f'(x),\quad \frac{d\,f(x)}{dx},\quad D_x y,\quad D_x f(x),$$

函数ノ dif. quo ヲ求ムル運算ヲ施スコトヲ, 其ノ函数ヲ differentiate スルト称ス.

§2. Differentiation ニ関スル一般ノ法則.

① (1) $\dfrac{dk}{dx} = 0$;(k: constant) $\quad y = f(x) = k \quad f(x+\Delta x) = k$.

(2) $\dfrac{d\,x^n}{dx} = n x^{n-1}$; ($n$: 正負ノ整数分数)

(3) $\dfrac{d\,a^x}{dx} = a^x \log a$

$$\lim_{h=0} \frac{a^{x+h} - a^x}{h} = \lim \frac{a^x(a^h - 1)}{h}$$

$$a^h - 1 = \frac{1}{z} \quad h = \log_a\left(1 + \frac{1}{z}\right)$$

$$\therefore = \lim_{h=0} \frac{a^x \frac{1}{z}}{\log_a\left(1+\frac{1}{z}\right)} = \lim_{z=\infty} \frac{a^x}{\log_a e}$$

$$= a^x \log a$$

(4) $\dfrac{de^x}{dx} = e^x$

(5) $\dfrac{d\log_a x}{dx} = \dfrac{1}{x\log_e a}$

$$\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{h} \qquad \begin{cases}\frac{x}{h}=z\\ \frac{1}{h}=\frac{z}{x}\end{cases}$$

$$= \lim_{z\to\infty}\frac{1}{x}\log_a\left(1+\frac{1}{z}\right)^z$$

(6) $\dfrac{d\log x}{dx} = \dfrac{1}{x}$

(7) $\dfrac{d\sin x}{dx} = \cos x$

(8) $\dfrac{d\cos x}{dx} = -\sin x$

(9) $\dfrac{d\tan x}{dx} = \dfrac{1}{\cos^2 x}$

(10) $\dfrac{d\cot x}{dx} = -\dfrac{1}{\sin^2 x}$

(11) $\dfrac{d\sec x}{dx} = \dfrac{\sin x}{\cos^2 x}$

(12) $\dfrac{d\,\mathrm{cosec}\,x}{dx} = -\dfrac{\cos x}{\sin^2 x}$

② Sum ~~of~~ differentiation

$y = y_1 + y_2$ トスレバ

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} \quad \left(\because \begin{aligned} y+\Delta y &= y_1+\Delta y_1 + y_2 + \Delta y_2 \\ y &= y_1 + y_2 \\ \Delta y &= \Delta y_1 + \Delta y_2 \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta y_1}{\Delta x} + \frac{\Delta y_2}{\Delta x}\end{aligned}\right.$$

両辺 lim ヲトレバ

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx})$$

例ヘバ 1. $y = \sin x + \cos x$

$$\frac{dy}{dx} = \cos x - \sin x$$

2. $y = \cancel{ax+b}\ x^3 + x$

$$\frac{dy}{dx} = 3x^2 + 1.$$

一般ニ $y = y_1 + y_2 + \cdots\cdots + y_n.$

ナルトキハ $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \cdots\cdots + \frac{dy_n}{dx}.$

③ Product ノ Diff.

$y = y_1 \cdot y_2$ トスレバ

$$\frac{dy}{dx} = y_2\frac{dy_1}{dx} + y_1\frac{dy_2}{dx}.$$

$$(\because \quad y + \Delta y = (y_1 + \Delta y_1)(y_2 + \Delta y_2)$$

$$y = y_1 y_2 \ \cancel{+y_2}$$

$$\Delta y = y_2 \Delta y_1 + (y_1 + \Delta y_1)\Delta y_2$$

$$\frac{\Delta y}{\Delta x} = y_2\frac{\Delta y_1}{\Delta x} + (y_1 + \Delta y_1)\frac{\Delta y_2}{\Delta x}.$$

$$\frac{dx}{dx} = y_2\frac{dy_1}{dx} + y_1\frac{dy_2}{dx}.)$$

例ヘバ 1. $y = x^2 \log x.$

$$\frac{dy}{dx} = \log x \cdot 2x + x.$$

2. $y = \sin x \cdot \cos x$

$$\frac{dy}{dx} = \cos^2 x + \sin x(-\sin x)$$
$$= \cos^2 x - \sin^2 x.$$

$$\rho = (\sqrt{\theta}+\sqrt{a})(\sqrt{\theta}+\sqrt{b}). \quad (\rho = f(\theta))$$
$$\frac{d\rho}{d\theta} = (\sqrt{\theta}+\sqrt{b})\frac{1}{2}\theta^{-\frac{1}{2}} + (\sqrt{\theta}+\sqrt{a})\frac{1}{2}\theta^{-\frac{1}{2}}.$$
$$= \frac{2\sqrt{\theta}+\sqrt{a}+\sqrt{b}}{2\theta^{\frac{1}{2}}}$$

Cor. $y_1 = C$ ナル時ハ

$$\frac{dy_1}{dx} = 0.$$
$$\therefore \quad \frac{dy}{dx} = c\,\frac{dy_2}{dx}.$$

ex. (1) $y = a x^n$
$$\frac{dy}{dx} = a_0 n x^{n-1}$$

(2) 一般ニ $y = a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-1} x + a_n$
$$\frac{\partial y}{\partial x} = n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \cdots\cdots + a_{n-1}$$

$y = y_1 \cdot y_2 \cdot y_3$
$$\frac{\partial y}{\partial x} = y_3 \frac{\partial(y_1 \cdot y_2)}{dx} + y_1 y_2 \frac{\partial y_3}{\partial x}$$
$$= y_3\left\{y_2 \frac{\partial y_1}{dx} + y_1 \frac{dy_2}{dx}\right\} + y_1 y_2 \frac{\partial y_3}{\partial x}.$$

$$= y_2 y_3 \frac{\partial y_1}{\partial x} + y_1 y_3 \frac{\partial y_2}{\partial x} + y_1 y_2 \frac{\partial y_3}{\partial x}$$

ex. $y = x^2 \cdot \log x \cdot \sin x$

$y' = 2x \cdot \log x \cdot \sin x + x \sin x + \cos x \cdot x^2 \cdot \log x.$

一般に. $y = y_1 y_2 y_3 \cdots\cdots y_n$ ならば

$$\frac{\partial y}{\partial x} = y_2 y_3 \cdots y_n \frac{\partial y_1}{\partial x} + y_1 y_3 \cdots y_n \frac{\partial y_2}{\partial x} + \cdots\cdots + y_1 y_2 \cdots y_{n-1} \frac{\partial y_n}{\partial x}.$$

④ quotient の diff.

$$y = \frac{y_1}{y_2}$$

$$\frac{\partial y}{\partial x} = \frac{y_2 \frac{\partial y_1}{\partial x} - y_1 \frac{\partial y_2}{\partial x}}{y_2^2}$$

$$(\because\ y + \Delta y = \frac{y_1 + \Delta y_1}{y_2 + \Delta y_2}.$$

$$y = \frac{y_1}{y_2}$$

$$\Delta y = \frac{y_2 \Delta y_1 - y_1 \Delta y_2}{y_2 (y_2 + \Delta y_2)}$$

$$\frac{\Delta y}{\Delta x} = \frac{y_2 \frac{\Delta y_1}{\Delta x} - y_1 \frac{\Delta y_2}{\Delta x}}{y_2 (y_2 + \Delta y_2)}$$

$$\frac{\partial y}{\partial x} = \frac{y_2 \frac{\partial y_1}{\partial x} - y_1 \frac{\partial y_2}{\partial x}}{y_2^2}$$

例へば. $y = \tan x = \dfrac{\sin x}{\cos x}.$

$$\frac{\partial y}{\partial x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

$$s = \frac{1-t^2}{1+t}$$

$$\frac{\partial s}{\partial t} = \frac{(1+t)(-2t)-(1-t^2)}{(1+t)^2} = \frac{-1-2t-t^2}{(1+t)^2} = -1.$$

Cor. $$y = \frac{c}{y_2}$$

$$\frac{\partial y_1}{\partial x} = 0.$$

$$\frac{\partial y}{\partial x} = \frac{-c\frac{\partial y_2}{\partial x}}{y_2^2}.$$

Ex. $$y = \sec x = \frac{1}{\cos x}.$$

$$\frac{\partial y}{\partial x} = \frac{-1\cdot -\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

(5) Differentiation of a fn of a fn.

$$z = f_1(y) \qquad y = f_2(x)$$

x = Δx ナル increment ヲ与フル時, z = corespond スル y ト z ノ incre.
夫々 Δy, Δz トスレバ.

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\cdot\frac{\Delta y}{\Delta x}.$$

lim ヲトレバ.

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}.$$

例ヘバ $$z = \tan^2 x$$

$$z = y^2 \qquad y = \tan x.$$

$$\frac{dz}{dx} = 2y\cdot\frac{1}{\cos^2 x} = 2\frac{\sin x}{\cos^3 x}.$$

$z = \log \cos x.$

$z = \log y \qquad y = \cos x.$

$$\frac{dz}{dx} = \frac{1}{y} \cdot -\sin x = -\tan x.$$

$z = \sin^n x$

$z = y^n \qquad y = \sin x$

$$\frac{dz}{dx} = n y^{n-1} \cos x = n \sin^{n-1} \cos x.$$

一般ニ，　$z = f_1(y_1) \quad y_1 = f_2(y_2), \quad \cdot \quad y_2 = f_3(y_3)$

$y_{n-1} = f_n(x)$　　十n個，

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y_1} \cdot \frac{\Delta y_1}{\Delta y_2} \cdot \ldots\ldots \frac{\Delta y_{n-1}}{\Delta x}$$

lim ニ移セバ

$$\frac{dz}{dx} = \frac{dz}{dy_1} \cdot \frac{dy_1}{dy_2} \cdot \ldots\ldots \frac{dy_{n-1}}{dx}.$$

Exa. $\rho = \cos^m k\theta$

$\rho = y_1^m \qquad y_1 = \cos y_2 \qquad y_2 = k\theta.$

$$\therefore \frac{d\rho}{d\theta} = m y_1^{m-1} (-\sin y_2) \cdot k. = -mk \cos^{m-1} \cancel{\cos} k\theta \sin k\theta.$$

$z = \log \cot^n x$

$z = \log y_1 \qquad y_1 = \cancel{\cot} y_2^n \qquad y_2 = \cot x.$

$$\frac{dz}{dx} = \frac{1}{y_1} \cdot n y_2^{n-1} \left(-\frac{1}{\sin^2 x}\right) = \frac{1}{\cot^n x} \, n \cot^{n-1} x \left(-\frac{1}{\sin^2 x}\right).$$

$$= -n\frac{1}{\cos x \sin x}.$$

$$z = \sqrt{a^2 - x^2}. \qquad z = y_1^{\frac{1}{2}},\ y_1 = a^2 - x^2,$$

$$\frac{dz}{dx} = \frac{1}{2}y_1^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{(a^2-x^2)^{\frac{1}{2}}}$$

(1) $y = (a+bx)^2$ $\qquad \frac{dy}{dx} = 2(a+bx)\cdot b.$

(2) $y = x(k+x)(l+x)$ $\qquad \frac{dy}{dx} = (k+x)(l+x) + x(k+x) + x\cdot(k+x)$ $= 3x^2 + 2(k+l)x + kl.$

(3) $y = \frac{k+lx}{m+nx}.$ $\qquad \frac{dy}{dx} = \frac{l(m+nx) - n(k+lx)}{(m+nx)^2}$

(4) $y = \frac{a+bx}{A+Bx+Cx^2}$ $\qquad y' = \frac{b(A+Bx+Cx^2) - (a+bx)(B+2Cx)}{A+Bx+Cx^2}$

(5) $y = a + \frac{b}{x^m} + \frac{c}{x^n}.$ $\qquad \text{〃} = \frac{-b\, m x^{m-1}}{x^{2m}} + \frac{-c x^{n-1}}{x^{2n}}$

(6) $y = \left(a + \frac{b}{x^m} + \frac{c}{x^n}\right)^2$ $\qquad \text{〃} = 2\left(a + \frac{b}{x^m} + \frac{c}{x^n}\right)(\qquad)$

(7) $y = \sqrt{(a+x)(b+x)}$ $\qquad \text{〃} = \frac{1}{2}\left(\frac{d\,(a+x)(b+x)}{dx}\right)^{-\frac{1}{2}}$

$$y = (a+x)^{\frac{1}{2}}(b+x)^{\frac{1}{2}}. \qquad = \frac{1}{2}(a+b+2x)^{-\frac{1}{2}}.$$

$$y' = \frac{1}{2}(a+x)^{-\frac{1}{2}}(b+x)^{\frac{1}{2}} + \frac{1}{2}(b+x)^{-\frac{1}{2}}(a+x)^{\frac{1}{2}}$$

$$y = y_1^{\frac{1}{2}} \qquad y_1 = (a+x)(b+x).$$

$$y = \frac{1}{2}y_1^{-\frac{1}{2}} \cdot (a+b+2x)$$

$\frac{d(a+x)}{dx} = \frac{\quad}{dx}$

$y_1 = a+x$

(8) $y = \sqrt{a+x} - \sqrt{b-x}$. $\quad y' = \frac{1}{2} \frac{1}{y_1^{\frac{1}{2}}} + \quad = \frac{1}{2\sqrt{a+x}} + \frac{1}{2\sqrt{b-x}}$

(9) $y = \sqrt[n]{a+bx+cx^2}$ $\quad y' = n(a+bx+cx^2)^{n-1}(b+2cx)$

(10) $y = x^m(a+bx^n)^p$ $\quad y' = \quad +$

$m(a+bx^n)^p x^{m-1} + x^m \frac{d(a+bx^n)^p}{dx}$

$= \quad +$

(11) $y = \frac{\sqrt{a-x}}{\sqrt{a}+\sqrt{x}}$ $\quad y' = \frac{\sqrt{a}+\sqrt{x}\frac{d\sqrt{a-x}}{dx} - \sqrt{a-x}\frac{d\sqrt{a}+\sqrt{x}}{dx}}{(\sqrt{a}+\sqrt{x})^2} = \frac{\quad}{(\sqrt{a}+\sqrt{x})^2}$

(12) $y = \frac{k}{\sqrt{2kx-x^2}}$ $\quad y' = \frac{\frac{1}{2}k(2kx-x^2)^{-\frac{1}{2}}(2k-2x)}{2kx-x^2} = \frac{\quad}{\quad}$

(13) $y = xe^{-x}$ $\quad y' = e^{-x} + xe^{-x}(-1)$

(14) $y = \left(\frac{1+\sqrt[3]{x}}{1-\sqrt[3]{x}}\right)^{\frac{1}{2}}$

(15) $y = x^m e^{-x^3}$ $\quad y' = e^{-x^3} m x^{m-1} + x^m \cdot -3x^2 \cdot e^{-x^3}$

$\frac{de^{-x^3}}{dx^3} = e^{-x^3}$. $\quad \frac{d-x^3}{dx} = -3x^2$ $\quad =$

(16) $y = a^{-\frac{1}{x}}$. $\quad y' = a^{-\frac{1}{x}} \log a \frac{1}{x^2}$

$\frac{d\,-\frac{1}{x}}{dx} = \frac{1}{x^2}$

(17) $y = \frac{x}{e^x-1}$ $\quad y = \frac{e^x-1 - x\frac{d(e^x-1)}{dx}}{(e^x-1)^2} = \frac{e^x-1-xe^x}{(e^x-1)^2}$.

c034-091挟込

⑥ Differentiation of inverse fns

$y = f(x)$ ヲ $x = \varphi(y)$ ノ inverse fn トス.

今 Δx 及び Δy ヲ夫々ノ increment トスレバ,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\Delta x}{\Delta y}}$$

lim ヲ取レバ $$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

1. 例ヘバ $\dfrac{d}{dx}a^x = a^x \log a$

($y = a^x$ ノ inverse fn ハ $\log_a x$ デアル)

$y = \log_a x$ $\quad x = a^y$

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{a^y \log a} = \frac{1}{x \log a}$$

2. $y = \sin^{-1} x$ $\qquad x = \sin y.$ $\qquad \cos y =$

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\cos y} = \pm\frac{1}{\sqrt{1-x^2}}.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad y : \left(-\frac{\pi}{2} — \frac{\pi}{2}\right)$$

$y = \cos^{-1} x$ $\qquad x = \cos y.$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = \mp\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad y(0 - \pi)$$

$y = \tan^{-1}x \quad x = \tan y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cancel{\cos}\ \sec^2 y} = \frac{1}{1+x^2}$$

$y = \cot^{-1}x \quad x = \cot y$

$$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

$1-\cos^2 y = 1 - \frac{1}{x^2}$

$y = \sec^{-1}x \quad x = \sec y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{\sin y}{\cos^2 y}} = \pm\frac{x}{x^2\sqrt{x^2-1}}$$

$$= \pm\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \quad y(0 - \pi)$$

$y = \operatorname{cosec}^{-1}x \quad x = \operatorname{cosec} y$

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}} \quad y\left(-\frac{\pi}{2} - \frac{\pi}{2}\right).$$

Ex. 1. $y = x\sin^{-1}x.$

$$y' = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

2. $y = (x - \tan^{-1}x)^3$

$$y' = 3(x-\tan^{-1}x)^2\left(1 - \frac{1}{1+x^2}\right)$$

$$= 3(x-\tan^{-1}x)^2 \frac{x^2}{1+x^2} \quad //$$

(7) $x = f_1(t)$　$y = f_2(t)$ ノ如ク. x ト y トガ t ナル parameter ニテ結ビツケラレタル時ハ

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$\Delta t, \Delta x, \Delta y$ ヲ相応スル increment トス. シカル時ハ

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}} = .$$

lim ヲ採レバ, 上ノ如シ.

$x = r\cos\theta$　　$y = r\sin\theta$.

$$\frac{dy}{dx} = \frac{r\cos\theta}{-r\sin\theta} = \frac{\cos\theta}{-\sin\theta}.$$

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1-\cos\theta) \end{aligned}\right\} \text{cycloid}$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a - a\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

⑧ Logarithmic differentiation

微分スベキf(x)ガ多クノf(x)ノ積及ビ商等ヨリ成リ立ツトキハ、先ヅ其ノ対数ヲトリテ而ル後、differ. スルヲ便トス。

Ex 1. $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$.

$$\log y = \frac{1}{2}\left\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4)\right\}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left\{ \qquad . \qquad \right\}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

$$y = \frac{(x-5)^{\frac{7}{3}}}{(x-2)^{\frac{5}{2}}(x-3)^{\frac{7}{4}}}$$

$$\log y = \frac{7}{3}\log(x-5) - \frac{5}{2}\log(x-2) - \frac{9}{4}\log(x-3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{7}{3}\frac{1}{x-5} - \frac{5}{2}\frac{1}{x-2} - \frac{9}{4}\frac{1}{x-3}$$

$$\frac{dy}{dx} = \left\{ \qquad \right\}\frac{(x-5)^{\frac{7}{3}}}{(x-2)^{\frac{5}{2}}(x-3)^{\frac{9}{4}}}$$

$$y = x^{x^2} \qquad \cancel{\frac{dy}{dx} = x^2 \cdot x^{x^2-1}}$$

$$=$$

$$\log y = x^2 \log x$$

$$\frac{1}{y}\frac{dy}{dx} = 2x\log x + x$$

$$\frac{dy}{dx} = x^{x^2} \cdot x(2\log x + 1)$$

⑨ Differentiation of implicit function

$$\frac{d}{dx}\{f(x,y)\} = 0.$$

ヲ $\frac{dy}{dx}$ ニ関シテトケバヨイ.

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} = m.$$

(2) $x^3 + y^3 - 3axy = 0.$

$$3x^2 + 3y^2\frac{dy}{dx} - 3ay - 3ax\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

上ノ二例ノ如ク $\frac{dy}{dx}$ 中ニハ y ヲ含ム.
之ニ x ヲ代入スレバ Explicit ト同様ニナル. 従ッテ其ノ必要ハ無イ.

⑩ Differ. of Determinant.

$$y = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

(f_1 等ハ x ノ fn トス)

$$\frac{\Delta y}{\Delta x} = \frac{\begin{vmatrix} f_1+\Delta f_1 & g_1+\Delta g_1 & h_1+\Delta h_1 \\ f_2+\Delta f_2 & g_2+\Delta g_2 & \cdots \\ f_3+\Delta f_3 & \cdots & \cdots \end{vmatrix} - \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & \cdots & \cdots \\ & \cdots & \cdots \end{vmatrix}}{\Delta x}$$

$$= \frac{\begin{vmatrix} f_1+\Delta f_1 & \cdots & \cdots \\ \cdots & & \end{vmatrix} - \begin{vmatrix} f_1 & g_1+\Delta g_1 & h_1+\Delta h_1 \\ f_2 & \cdots & \\ f_3 & \cdots & \end{vmatrix}}{\Delta x}$$

$$+ \frac{\begin{vmatrix} f_1 & g_1+\Delta g_1 & h_1+\Delta h_1 \\ f_2 & \cdots & \cdots \\ f_3 & \cdots & \cdots \end{vmatrix} - \begin{vmatrix} f_1 & g_1 & h_1+\Delta h_1 \\ f_2 & g_2 & \cdots \\ f_3 & g_2 & \cdots \end{vmatrix}}{\Delta x}$$

$$+ \frac{\begin{vmatrix} f_1 & g_1 & h_1+\Delta h_1 \\ f_2 & g_2 & \cdots \\ f_3 & g_2 & \cdots \end{vmatrix} - \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & \cdots & \cdots \\ f_3 & \cdots & \cdots \end{vmatrix}}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \begin{vmatrix} \frac{\Delta f_1}{\Delta x} & g_1+\Delta g_1 & h_1+\Delta h_1 \\ \frac{\Delta f_2}{\Delta x} & \cdots & \cdots \\ \frac{\Delta f_3}{\Delta x} & \cdots & \cdots \end{vmatrix} + \cdots\cdots$$

$$\frac{dy}{dx} = \begin{vmatrix} \frac{df_1}{dx} & g_1 & h_1 \\ \frac{df_2}{dx} & g_2 & h_2 \\ \frac{df_3}{dx} & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & \frac{dg_1}{dx} & h_1 \\ f_2 & \frac{dg_2}{dx} & h_2 \\ f_3 & \frac{dg_3}{dx} & h_3 \end{vmatrix}$$

$$+ \begin{vmatrix} f_1 & g_1 & \frac{dh_1}{dx} \\ f_2 & g_2 & \frac{dh_2}{dx} \\ f_3 & g_3 & \frac{dh_3}{dx} \end{vmatrix}$$

一般ニ nth order ニツイテ成立. 即 row ト column ノ interchange ニヨルカラ. イヅレニツイテヤツテモ同様ナリ.

Ex. $$y = \begin{vmatrix} 1 & \sin x & \cos x \\ \sin x & 1 & \sin 2x \\ \cos x & \sin 2x & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \sin x & \cos^2 x & \sin x \cos x \\ \cos x & \sin x \cos x & \sin^2 x \end{vmatrix} = \begin{vmatrix} \cos^2 x & \sin x \cos x \\ \sin x \cos x & \sin^2 x \end{vmatrix}$$

$$y' = \begin{vmatrix} 0 & \sin x & \cos x \\ \cos x & 1 & \sin 2x \\ -\sin x & \sin 2x & 1 \end{vmatrix} = 0.$$

$$+ \begin{vmatrix} 1 & \cos x & \cos x \\ \sin x & 0 & \sin 2x \\ \cos x & 2\cos 2x & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & \sin x & -\sin x \\ \sin x & 1 & 2\cos 2x \\ \cos x & \sin 2x & 0 \end{vmatrix}$$

$$= -\sin x \begin{vmatrix} \cos x & \sin 2x \\ -\sin x & 1 \end{vmatrix}$$

$$+ \cos x \begin{vmatrix} \cos x & 1 \\ -\sin x & \sin 2x \end{vmatrix}$$

$$-\sin x \begin{vmatrix} \cos x & \cos x \\ 2\cos 2x & 1 \end{vmatrix} - \sin 2x \begin{vmatrix} 1 & \cos x \\ \cos x & 2\cos 2x \end{vmatrix}$$

$$+\cos x \begin{vmatrix} \sin x & -\sin x \\ 1 & 2\cos 2x \end{vmatrix} - \sin 2x \begin{vmatrix} 1 & -\sin x \\ \sin x & 2\cos 2x \end{vmatrix}$$

$$= \cancel{-\sin x \cos x} - \sin^2 x \sin 2x + \cos^2 x \sin 2x + \cancel{\cos x \sin x}$$

$$\cancel{-\sin x \cos x} + 2\sin x \cos x \cos 2x - 2\sin 2x \cos 2x + \cos^2 x \sin 2x$$

$$+ 2\sin x \cos x \cos 2x \cancel{+\sin x \cos x} - 2\sin 2x \cos 2x - \sin^2 x \sin 2x$$

$$= -2\sin^2 x \sin 2x + 2\cos^2 x \sin^2 x - 2\sin 2x \cos 2x .$$

$$= 2\sin 2x (\cos^2 x - \sin^2 x - \cos 2x$$

$$=$$

(69) $\log y = \log e^{x}\sqrt{\frac{1+x}{1-x}}$

[illegible]

(17) $y=(e^{\frac{x}{a}}-e^{-\frac{x}{a}})^2 \quad y'=2(e^{\frac{x}{a}}-e^{-\frac{x}{a}})(e^{\frac{x}{a}}+e^{-\frac{x}{a}})\frac{1}{a}$

(18) $y=\dfrac{x}{e^x-1} \quad y'=\dfrac{(e^x-1)-x\frac{d(e^x-1)}{dx}}{(e^x-1)^2}=\dfrac{e^x-1-xe^x}{(e^x-1)^2}$,

(19) $\log y = \log e^x\sqrt{\dfrac{1+x}{1-x}}$

$$\frac{d\log y}{dy}\cdot\frac{dy}{dx}=\frac{d\log e^x\sqrt{\frac{1+x}{1-x}}}{dx}=\frac{d(\log e^x+\log\sqrt{\frac{1+x}{1-x}})}{dx}$$

$\dfrac{1}{y}$

$$=\frac{d\log e^x}{dx}+\frac{d\log\sqrt{\frac{1+x}{1-x}}}{dx}=\frac{1}{e^x}e^x+\frac{1}{\sqrt{\frac{1+x}{1-x}}}\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{-2x}{(1-x)^2}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d\log e^x\sqrt{\frac{1+x}{1-x}}}{dx}=1+$$

(20) $y=x\log x-x \quad \log x+1-1=\log x.$

(21) $y=\log(a+bx^2) \quad y'=\dfrac{1}{a+bx^2}\cdot 2bx.$

(22) $y=\log\dfrac{1+x+x^2}{1-x+x^2} \quad y'=\dfrac{1-x+x^2}{1+x+x^2}\,\dfrac{(1-x+x^2)(1+2x)+(1+x+x^2)(1-2x)}{(1-x+x^2)^2}$

$$y'=\frac{2(1+x^2)+2x^2}{(1+x+x^2)(1-x+x^2)}=\frac{2}{(1+x+x^2)(1-x+x^2)}$$

$(1+x)^{\frac{1}{2}+\frac{1}{2}}\cdot(1-x)$.

(23) $y=\log\sqrt{\dfrac{1-x}{1+x}}$ $\quad y'=\sqrt{\dfrac{1+x}{1-x}}\cdot\dfrac{1}{2}\left(\sqrt{\dfrac{1-x}{1+x}}\right)^{-\frac{1}{2}}\cdot\dfrac{-2}{(1+x)^2}$

$=-\dfrac{1}{1-x^2}$.

(24) $y=\log(\sqrt{a+bx}-\sqrt{a})$

$y'=\dfrac{1}{\sqrt{a+bx}-\sqrt{a}}\cdot\dfrac{1}{2}(a+bx)^{-\frac{1}{2}}(b)$

(25) $y=\log(\sqrt{ae^x+b}+\sqrt{ae^x})$

$y'=\dfrac{1}{\sqrt{ae^x+b}+\sqrt{ae^x}}\cdot\dfrac{1}{2}\left\{(ae^x+b)^{-\frac{1}{2}}\cdot ae^x+(ae^x)^{-\frac{1}{2}}ae^x\right.$

$=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{ae^x+b}+\sqrt{ae^x}}\cdot ae^x\left\{(ae^x+b)^{-\frac{1}{2}}+(ae^x)^{-\frac{1}{2}}\right\}$.

$=\dfrac{1}{2}\dfrac{\sqrt{ae^x}}{\sqrt{ae^x+b}}$.

(26) $y=(\sin mx)^2$

$y'=2\sin mx\cdot\cos mx\cdot m$.

(27) $y=\cos(a+bx)$

$y'=-\sin(a+bx)\cdot b$

(28) $y=\tan x-\cot x$

$y'=\dfrac{1}{\cos^2 x}+\dfrac{1}{\sin^2 x}=\dfrac{4}{\sin^2 2x}$. $\quad\dfrac{\sin x\cos x+x}{\cos^2 x}$

(29) $y=(x\tan x)^2$ $\quad y'=2x\tan x\left(\tan x+\dfrac{x}{\cos^2 x}\right)$

(30) $y = x^m \sin kx$ $\quad y' = \sin kx \cdot m \cdot x^{m-1} + x^m \dfrac{d \sin kx}{dx}$

$= x^{m-1}(\sin kx \cdot m + kx \cdot \cos kx)$

(31) $y = e^{ax} \cos bx$ $\quad y' = \cos bx \cdot e^{ax} \cdot a - e^{ax} \cdot b \sin bx.$

$= e^{ax}(a \cos bx - b \sin bx)$

(32) $y = \log \cos px$ $\quad y' = \dfrac{1}{\cos px} \cdot - \sin px \cdot p$

$= - \tan px \cdot p.$

(33) $y = \dfrac{\sin x}{1 + \tan x}$ $\quad y' = \dfrac{(1+\tan x)\cos x - \sin x \frac{1}{\cos^2 x}}{(1+\tan x)^2}$

$= \dfrac{\cos x + \sin x \left(\frac{\cos^2 x - 1}{\cos^2 x}\right)}{(1+\tan x)^2}$

$= \dfrac{\cos x + \tan^2 x}{(1+\tan x)^2}$

(34) $y = x \sin^{-1} x$ $\quad \dfrac{dy}{dx} = \sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}}.$

(35) $y = (x - \tan^{-1} x)^2$

$\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1+x^2}$ $\quad y' = 2(x - \tan^{-1} x) \cdot \dfrac{\tan^{-1} x - \frac{x}{1+x^2}}{(\tan^{-1} x)^2}$

(36) $y = \cot^{-1} \dfrac{x}{\sqrt{1+x^2}}$ $\quad y' = - \dfrac{1}{1 + \frac{x^2}{1+x^2}} \cdot \dfrac{\sqrt{1+x^2} - x \cdot}{1+x^2}$

$= \dfrac{-1+x^2}{1+2x^2} \cdot \dfrac{1+x^2 - x^2}{1+x^2 \sqrt{1+x^2}} = \dfrac{-1}{1+2x^2 \sqrt{1+x^2}}$ $\quad \left(\frac{1}{2} \frac{2x}{\sqrt{1+x^2}}\right.$

(37) $y = \cos^{-1}\dfrac{x}{\sqrt{1+x^2}}$

$$y' = -\frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\cdot\frac{\sqrt{1+x^2} - x\cdot\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}$$

$$= -\frac{\sqrt{1+x^2}\cdot\{1+x^2-x^2\}}{(1+x^2)\sqrt{1+x^2}} = -\frac{1}{1+x^2}.$$

(38) $y = \sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}}$

$y' = \dfrac{1}{x\sqrt{x^2-1}}$

$$y' = \frac{1}{\frac{a}{\sqrt{a^2-x^2}}\sqrt{\frac{a^2}{a^2-x^2}-1}}$$

$$= \frac{\left(\sqrt{a^2-x^2}\right)^2}{a\cdot\sqrt{x^2}}$$

$$= \frac{a^2-x^2}{a\cdot x}.$$

(39) $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$\{(1-\cos x)\sin x\}$

$$y' = \frac{1}{1+\frac{1-\cos x}{1+\cos x}}\cdot\sqrt{\frac{1+\cos x}{1-\cos x}}\cdot\frac{1}{2}\cdot\frac{(1+\cos x)\sin x +}{(1+\cos x)^2}$$

$$= \frac{\cancel{(1+\cos x)}\ \sin x\ 1}{2\sqrt{1-\cos x}\left(\sqrt{1+\cos x}\right)^2} = \frac{1}{2}$$

40. $y = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}$.

$$y' = \sqrt{a^2-x^2} + \frac{1}{2}\cdot\frac{x}{\sqrt{a^2-x^2}}(-2x)$$

$$+ a^2 \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$= \sqrt{a^2-x^2} - x^2\frac{1}{\sqrt{a^2-x^2}} + \cancel{2a\cdot\sin^{-1}\frac{x}{a}}$$

$$+ \frac{a^2}{\sqrt{a^2-x^2}} = 2\{\sqrt{a^2-x^2} + \cancel{2a\sin^{-1}\frac{x}{a}}\}$$

41. $y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$

$$y' = \frac{\cancel{\sqrt{1-x^2}}\{x\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x\}}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\frac{1}{2}\frac{-2x}{\sqrt{1-x^2}}$$

$$= \frac{\sin^{-1}x\sqrt{1-x^2}}{1-x^2\sqrt{1-x^2}} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}.$$

42. $y = \log(x+\sqrt{x^2-a^2}) + \sec^{-1}\frac{x}{a}$

$$y' = \frac{1+\frac{2x}{2\sqrt{x^2-a^2}}}{x+\sqrt{x^2-a^2}} + \frac{1}{a}\frac{1}{\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1}}$$

$$= \frac{\cancel{\sqrt{x^2-a^2}+x}}{\cancel{(x+\sqrt{x^2-a^2})}\sqrt{x^2-a^2}} + \frac{a}{x\sqrt{x^2-a^2}}$$

$$= \frac{x+a}{x\sqrt{x^2-a^2}} = \frac{1}{x}\sqrt{\frac{x+a}{x-a}}$$

$\frac{dy}{dx} = \text{hem}$

43. $y = \sin^{-1}\sqrt{\frac{a^2-x^2}{b^2-x^2}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{a^2-x^2}{b^2-x^2}}} \cdot \frac{-\cancel{2}x(b^2-x^2)+\cancel{2}x(a^2-x^2)}{\cancel{2}\sqrt{\frac{a^2-x^2}{b^2-x^2}}\,(b^2-x^2)^2}$$

$\frac{1}{\sqrt{b^2-x^2}}\sqrt{b^2-a^2}$

$$= \frac{x(a^2-b^2)}{\sqrt{b^2-a^2}\cdot\sqrt{a^2-x^2}\,(b^2-x^2)}$$

$$= \frac{-x\cdot\sqrt{b^2-a^2}}{\sqrt{a^2-x^2}\,(b^2-x^2)}$$

44. $y = e^{ax}\sin kx$

$y' = \sin kx \cdot a \cdot e^{x} + e^{ax}\cdot \cos kx \cdot k.$

45. $y = a^{\sqrt{a^2-x^2}}$

$\log y = \sqrt{a^2-x^2}\cdot \log a$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}\log a, \quad \frac{dy}{dx} = -a^{\sqrt{a^2-x^2}}\log a\,\frac{x}{\sqrt{a^2-x^2}},$

46. $y = \log\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}} = \frac{1}{2}\cdot\left\{\log(\sqrt{1+x^2}+x) - \log(\sqrt{1+x^2}-x)\right\}$

$$y' = \frac{1}{2}\cdot\left\{\frac{\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x} - \frac{\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}-1}{\sqrt{1+x^2}-x}\right\}$$

$$= \frac{1}{2}\cdot\frac{1+x^2-x^2+1+x^2-x^2}{(1+x^2-x^2)\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}},$$

(47) $y = \log \cdot \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$y' = \frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} \frac{1}{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \frac{1}{\cos x}$$

$$\cancel{= \frac{\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2}}$$

$$\cancel{= \frac{1}{4}\sin\left(\frac{\pi}{2}+x\right) = \frac{1}{4}\cos x}$$

(48) $y = \log\left\{2x-1+2\sqrt{x^2-x-1}\right\}$

$$y' = \frac{2+\frac{2x-1}{\sqrt{x^2-x-1}}}{2x-1+2\sqrt{x^2-x-1}} = \frac{1}{\sqrt{x^2-x-1}}.$$

(49) $y = \tan a^{\frac{1}{x}}$

$$\frac{dy}{dx} = \frac{1}{\cos^2 a^{\frac{1}{x}}} \cdot a^{\frac{1}{x}} \cdot \log a \cdot \frac{-1}{x^2}$$

$x^{x} = -2 \cdot x$
$= -1\ x$

(50) $y = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{1}{1+\frac{x^2}{1-x^2}} \cdot \frac{\sqrt{1-x^2}+\frac{1}{2}\frac{2x^2}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{(1-x^2)\cancel{(1+x^2)}}{1-x^2 \cdot \sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$\frac{1+1-x}{1+x^2}$

(51) $y = \sin^{-1}\frac{1-x^2}{1+x^2}$ $\quad (1+x^2)-(1-x^2)$

$$y' = \frac{\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}} = \frac{-2x\cdot 2}{2x\cdot(1+x^2)}$$

$$= \frac{-2}{1+x^2}.$$

(52) $y = \cos^{-1}\frac{3+5\cos x}{5+3\cos x}$

$$= \frac{\frac{(5+3\cos x)\cdot 5\sin x-(3+5\cos x)\cdot 3\sin x}{(5+3\cos x)^2}}{\sqrt{\frac{25+30\cos x+9\cos^2 x-9-30\cos x-25x^2}{(5+3\cos x)^2}}}$$

$$= \frac{16\sin x}{(5+3\cos x)\,4\sin x}.$$

$$= \frac{4}{(5+3\cos x)}.$$

(53) $y = x^{\sin^{-1}x}$

$\log y = \sin^{-1}x\log x.$

$$\frac{1}{y}\frac{dy}{dx} = \log x\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x\frac{1}{x}.$$

$$\frac{dy}{dx} =$$

(54) $y = e^{e^x}$. $\quad \log y = e^x\cdot\log e.$

$$\frac{dy}{dx} = e^{e^x}\cdot\log e\cdot e^x.$$

(55) $y = x^{e^x}$, $\quad \frac{dy}{dx} = x^{e^x}\cdot\left\{e^x\cdot\frac{1}{x}+e^x\log x\right\}$

$\log y = e^x\log x.$

$y^2 = e^{\frac{x-y}{y}}\, y - x\frac{dy}{dx}$

(56) $x = e^{\frac{x-y}{y}}$

$1 = e^{\frac{x-y}{y}} \cdot \dfrac{d\frac{x-y}{y}}{dx} = e^{\frac{x-y}{y}} \cdot \dfrac{y - x\frac{dy}{dx}}{y^2}$

$\dfrac{e^{\frac{y}{x-y}} \cdot y^2 - y}{-x} \qquad \dfrac{dy}{dx} = \dfrac{e^{\frac{x-y}{y}} \cdot y}{x}$

(57) $\log\sqrt{x^2+y^2} = \tan^{-1}\dfrac{y}{x}$

$\dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{2x+2y\frac{dy}{dx}}{2\sqrt{x^2+y^2}} = \dfrac{1}{1+\frac{y^2}{x^2}} \cdot \dfrac{x\frac{dy}{dx}-y}{x^2}$

$\dfrac{x+y\frac{dy}{dx}}{x^2+y^2} = \dfrac{x\frac{dy}{dx}-y}{x^2+y^2}$,

(58) $\sin(xy) = \sin mx$.

$\cos(xy)\left\{x\dfrac{dy}{dx}+y\right\} = \cos(mx)\cdot m$.

$\dfrac{dy}{dx} = \dfrac{m\cos mx - y\cos yx}{x\cos xy}$.

(59) $ax^3 + x^3y = ay^3$

$3ax^2 + 3yx^2 + x^3\dfrac{dy}{dx} = 3ay^2\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = \dfrac{3ax^2+3yx^2}{3ay^2-x^3}$

(61)
$$\frac{dy}{dx} = \frac{d}{dx}\left\{\sin x\cos x\sqrt{1-k^2\sin^2 x}\right\}$$

$$= \sqrt{1-k^2\sin^2 x}\cdot\cos x \cdot \sqrt{1-k^2\sin^2 x}\cdot\sin x\cos x$$

$$-\frac{1}{2}\sin x\cos x\,\frac{1}{\sqrt{1-k^2\sin^2 x}}\left(-2k^2\sin x\cdot\cos x\right)$$

$$= \frac{k^2\sin^2 x\cos^2 x}{\sqrt{1-k^2\sin^2 x}} = (-k^2)\sin^2 x+\sin^2 x$$

$$\cos x - k^2\sin^2 x\cos x - \overset{1-\sin^2 x}{\sin^2 x} + k^2\sin^3 x - \frac{1}{2}\sin x\cos x$$

$$= \cancel{-2k^2\sin x\cos x} + k^2\sin^2 x\cos^2 x$$

(60) $\sin nx = 2^{n-1}\sin x$

$$\frac{dy}{dx} = n\cos nx = 2^{n-1}\cos x$$

$$\therefore \frac{dy}{dx}\Big/\sin nx = \cot x +$$

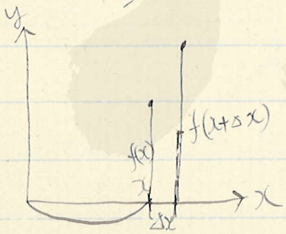
§3. Differential 微分。

$y = f(x)$ ヲ (α, β) ニ於テ continuous ナリトシ、
Δx ヲ x ノ任意ノ increment トスル時、$f'(x)\cdot\Delta x$ ヲ y ノ differential ト称ヘテ之ヲ $dy = f'(x)\Delta x$ トシテアラハス。
然ルニ若シ fn ガ $y = x$ ナル時ハ、$f'(x) = 1$.
従ッテ $dy = dx$.
($\because dy = f'(x)\Delta x \quad \therefore dx = \Delta x. \quad \therefore dy = dx$)
即チ一ツノ fn ノ derived fn $f'(x)$ ハ其ノ fn ノ differential ト independent variable ノ differential トノ比ナリ。∴ $\frac{dy}{dx}$ ハ dy ヲ dx デ割ッタ商又ハ $\frac{\Delta y}{\Delta x}$ ノ limit ト考ヘテモヨイコトニナル

Derivative ノ Application

§1. Increasing or Decreasing of a function.

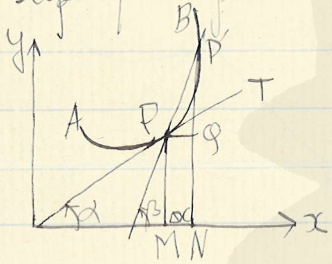
Δx ヲ >0 トス。 $\frac{dy}{dx} = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = L_{\Delta x = 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$



コノ値ガ >0 ナル時ハ
$f(x+\Delta x) > f(x)$
<0 ナル時ハ
$f(x+\Delta x) < f(x)$
∴ x ガ increase スルモノト考ヘル時、x ノ fn ハ其ノ differ quotient ガ posi, neg ナルニ従ヒテ fn ノ値ガ increase

or decrease スル

§2. Slope of Tangent



AP・P'B ; $y = f(x)$

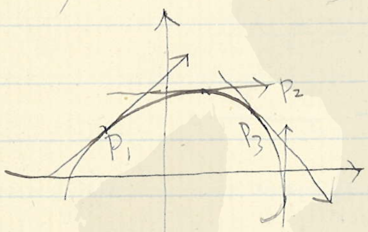
$MN = \Delta x$.

$P'Q = \Delta y$

$$\frac{\Delta y}{\Delta x} = \frac{P'Q}{MN} = \frac{P'Q}{PQ} = \tan\beta$$; slope of secant line PP'.

$$\frac{dy}{dx} = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \underset{(m)}{\tan\alpha}.$$; slope of tangent line PT.

任意ノ点ニ於ケル curve ノ方向ハ其ノ点ニ於ケル curve ノ切線ノ方向ヲ以ツテ現ハス。コノ時ハ $\frac{dy}{dx} = \tan\alpha$ ハ任意ノ点ニ於ケル curve ノ方向ヲ与ヘル。



P_1 ナル点ニ於テハ curve ハ rising

α ハ鋭角 ∴ $\tan\alpha = \frac{dy}{dx} > 0$.

P_2 ニ於テハ curve ガ x 軸ニ平行デアル.

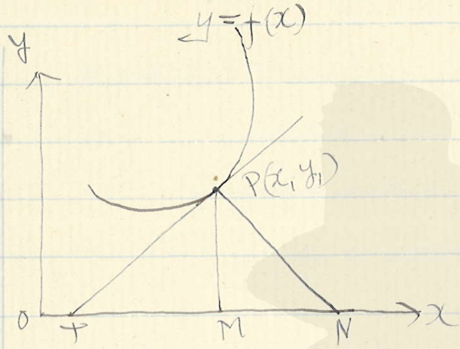
∴ $\tan\alpha = \frac{dy}{dx} = 0$.

P_3 ニ於テハ curve ハ falling

∴ $\tan\alpha = \frac{dy}{dx} < 0$.

§3. Equations of Tangent and normal

$y = f(x)$



P＝於ケル切線ノ方程式ハ

$$y-y_1=f'(x_1)(x-x_1)$$

$$\text{or} =\left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}(x-x_1)$$

従ッテ normal ノ equation ハ

$$y-y_1=-\frac{1}{f'(x_1)}(x-x_1)$$

例ヘバ　$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\left(\frac{dy}{dx}\right)_{x_1y_1}=\frac{b^2x_1}{a^2y_1}.$$

tang at (x_1y_1) : $y-y_1=\dfrac{b^2x_1}{a^2y_1}(x-x_1).$

$$\frac{y_1y}{b^2}-\frac{y_1^2}{b_2}=\frac{x_1x}{a^2}-\frac{x_1^2}{a^2}$$

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1.$$

Ex 1.　$y=f(x)=x^4-8x^3+22x^2-24x+4$

ガ　x ノ如何ナル interval ＝於テ increasing デアル.

$$\frac{dy}{dx}=4x^3-24x^2+44x-24$$

$$=4(x^3-6x^2+11x-6)>0.$$

$$(x-1)(x-2)(x-3)>0$$

$1<x<2.$　　　$x>3.$

其ノ他ノ値デハ decreasing.

Ex 2. $y^2=4\alpha x$.

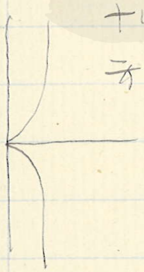
$$2y\frac{dy}{dx}=4\alpha$$

$$\frac{dy}{dx}=\frac{2\alpha}{y}$$

∴ $x_1\, y_1$ ナル点ニ於ケル tang.: $y-y_1=\frac{2\alpha}{y_1}(x-x_1)$

$$y_1 y = 2\alpha(x+x_1)$$

Ex 3. $y^2=\frac{x^3}{2a-x}$.



ナル curve 上ノ一点 (a,a) ナル点ニ於ケル切線ノ方程式ヲ求ム.

$$2y\frac{dy}{dx}=\frac{3x^2(2a-x)+x^3}{(2a-x)^2}$$

$$\frac{dy}{dx}=\frac{3x^2(2a-x)+x^3}{2y(2a-x)^2}$$

$$\left(\frac{dy}{dx}\right)_{a,a}=\frac{3a^2\cdot a+a^3}{2a\cdot a^2}=\frac{4a^3}{2a^3}=2.$$

∴ 於ケル equation ハ $y-a=2(x-a)$

$$y-2x+a=0$$

Ex 4. $x^2(x+y)=a^2(x-y)$

ナル曲線ノ Origin ニ於ケル tang. ノ方程式.

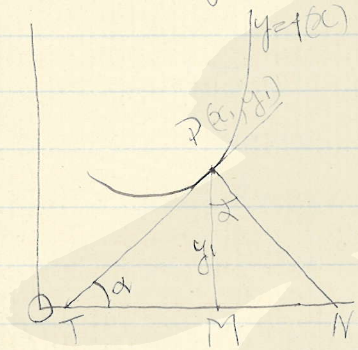
$$\cancel{\frac{dy}{dx}}\quad 2x(x+y)+x^2\left(1+\frac{dy}{dx}\right)=a^2\left(1-\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{a^2 - 3x^2 - 2xy}{x^2 + a^2}$$

$$\left(\frac{dy}{dx}\right)_{0,0} = 1.$$

$$y = x.$$

§4.



PT : length of tangent

$$= y_1 \operatorname{cosec}\alpha. = y_1\sqrt{1+\frac{1}{[f'(x_1)]^2}}$$

PN : length of normal :

$$= y_1 \sec\alpha = y_1\sqrt{1+[f'(x_1)]^2}$$

TM : subtangent

$$= y_1 \cot\alpha = y_1 \times \frac{1}{f'(x_1)}$$

MN ; subnormal

$$= y_1 \tan\alpha = y_1 f'(x_1)$$

例へバ、

Ex1. $y^2 = \dfrac{x^3}{2a-x}$

The curve ニ於テ. (a, a) ナル点ニ於ル四ツノ長サヲ計算スル。

$$\left(\frac{dy}{dx}\right)_{(a,a)} = 2.$$

length of Tang : $a\sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}a.$

〃 norm : $a\sqrt{1+4} = \sqrt{5}a$

subtang : $a/2.$ subnorm : $2a.$

Ex 2. $xy = a^2$

上ノ一点 (x_1, y_1)

ニ関スル四ツノ長サ.

$$y + x\frac{dy}{dx} = 0.$$

$$\therefore \left(\frac{dy}{dx}\right)_1 = -\frac{y_1}{x_1}.$$

L of tang; $y_1\sqrt{1+\frac{x_1^2}{y_1^2}} = \sqrt{x_1^2+y_1^2}.$

〃 norm; $y_1\sqrt{1+\frac{y_1^2}{x_1^2}} = \frac{y_1}{x_1}\sqrt{x_1^2+y_1^2}$

Sub tang; $\left|y_1 \times \left(-\frac{x_1}{y_1}\right)\right| = |x_1|$

Sub norm; $\left|y_1 \times \left(-\frac{y_1}{x_1}\right)\right| = \frac{y_1^2}{|x_1|}$

Ex 3. $y^2 = 4dx.$

ノ上ノ一点ニ~~於ル~~関スル四ツノ長サヲ求ム.

$$\frac{dy}{dx} = \frac{2d}{y}.$$

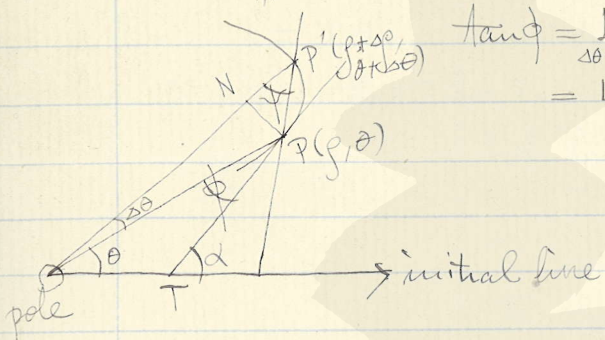
L of tang; $y\sqrt{1+\frac{y^2}{4d^2}} = \frac{y}{2d}\sqrt{4d^2+y^2}$

〃 normal; $y\sqrt{1+\frac{4d^2}{y^2}} = \sqrt{4d^2+y^2}.$

sub tan; $y \cdot \frac{y}{2\cancel{4}d} = \frac{y^2}{2d}$

〃 norm; $2d.$

§5. Curve上ノ一点ニ引キタル radius vector ト其ノ点ニ於ケル tangent トノナス角



$$\tan\phi = L_{\Delta\theta\to 0} \tan\psi$$

$$= L \frac{PN}{P'N} = L \frac{\rho\sin\Delta\theta}{\rho+\Delta\rho-\rho\cos\Delta\theta}$$

$$= L \frac{\rho\frac{\sin\Delta\theta}{\Delta\theta}}{\frac{\Delta\rho}{\Delta\theta}+\rho\frac{1-\cos\Delta\theta}{\Delta\theta}}$$

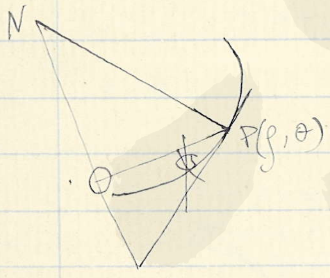
$$= \frac{\rho}{\frac{d\rho}{d\theta}}.$$

$$\tan\alpha = \tan(\phi+\theta)$$

$$= \frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}$$

$$= \frac{\frac{\rho}{\frac{d\rho}{d\theta}}+\frac{\sin\theta}{\cos\theta}}{1-\frac{\rho}{\frac{d\rho}{d\theta}}\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\rho\cos\theta+\frac{d\rho}{d\theta}\sin\theta}{\frac{d\rho}{d\theta}\cos\theta-\rho\sin\theta}.$$



PT: Length of Polar tangent

$$= \rho\sec\phi = \rho\sqrt{1+\frac{\rho^2}{\left(\frac{d\rho}{d\theta}\right)^2}}.$$

PN: Length of Polar Normal.

$$= \rho\,\mathrm{cosec}\,\phi = \rho\sqrt{1+\frac{1}{\rho^2}\left(\frac{d\rho}{d\theta}\right)^2}$$

OT: ~~Length of~~ Polar subtang.

$$= \rho\tan\phi = \rho^2\Big/\frac{d\rho}{d\theta}$$

ON: ~~Length of~~ Polar subnormal.

$= \rho \cot \phi = \frac{d\rho}{d\theta}$.

Ex: $\rho = a\theta$. a: const. 上ノ(ρ_0, θ_0)

ニ於テ四ツノ長サヲ求メヨ.

$\frac{d\rho}{d\theta} = a$.

L. of P. T. $= \rho_0\sqrt{1+\frac{\rho_0^2}{a^2}} = \frac{\rho_0}{a}\sqrt{a^2+\rho_0^2}$

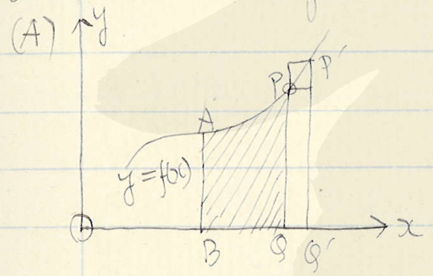
P. N. $= \rho_0\sqrt{1+\frac{1}{\rho_0^2}a^2} = \sqrt{\rho_0^2+a^2}$

P. sub T. $= \rho_0^2/a$

P. sub N. $= a$

§6. Derivative of Area.

(A)



AB: fixed ordinate

PQ: variable

面積 $B\overset{\frown}{AP}Q$ ヲ u ニテアラハス.

次ニ x ガ Δx ナル increment ヲ受ケシ時、

ヨリテ新シキ面積 u ハ Δu ナル

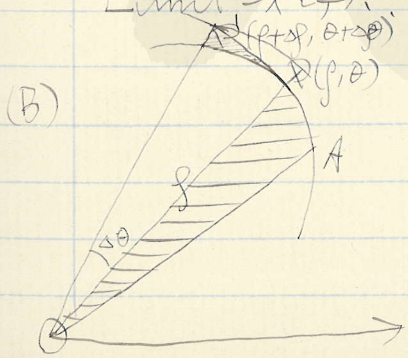
increment ヲ受ケトス.

コレハ明ニ、 $y\Delta x < \Delta u < (y+\Delta y)\Delta x$.

$y < \frac{\Delta u}{\Delta x} < y + \Delta y$.

Limit ニ於テハ、 $\frac{du}{dx} = y = f(x)$.

(B)



OA: fixed radius vector

OP: variable

面積 $O\overset{\frown}{AP}O$ ハ u

θガ$\Delta\theta$ナルincrementヲ得タルトキ、uノナルincrementヲΔuトスル.

面積ハ、

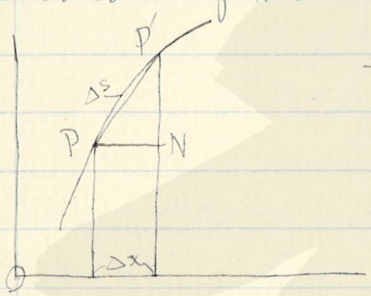
$$\frac{1}{2}\rho^2\Delta\theta < \Delta u < \frac{1}{2}(\rho+\Delta\rho)^2\Delta\theta.$$

$$\frac{1}{2}\rho^2 < \frac{\Delta u}{\Delta\theta} < \frac{1}{2}(\rho+\Delta\rho)^2$$

Limitヲトルト、

$$\frac{du}{d\theta} = \frac{1}{2}\rho^2.$$

Derivative of Arc.



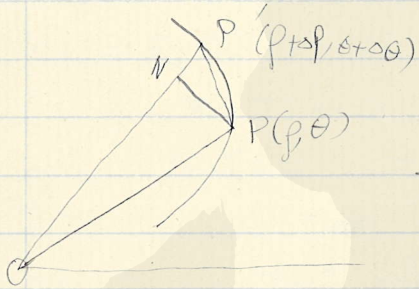
$$\frac{ds}{dx} = \underset{\Delta x\to 0}{L}\frac{\widehat{PP'}}{\Delta x} = \underset{\Delta x\to 0}{L}\left\{\frac{\widehat{PP'}}{PP'}\cdot\frac{PP'}{\Delta x}\right\}$$

$$= L\frac{\widehat{PP'}}{PP'}\,L\frac{PP'}{\Delta x}.$$

コレニ $L\frac{\widehat{PP'}}{PP'} = 1.$ (假定)トスレバ、

$$= L\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x} = L\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}.$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2},$$



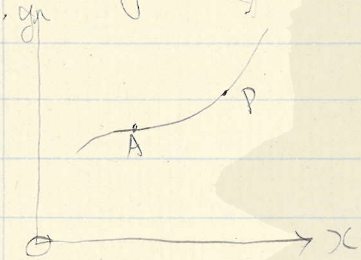
$$\frac{ds}{d\theta} = L\frac{\Delta s}{\Delta\theta} = L\frac{PP'}{\Delta\theta} = L\frac{\sqrt{PN^2+P'N^2}}{\Delta\theta}$$

$$= L\frac{\sqrt{(\rho\sin\Delta\theta)^2+(\rho+\Delta\rho-\rho\cos\Delta\theta)^2}}{\Delta\theta}.$$

$$= L\sqrt{\rho\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2+\left(\frac{\Delta\rho}{\Delta\theta}+\rho\frac{1-\cos\Delta\theta}{\Delta\theta}\right)^2}$$

$$= \sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}$$

§8. Velocity (plane 上ノ motion ヲ考フ)



一ツノ plane curve ニ沿ッテ動ク P 点ノ motion ヲ考フ。Curve 上ノ一点 A ヲ定メテ以テ基トシ、A ヨリ点 P ニ至ル Curve ニ沿フ長サヲ s ニテ表ハシ、A ヨリ P ニ至ル迄ニ要スル時間ヲ t ニテ表ハサン。

t ノ各ノ値ニ対シ点 P ノ一ツノ位置ガ対応スルコトナリ。従ッテ又 s ノ一ツノ値ガ対応スル。∴ s ハ t ノ fn ト考ヘラル。

即チ $s = f(t)$

次ニ t ガ Δt ナル increment ヲナシタル時 s ノナル increment ヲ Δs ニテ表ハサン。$\dfrac{\Delta s}{\Delta t}$ ハ Δt ナル時間ニ於ケル平均速度ヲ与フベシ。従ッテ P ニ於ケル velocity $v = \lim_{\Delta t=0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$.

而シテ点 P ノ coordinates ヲ x 及ビ y ニテ表ハサン。x, y ハ共ニ t ノ fn ナリ。∴ $x = f_1(t)$, $y = f_2(t)$ ナル方程式ニテ motion ノ模様ガ知ラル。ヨッテ velocity ノ x-component $v_x = \dfrac{dx}{dt}$. $v_y = \dfrac{dy}{dt}$.

従ッテ resultant $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$

Successive Differentiation

逐次微分法

$y=f(x)$ ナル微分シ得ル fn ガ与ヘラレタリトス. シカル時ハ
其ノ derived fn $\frac{dy}{dx}$ ハ一般ニ x ノ fn ニシテ尚モ
微分シ得可キモノナリ. ヨッテ之ヲ微分シタルモノ $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ ヲ
$\left(\frac{d}{dx}\right)^2 y$. or $\frac{d^2y}{dx^2}$ ナニテ表ハス.

二次元ノ fn ノ second derived function トス.
Second Derived fn ヲ更ニ微分シタルモノ $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ ヲ $\frac{d^3y}{dx^3}$.
ニテ表ハス, 三次元ノ fn ノ 3rd derived fn トス, 以下之ニ
準ズ. シカシテ $y=f(x)$ ナル fn ガ与ヘラレシ時. 其ノ derivative
$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... $\frac{d^ny}{dx^n}$, ハ 其々.

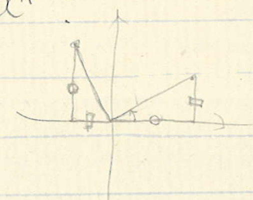
y', y'', y''', ... $y^{(n)}$,
$f'(x)$, $f''(x)$, $f'''(x)$, ... $f^{(n)}(x)$
トシテ表ハスコトアリ.

Ex: (1) $y=x^k$, Successive Derived fn ヲ求ム.

$y'=kx^{k-1}$. $y''=k(k-1)x^{k-2}$. $y'''=k(k-1)(k-2)x^{k-3}$.

...... $y^{(n)}=k(k-1)(k-2)\cdots(k-n+1)x^{k-n}$.

(2) $y=\sin(x+A)$



$y'=\cos(x+A)=\sin\left(x+A+\frac{\pi}{2}\right)$

$y''=\cos\left(x+A+\frac{\pi}{2}\right)=\sin\left(x+A+2\frac{\pi}{2}\right)$

$y'''=\cos\left(x+A+2\frac{\pi}{2}\right)=\sin\left(x+A+3\frac{\pi}{2}\right)$

- - - - - - - - - -

$y^{(n)}=\sin\left(x+A+\frac{n\pi}{2}\right)$.

$y' = -\sin x = \cos\left(x+\frac{\pi}{2}\right)$

$A=0$ ナルトキハ,　$y=\sin x$ ∴ $y^{(n)} = \sin\left(x+\frac{n\pi}{2}\right)$.

$A=\frac{\pi}{2}$ ナルトキハ,　$y = \left(\sin\left(x+\frac{\pi}{2}\right) = \right)\cos x$.

∴ $y^{(n)} = \left(\sin\left(x+\frac{(n+1)\pi}{2}\right) = \right)\cos\left(x+\frac{n\pi}{2}\right)$.

(3) $\left.\begin{array}{l} x = f_1(t) \\ y = f_2(t) \end{array}\right\}$ $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ヲ求メル.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ 1st derivative.

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

$$\frac{d^3y}{dx^3} = \frac{\left(\frac{d^3y}{dt^3}\frac{dx}{dt} - \frac{d^3x}{dt^3}\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^3 - 3\left(\frac{dx}{dt}\right)^2\frac{d^2x}{dt^2}\left(\frac{d^2y}{dx^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^7}$$

$$= \frac{\left(\frac{d^3y}{dt^3}\frac{dx}{dt} - \frac{d^3x}{dt^3}\frac{dy}{dt}\right)\frac{dx}{dt} - 3\frac{d^2x}{dt^2}\left(\frac{d^2y}{dx^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^5}$$

(4) Implicit function

$$x^3+y^3-3kxy=0$$

$$3x^2+3y^2\frac{dy}{dx}-3ky-3kx\frac{dy}{dx}=0.$$

$$\frac{dy}{dx} = \frac{ky-x^2}{y^2-kx}.$$ 1st der.

$$\frac{d^2y}{dx^2} = \frac{\left(k\frac{dy}{dx}-2x\right)(y^2-kx) - \left(2y\frac{dy}{dx}-k\right)(ky-x^2)}{(y^2-kx)^2}$$

$$= \frac{\left(k\frac{ky-x^2}{y^2-kx}-2x\right)(y^2-kx) - \left(2y\frac{ky-x^2}{y^2-kx}-k\right)(ky-x^2)}{}$$

$$(ky - x^2)$$

$$= \frac{\{k(ky-x^2) - 2x(y^2-kx)\}(y^2-kx) - \{2y(ky-x^2) - k(y^2-kx)\}}{(y^2-kx)^3}$$

$$= \frac{2k(ky-x^2)(y^2-kx) - 2x(y^2-kx)^2 - 2y(ky-x^2)^2}{(y^2-kx)^3}.$$

$$= \frac{\overset{2yx^4}{\cancel{(2k^2-2k^2)x^?}} \mp \overset{4k^2y^2}{\cancel{(2ky^2-2ky^2)x^2}} -}{}$$

$$= \frac{2\{-2k^2x^2y^2 - k^3xy - 2y^3x + \cdots}{}$$

§2. 一ノ函数ガ他ノ二函数ノ product ヨリ成立ツ場合ニ之ヲ逐次微分スルニハ Leibniz ノ公式ニヨルヲ便トス。

Theo. $y = u.v.$

$$y' = u'v + uv'$$

$$y'' = u''v + 2u'v' + uv''.$$

$$y''' = u'''v + 3u''v' + 3u'v'' + uv'''.$$

一般ニ $$y^{(n)} = u^{(n)}v + {}_nC_1 u^{(n-1)}v' + {}_nC_2 u^{(n-2)}v'' + {}_nC_3 u^{(n-3)}v''' + \cdots \cdots + u v^{(n)}$$

コノ公式ヲ一般ニ証スルタメニ両辺ヲ更ニ微分スル。

$$y^{(n+1)} = u^{(n+1)}v + u^{(n)}v' + {}_nC_1 u^{(n)}v' + {}_nC_1 u^{(n-1)}v'' + {}_nC_2 u^{(n-1)}v'' + {}_nC_2 u^{(n-2)}v''' + \cdots + u'v^{(n)} + uv^{(n+1)}$$

$$= u^{(n+1)}v + {}_{n+1}C_1 u^{(n)}v' + {}_{n+1}C_2 u^{(n-1)}v'' + {}_{n+1}C_3 u \cdots + uv^{(n+1)}$$

$$\frac{n(n-1)\cdots(n-r+1)}{r!} + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!} = \frac{n(n-1)\cdots(n-r+2)}{r!}(n-r+1+r)$$

$$(\because {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r.)$$

即チ n ノ時ニ公式ガ成立スルトセバ $n+1$ ノ時ニモ又成立スルコトナル。然ルニ $n=2$ ハ実際ノ計算ニヨリ、其ノ成立ヲ知ルノデ一般ニ真ナリ。コノ公式ヲ Leibniz ノ公式ト名ヅク。

Ex 1. $y = e^x \cdot \log x.$

Leib. ノ公式ニヨッテ $\frac{d^4y}{dx^4}$ ヲ求メルコト。

$u = e^x$　　$v = \log x.$

$u' = e^x \qquad v' = \frac{1}{x}$

$u'' = e^x \qquad v'' = -\frac{1!}{x^2}$

$u''' = e^x \qquad v''' = \frac{2!}{x^3}$

$u^{IV} = e^x \qquad v^{IV} = -\frac{3!}{x^4}$

$$\frac{d^4y}{dx^4} = e^x \log x + 4e^x \cdot \frac{1}{x} + 6e^x \times \left(-\frac{1}{x^2}\right) + 4e^x \times \frac{2!}{x^3} - e^x \frac{3!}{x^4}.$$

$$= e^x \left(\log x + \frac{4}{x} - \frac{6}{x^2} + \frac{8}{x^3} - \frac{6}{x^4}\right)$$

Ex 2. $y = x^{n-1} \log x$.

n th derivativeを求む.

$u = x^{n-1} \qquad v = \log x$

$u' = (n-1)x^{n-2} \qquad v' = \frac{1}{x}$

$u'' = (n-1)(n-2)x^{n-3} \qquad v'' = -\frac{1}{x^2}$

$u^3 = (n-1)(n-2)(n-3)x^{n-4} \qquad v''' = \frac{2!}{x^3}$

$\cdots\cdots \qquad v^{IV} = -\frac{3!}{x^4}$

$\cdots\cdots$

$u^{(n-2)} = (n-1)(n-2)\cdots 4\cdot 3\cdot 2x$

$u^{(n-1)} = (n-1)(n-2)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1. \qquad v^{(n-1)} = \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$

$u^{(n)} = 0 \qquad v^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$

$$y^{(n)} = 0 + n(n-1)!\frac{1}{x} + \frac{n(n-1)}{2!}(n-1)!\,x \times \left(-\frac{1}{x^2}\right) + \frac{n(n-1)(n-2)}{3!}(n-1)!\,x^2 \times \frac{1}{x^3}$$

$$\cdots\cdots + x^{n-1} \times \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$y^{(n)} = \frac{(n-1)!}{x}\left\{ 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \cdots + (-1)^n \frac{1}{} \right\}$$

$$= \frac{(n-1)!}{x}$$

$$\left(\because\ (1-1)^n = 1 - n + \frac{n(n-1)}{2!} - \cdots + (-1)^n\right)$$

Ex 3. $y = \sin(m \sin^{-1} x)$ ヲ与ヘテ. (1)

$(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx} - m^2 y$ ヲ証セヨ. (2)

更ニ

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} = (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2-m^2)\frac{d^ny}{dx^n}. \quad (3)$$

from (1) $\frac{dy}{dx} = \cos(m\sin^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$

$$\frac{d^2y}{dx^2} = -\sin(m\sin^{-1}x) \times \frac{m^2}{1-x^2} + \cos(m\sin^{-1}x) \times \frac{mx}{(1-x^2)^{3/2}}$$

$$(1-x^2)\frac{d^2y}{dx^2} = -m^2 y + x\frac{dy}{dx}.$$

更ニ (2)ヲ n times 微分スレバ次ノ如シ.

$$\begin{cases} u = (1-x^2) & v = \frac{d^2y}{dx^2} \\ u' = -2x & v' = \frac{d^3y}{dx^3} \\ u'' = -2 & v'' = \\ u''' = 0 & \end{cases}$$

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} + n(-2x)\frac{d^{n+1}y}{dx^{n+1}} + \frac{n(n-1)}{2!}(-2)\frac{d^n y}{dx^n}$$

$$= x\frac{d^{n+1}y}{dx^{n+1}} + n\frac{d^n y}{dx^n} - m^2\frac{d^n y}{dx^n}.$$

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} = (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2-m^2)\frac{d^n y}{dx^n}$$

(2) 及ビ (3) ニヨリ、f_n ノ differential quotient ヲ equation ヲ differential equation ト知ルベシ. 今 (1) ナル f_n ハ三次ノ微分方程式ニ適スルト解ス. (2), (3) ナル微分方程式ヲ解ケトイフコトハ (1) ナル f_n ヲ求ムルコトデアル.

$x=0$ $\quad \dfrac{d^{n+2}y}{dx^{n+2}} = n^2 \dfrac{d^n y}{dx^n} = n^2\cdot(n-2)^2 \dfrac{d^{n-2}y}{dx^{n-2}} = \cdots\cdots$

$n=2k \quad = (2k)^2\cdot(2k-2)^2 \cdots\cdots 4\cdot 2 \dfrac{d^2y}{dx^2}$

$= 0$

$n=2k+1 \quad = (2k+1)^2(2k-1)^2 \cdots\cdots 3\cdot 1$

$$(1-x^2)\frac{d^{n+3}y}{dx^{n+3}} - 2x\frac{d^{n+2}y}{dx^{n+2}} = (2n+1)x\frac{d^{n+2}y}{dx^{n+2}}$$

$$+(2n+1)\frac{d^{n+1}y}{dx^{n+1}} + n^2\frac{d^{n+1}y}{dx^{n+1}}.$$

$$(1-x^2)\frac{d^{n+3}y}{dx^{n+3}} - (2n+3)x\frac{d^{n+2}y}{dx^{n+2}} - (n+1)^2\frac{d^{n+1}y}{dx^{n+1}} = 0.$$

$x=0$: $$\frac{d^{n+2}y}{dx^{n+2}} = n^2\frac{d^ny}{dx^n}$$

if $\frac{d^{2k}y}{dx^{2k}}=0$, $\frac{d^{2k+2}y}{dx^{2k+2}}=0$　$k=1$ $\left(\frac{d^2y}{dx^2}\right)_0 = 0$

if $\frac{d^{2k+1}y}{dx^{2k+1}} = 1^2\cdot 3^2\cdot 5^2\cdots\cdots(2k-1)^2$,

$$\frac{d^{2k+3}y}{dx^{2k+3}} = 1^2\cdot 3^2\cdots\cdots(2k-1)^2(2k+1)^2$$

$k=1$　$\left(\frac{d^3y}{dx^3}\right)_0 = 1.$

(8) $y = a^{\sin^{-1}x}$; $\dfrac{dy}{dx} = a^{\sin^{-1}x} \cdot \log a \cdot \dfrac{1}{\sqrt{1-x^2}}$

$$\frac{d^2y}{dx^2} = a^{\sin^{-1}}(\log a)^2 \frac{1}{1-x^2} + a^{\sin^{-1}x} \log a \frac{x}{(\sqrt{1-x^2})^3}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = y(\log a)^2.$$

(9) $y = e^{-kx}(A\cos mx + B\sin mx)$

$$\frac{dy}{dx} = e^{-kx}\{-k(A\cos mx + B\sin mx) + m(A(-\sin mx) + B\cos mx)\}$$

$$\frac{d^2y}{dx^2} = -k\frac{dy}{dx} + e^{-kx}\{-km(A(-\sin mx) + B\cos mx)$$

$$\boxed{+k^2(A\cos mx + B\sin mx) - k^2(A\cos mx + B\sin mx)} - m^2(A\cos mx + B\sin mx)\}$$

$$\therefore \frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + (k^2+m^2)y = 0.$$

(10) $y = \sin^{-1}x$. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

$$\frac{d^2y}{dx^2} = \frac{x}{(\sqrt{1-x^2})^3}.$$

$n=0$ $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} - n^2\frac{d^ny}{dx^n} = 0.$$

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} = (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n^2\frac{d^{n+1}y}{dx^{n+1}}$$

(5) $$y = \frac{ax+b}{x^2-c^2} = \frac{\frac{1}{2}(a-\frac{b}{c})}{x+c} + \frac{\frac{1}{2}(a+\frac{b}{c})}{x-c}$$

$$y' = \frac{1}{2}(a-\frac{b}{c})\{-(x+c)^{-2}\} + \frac{1}{2}(a+\frac{b}{c})\{-(x-c)^{-2}\}$$

$$y'' = \frac{1}{2}(a-\frac{b}{c})\{2\cdot(x+c)^{-3}\} + \frac{1}{2}(a+\frac{b}{c})\{2(x-c)^{-3}\}$$

$$y^{(n)} = \frac{1}{2}(a-\frac{b}{c})(-1)^n n!(x+c)^{-(n+1)} + \frac{1}{2}(a+\frac{b}{c})(-1)^n n!(x-c)^{-(n+1)}$$

$$= \frac{(-1)^n n!}{2}\left\{(a-\frac{b}{c})(x+c)^{-(n+1)} + (a+\frac{b}{c})(x-c)^{-(n+1)}\right\}$$

(6) $$y = x^2 \sin x$$

$u = x^2$ $\quad$ $v = \sin x$

$u' = 2x$ $\quad$ $v^{(n-2)} = \sin(x + \frac{n-2}{2}\pi)$

$u'' = 2.$ $\quad$ $v^{(n-1)} = \sin(x + \frac{n-1}{2}\pi)$

$v^{(n)} = \sin(x + \frac{n\pi}{2})$

$$y^{(n)} = x^2 \sin(x + \frac{n\pi}{2}) + 2nx \sin(x + \frac{n-1}{2}\pi) + 2\frac{n(n-1)}{2}\sin(x + \frac{n-2}{2}\pi)$$

(7) $$y = e^{ax}\cos kx.$$

$u = e^{ax}$ $\quad$ $v = \cos kx = \sin(kx + \frac{\pi}{2})$

$u' = ae^{x}$ $\quad$ $v' = k(\sin(kx + \frac{2\pi}{2}))$

$u'' = ae^{x}$ $\quad$ $v'' = k^2(\sin kx + \frac{3\pi}{2})$

$$y^{(n)} = e^{ax}\{k^n(\sin kx + \frac{n-1}{2}\pi) + ank^{n-1}(\sin kx + \frac{n-2}{2}\pi$$

(1) $y = \sin kx \qquad y' = k\cos kx = k\sin\left(kx + \frac{\pi}{2}\right)$

$\therefore\ y^{(n)} = k^n \sin\left(kx + \frac{n\pi}{2}\right).$

(2) $y = \sin^3 kx \qquad y' = 3\sin^2 kx \cdot k\cos kx.$

$= 3k(\cos kx - \cos^3 kx)$

$= 3k\left\{\sin\left(kx+\frac{\pi}{2}\right) - \sin^3\left(kx+\frac{\pi}{2}\right)\right\}$

$y'' = 3k^2\sin\left(kx+\frac{2\pi}{2}\right) - 9k^2\sin\left(kx+\frac{2\pi}{2}\right) + 9k^2\sin^3\left(kx+\frac{2\pi}{2}\right)$

$= -3k^2\left(2\sin\left(kx+\frac{2\pi}{2}\right) - 3\sin^3\left(kx+\frac{2\pi}{2}\right)\right)$

$y''' = -3k^2\left(2k\sin\left(kx+\frac{3\pi}{2}\right) - 9k\sin\left(kx+\frac{3\pi}{2}\right) + 9k\sin^3\right)$

$= 3k^3\left(7\sin\left(kx+\frac{3\pi}{2}\right) - 9\sin^3\left(kx+\frac{3\pi}{2}\right)\right)$

$y^{(n)} = (-1)^{n-1} 3k^n (2_0 \sin$

$-9-2$

$7 - 27 = -20. \quad 27$

27

(2) $y = \sin^3 kx = \frac{1}{4}(3\sin kx - \sin 3kx)$

$y' = \frac{3k}{4}\left(\sin\left(kx+\frac{\pi}{2}\right) - \sin\left(3kx+\frac{\pi}{2}\right)\right)$

$y^{(n)} = \left(\frac{3k}{4}\right)^n\left(\sin\left(kx+\frac{n\pi}{2}\right) - \sin\left(3kx+\frac{n\pi}{2}\right)\right)$

(3) $y = a^{kx} \qquad y' = k a^{kx}\log a \qquad y'' = k^2\log^2 a\, a^{kx}$

$y^{(n)} = k^n \log^n a \cdot a^{kx}$

(4) $y = \frac{1-x}{1+x} \quad y' = \frac{-(1+x) - (1-x)}{(1+x)^2} = -\frac{2}{(1+x)^2} = -2(1+x)^{-2}$

$y'' = -2 \cdot -2\,(1+x)^{-3}. \qquad y''' = -2 \cdot -2 \cdot -3\,(1+x)^{-4}.$

$y^{(n)} = (-1)^n\, 2 \cdot n! \cdot (1+x)^{-(n+1)}$

$$y = \frac{ax+b}{x^2-c^2}$$

$$u = ax+b \qquad v = \frac{1}{x^2-c^2}$$

$$u' = a \qquad v' = \frac{-1}{(x^2-c^2)^2} \cdot 2x$$

$$u'' = 0 \qquad v'' = -2\left\{\frac{1}{(x^2-c^2)^2} - x\frac{2}{(x^2-c^2)^3}\right\}$$

$$v''' = \frac{-2 \cdot -3!}{(x^2-c^2)^4}$$

$$v^{(n-2)} = \frac{(-1)^{n-2}\, n-2!}{(x^2-c^2)^{n-1}}$$

$$v^{(n-1)} = \frac{(-1)^{(n-1)}\, n-1!}{(x^2-c^2)^{n}}$$

$$v^{(n)} = \frac{(-1)^n\, 2\; n!}{(x^2-c^2)^{n+1}}$$

$$y^{n)} = (ax+b)\frac{(-1)^n \times 2 \times n!}{(x^2-c^2)^{n+1}} + na\frac{(-1)^{n-1}\, 2 \cdot (n-1)!}{(x^2-c^2)^n}$$

$$= \frac{(-1)^n \cdot 2\, n!}{(x^2-c^2)^{n+1}}\left(ax+b - a(x^2-c^2)\right)$$

$$= \frac{(-1)^{n+1}\, 2 \cdot n!\,(ax^2 - ax + b + ac^2)}{(x^2-c^2)^{n+1}}$$

$$y = \frac{a-\frac{b}{c}}{2(x-c)} + \frac{a+\frac{b}{c}}{2(x+c)}$$

$$y' = \frac{a-\frac{b}{c}}{2}\,\frac{-}{(x-c)^2}$$

17. $\tan\alpha = \frac{dy}{dx}$　　$y = \rho\sin\theta$　$x = \rho\cos\theta$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\rho\cos\theta + \sin\theta\frac{d\rho}{d\theta}}{-\rho\sin\theta + \cos\theta\frac{d\rho}{d\theta}}$$

$$a^{2m}xy^{m-2}-a^{m}xy^{m-2}x_1^{m}=a^{2m}y^{m-1}x_1^{m-1}-b^{m}x^{m}x_1^{m-1}$$
$$+b^{m}x^{m-1}x_1^{m}.$$

(15) $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1.$ $\qquad \frac{dy}{dx}=-\frac{b^m x^{m-1}}{a^m y^{m-1}}.$

$$y-y_1=\frac{dy_1}{dx_1}(x-x_1) \qquad y-y_1=-\frac{b^m x_1^{m-1}(x-x_1)}{a^m y_1^{m-1}}$$

$$-\frac{dy_1}{dx_1}y=x, \qquad \frac{y^m}{b^m}-\frac{y_1^{m-1}y}{b^m}=-\frac{x^m}{a^m}+\frac{x_1^{m-1}x}{a^m}$$

$$\frac{b^m x_1^{m-1}}{a^m y_1^{m-1}}y=x, \qquad \frac{x_1^m}{a^m}+\frac{y_1^m}{b^m}=\frac{y_1^{m-1}}{b^m}y+\frac{x_1^{m-1}}{a^m}x$$

$$1=$$

$$\frac{b^m}{y_1^{m-1}}y-\frac{a^m}{x_1^{m-1}}x=0$$

$$\frac{y_1^{m-1}}{b^m}=\frac{x_1^{m-1}y}{a^m x}$$

(16) $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$

$$\rho^2=x^2+y^2 \qquad \tan\theta=\frac{y}{x}.$$

$$x=\rho\sin\theta\ (\cos), \quad y=\rho\cos\theta\ (\sin).$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cos\theta\frac{d\rho}{d\theta}-\rho\sin\theta}{\sin\theta\frac{d\rho}{d\theta}+\rho\cos\theta}$$

$$\frac{ds}{d\theta}=\frac{ds}{dx}\cdot\frac{dx}{d\theta}=\sqrt{1+\left(\frac{\cos\theta\frac{d\rho}{d\theta}-\rho\sin\theta}{\sin\theta\frac{d\rho}{d\theta}+\rho\cos\theta}\right)^2}\cdot\sin\theta\frac{d\rho}{d\theta}+\rho\cos\theta,$$

$$=\sqrt{\sin^2\theta\left(\frac{d\rho}{d\theta}\right)^2+\rho^2\cos^2\theta+\cos^2\theta\left(\frac{d\rho}{d\theta}\right)^2+\rho^2\sin^2\theta}$$

$$=\sqrt{\left(\frac{d\rho}{d\theta}\right)^2+\rho^2}$$

(13) $\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3=1$.　$\frac{3x^2}{a^3}+\frac{3y^2}{b^3}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-b^3x^2}{a^3y^2}$$

$$\frac{lx^2}{a^3}+\frac{my^2}{b^3}=1.$$

$$(y-y')=-\frac{b^3x^2}{a^3y^2}(x-x')$$

$$(m-y')=-\frac{b^3x^2}{a^3y^2}(l-x')$$

$$-\frac{my^2}{b^3}+\frac{y'y^2}{b^3}=\frac{lx^2}{a^3}-\frac{x'x^2}{a^3}.$$

$$\frac{dy}{dx}$$

(14)

(a,b)

$$y-y_1=\frac{dy}{dx}(x-x_1)$$

$$\frac{a\frac{dy}{dx}-b-(x_1\frac{dy}{dx}-y_1)}{\varepsilon\sqrt{\left(\frac{dy}{dx}\right)^2+1}}$$

(15) $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1$.　$\frac{dy}{dx}=\frac{-b^m x^{m-1}}{a^m y^{m-1}}$

$$y-y_1=\frac{dy_1}{dx_1}(x-x_1)\qquad a^m y^m-a^m y_1 y^{m-1}=-b^m x^m+b^m x_1 x^{m-1}$$

$$-\frac{dy}{dx_1}y=x_1\qquad -b^m x_1^{m-1}y=a^m y_1^{m-1}x.$$

$$\frac{y_1}{b}=\frac{y}{b}+\frac{b^{m-1}x^m}{a^m y^{m-1}}-\frac{b^{m-1}x_1x^{m-1}}{a^m y^{m-1}}$$

$$\frac{y_1^{m-1}}{b^{m-1}}=\frac{-b\,x_1^{m-1}y}{a^m x}$$

$$1-\left(\frac{x_1}{a}\right)^m=\frac{-x_1^{m-1}y}{a^m x}-\frac{b^m x^{m-1}x_1^{m-1}}{a^{2m}y^{m-2}}+\frac{b^m x_1^m x^{m-2}}{a^{2m}y^{m-2}}$$

(11) $\tan\phi = \dfrac{\rho}{\frac{d\rho}{d\theta}} = \dfrac{\rho}{\rho\cdot a} = \dfrac{1}{a},$

(12) $\rho = a(1+\cos\theta) \quad \rho = b(1-\cos\theta),$

然 $a + a\cos\theta = b - b\cos\theta$

$\cos\theta = \dfrac{-(a-b)}{a+b} \quad \rho = 2ab.$

$\dfrac{d\rho}{d\theta} = -a\sin\theta = -a\sqrt{1-\dfrac{(a-b)^2}{(a+b)^2}} = -a\sqrt{4ab}/(a+b)$

$\therefore \tan\phi = \dfrac{\rho}{\frac{d\rho}{d\theta}} = \dfrac{-\sqrt{b}}{\sqrt{a}}(a+b) = \dfrac{-2b}{\sin\theta}$

$\dfrac{d\rho}{d\theta} = b\sin\theta = b\sqrt{4ab}/(a+b)$

$\therefore \tan\phi' = \dfrac{\rho}{\frac{d\rho}{d\theta}} = \dfrac{\sqrt{a}}{\sqrt{b}}(a+b) = \dfrac{2a}{\sin\theta}$

$\tan A = \dfrac{\frac{\sqrt{a}}{\sqrt{b}}(a+b) + \frac{\sqrt{b}}{\sqrt{a}}(a+b)}{1-(a+b)^2} = \dfrac{(a+b)^2}{\sqrt{ab}\,(1-(a+b)^2)}.$

$1 + \tan\phi\tan\phi' = 1 - \dfrac{4ab}{\sin^2\theta} = \dfrac{1-\left(\frac{a-b}{a+b}\right)^2 - 4ab}{\sin^2\theta}.$

In the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ find the length of the ⊥ar from the origin on the tangent at (x, y); also find the length of that part of tangent which is intercepted between the 2 axes.

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{dy}{dx}\frac{2}{3}y^{-\frac{1}{3}}=0.$$

$$\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

$$\frac{y+x\cdot\left(\frac{y}{x}\right)^{\frac{1}{3}}}{\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}}=\frac{y^{\frac{1}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)}{x^{\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}$$

$$=\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}.$$

$$\left(x-\frac{y}{\frac{dy}{dx}}\right)^2+\left(y-x\frac{dy}{dx}\right)^2$$

$$=\left(x+y^{\frac{2}{3}}\cdot x^{\frac{1}{3}}\right)^2+\left(y+x^{\frac{2}{3}}\cdot y^{\frac{1}{3}}\right)^2.$$

$$=\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)$$

$$=\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3$$

$$=x^2+3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}+y^2.$$

(8) $a_1 x^2 + b_1 y^2 = 1$, $\qquad \frac{dy}{dx} = -\frac{a_1 x}{b_1 y}$

$a_2 x^2 + b_2 y^2 = 1$, $\qquad \frac{dy}{dx} = -\frac{a_2 x}{b_2 y}$

(x_1, y_1) で交はる，とする.

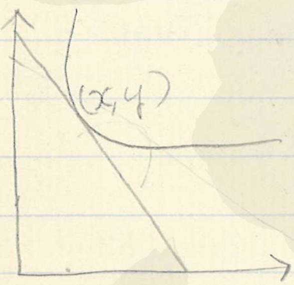
$$\frac{a_1 x_1}{b_1 y_1} = -\frac{b_2 y_1}{a_2 x_1}$$

$$a_1 a_2 x_1^2 + b_1 b_2 y_1^2 = 0.$$

$$\rightarrow \quad x_1^2 + y_1^2 = 0,$$

$$a_1 a_2 = b_1 b_2.$$

(9)



$$(Y - y) = \frac{dy}{dx}(X - x)$$

$Y = 0 \quad X = x - \frac{y}{\frac{dy}{dx}}$

$X = 0 \quad Y = y - \frac{dy}{dx} y$

(10)

$$(Y - y) = \frac{dy}{dx}(X - x)$$

$$\frac{y - x\frac{dy}{dx}}{\varepsilon\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}$$

(5) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0.$$

$$\frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

$$=\frac{y^2}{b^2}x\cdot\frac{b^2xy}{b^2-y^2}=\frac{\frac{x^2y^2}{b^2-y^2}}{}$$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$

$$\text{sub tan}=y\cdot\frac{ya^2}{xb^2}=\frac{a^2}{x}\left(1-\frac{x^2}{a^2}\right)$$

$$\frac{a^2}{x^2}=\frac{1}{1-\frac{y^2}{b^2}}$$

$$=\boxed{\frac{a^2}{x}-x}$$

$$=\frac{b^2xy}{b^2-y^2}$$

(6) $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$

$$\frac{dy}{dx}=\frac{e^{\frac{x}{a}}}{2}-\frac{e^{-\frac{x}{a}}}{2}.$$

$$e^{\frac{2x}{a}}$$

$$\text{Norm: }\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\sqrt{1+\frac{1}{4}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2}$$

$$=\frac{y^2}{a}$$

(7) $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

$$\frac{nx^{n-1}}{a^n}+\frac{ny^{n-1}}{b^n}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^nx^{n-1}}{a^ny^{n-1}}$$

$$\frac{dy}{dx}_{a,b},=\frac{b}{a}$$

Examples.

(1) $y(x-1)(x-2)=x-3$

$$\frac{dy}{dx}=\frac{(x-1)(x-2)-(x-3)(x-1)-(x-3)(x-2)}{(x-1)^2(x-2)^2}$$

$$\left(\frac{dy}{dx}\right)_{x=3+\sqrt{2}}=\frac{(2+\sqrt{2})(1-\sqrt{2})-\sqrt{2}(2+\sqrt{2})-\sqrt{2}(1+\sqrt{2})}{(2+\sqrt{2})^2(1+\sqrt{2})^2}$$

$$=0.$$

$x+\frac{1}{x}$

(2) $y^2=4dx$

$2y\frac{dy}{dx}=4d$ $\quad \frac{dy}{dx}=\frac{4d}{2y}=1$

$x(1+\frac{1}{x})$

$\begin{cases} y=2d \\ x=d \end{cases}$

$e^{x}(1+e^{-2x})$

(3) $e^x-e^{-x}-2\sin x=f(x)$

$f(0)=0 \quad f'(x)=e^x+e^{-x}-2\cos x,$

$(e^{\frac{x}{2}}-e^{-\frac{x}{2}})^2$

$e^x+e^{-x}\geqq 2 \quad (\text{for } x\geqq 0)$

$2\cos x\leqq 2$

$\therefore f'(x)\geqq 0 \quad (\text{for } x\geqq 0)$

$\therefore x\geqq 0$ で increasing

(4) $y=x^5-5ax^4+5a^2x^3 \quad a>0.$

$$\frac{dy}{dx}=5x^4-20ax^3+15a^2x^2$$

$$=5x^2(x^2-4ax+3a^2)=5x^2(x-3a)(x-a)$$

$3a>x>a$