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A Method of Quantization of Nonlinear Fields

One particle probability amplitude
$\psi(x, y, z, t)$

Probability amplitude for a particle of the kind α
$$\psi_\alpha(x, y, z, t) \quad \text{(more briefly } \alpha \text{ mode function)}$$
or the wave function for the mode α of the fundamental entity of matter

Question 1. What is the law of which determines the totality of mode of the entity?

Probability amplitude c_α that the mode α is realized.
The ~~amplitude~~ totality of the amplitudes c_α ($\alpha = 1, 2, \ldots$) constitutes a state
$$\Psi = c_0 \Psi_0 + \sum_\alpha c_\alpha \Psi_\alpha$$
where
Ψ_0: the state in which no mode is realized, i.e. the state of vacuum
Ψ_α: the state in which the mode α is realized

~~We~~ The possibility that two or more modes are realized simultaneously is precluded.

We introduce the operators such that
$$\Psi_\alpha = a_\alpha^* \Psi_0$$
$$a_\alpha \Psi_\alpha = \Psi_0 \qquad (a_\alpha \Psi_0 = 0)$$
$$a_\beta \Psi_\alpha = 0 \qquad \alpha \neq \beta$$
$$a_\beta^* \Psi_\alpha = a_\beta^* a_\alpha^* \Psi_0 = 0$$
$$a_\alpha a_\beta \Psi = 0$$
$$\Psi = (c_0 + c_\alpha a_\alpha^*) \Psi_0$$

$$\Psi = \begin{pmatrix} c_0 \\ \vdots \\ c_\alpha \\ \vdots \\ c_\beta \\ \vdots \end{pmatrix} \qquad a_\alpha = \begin{array}{c|c} & \begin{array}{ccccc} 0 & & & \alpha & \end{array} \\ \hline & \begin{pmatrix} 0 & \cdots & \cdots & 1 & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & 0 & & \ddots \end{pmatrix} \end{array}$$

$$\Psi_0 = \begin{array}{r} 0 \rightarrow \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \qquad a_\alpha^* = \begin{array}{r} 0 \\ \vdots \\ \vdots \\ \alpha \end{array} \begin{pmatrix} 0 & & & \\ 0 & & & \\ & 0 & & \\ & & \ddots & \\ 1 & & & \ddots \end{pmatrix}$$

$$\Psi_\alpha = \begin{array}{r} \\ \\ \\ \alpha \rightarrow \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$a_\alpha a_\alpha^* \Psi_0 = \Psi_0 \qquad a_\alpha^* a_\alpha \Psi_0 = 0$$

$$a_\alpha a_\beta^* \Psi_0 = \delta_{\alpha\beta} \Psi_0$$

$$a_\alpha \Psi = a_\alpha (c_0 + c_\beta a_\beta^*) \Psi_0 = c_\alpha \Psi_0$$

$$a_\alpha^* \Psi = a_\alpha^* (c_0 + c_\beta a_\beta^*) \Psi_0 = c_0 \Psi_\alpha$$

$$\left.\begin{array}{l} a_\alpha a_\alpha^* \Psi = c_0 \Psi_0 \\ a_\alpha^* a_\alpha \Psi = c_\alpha \Psi_\alpha \end{array}\right\} [a_\alpha, a_\alpha^*] \Psi = c_0 \Psi_0 - c_\alpha \Psi_\alpha$$

$$a_\alpha a_\beta^* \Psi = c_0 \cancel{a_\alpha} \Psi_\beta = c_0 \Psi_0 \delta_{\alpha\beta}$$

$$a_\beta^* a_\alpha \Psi = c_\alpha a_\beta^* \Psi_0 = c_\alpha \Psi_\beta$$

$$[a_\alpha, a_\beta^*] \Psi = c_0 \Psi_0 \delta_{\alpha\beta} - c_\alpha \Psi_\beta$$

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arbitrary operator $\boxed{O = O_{\alpha\beta} a_\alpha^* a_\beta}$

unit operator $\sum a_\alpha^* a_\alpha$

$$\sum_{\alpha=0}^{\infty} a_\alpha^* a_\alpha \Psi = \Psi$$

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Formal
Construction of states and operators

$\Psi = \sum_\alpha c_\alpha \Psi_\alpha$? $\alpha = 0, 1, 2, \cdots$

$(\Psi_\alpha, \Psi_\beta) = 0$? In what sense ?

$a_\alpha \Psi_\beta = \cancel{(\delta_{\alpha\beta} - \delta_{\alpha 0}\delta_{\beta 0})\Psi_0}\ \delta_{\alpha\beta}\Psi_0$ $\alpha, \beta = 0, 1, 2, \cdots$

$a_\alpha^* \Psi_\beta = \delta_{\beta 0} \Psi_\alpha$

$a_\alpha^* a_\beta \Psi_\gamma = a_\alpha^* \delta_{\beta\gamma}\Psi_0 = \delta_{\beta\gamma}\Psi_\alpha$

$a_\beta a_\alpha^* \Psi_\gamma = a_\beta \delta_{\gamma 0}\Psi_\alpha = \delta_{\gamma 0}\delta_{\beta\alpha}\Psi_0$

$[a_\beta, a_\alpha^*]\Psi_\gamma = \delta_{\beta\alpha}\delta_{\gamma 0}\Psi_0 - \delta_{\beta\gamma}\Psi_\alpha$

$a_\alpha a_\beta \Psi_\gamma = \delta_{\beta\gamma} a_\alpha \Psi_0 = \delta_{\beta\gamma}\delta_{\alpha 0}\Psi_0$

$a_\alpha^* a_\beta^* \Psi_\gamma = \delta_{\gamma 0} a_\alpha^* \Psi_\beta = \delta_{\gamma 0}\delta_{\beta 0}\Psi_\alpha$

Superposition of non-orthogonal eigenstates ?
or new definition of inner product $(\Psi_\alpha, \Psi_\beta)$
(metric of Hilbert space) ?

$\cancel{\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \kappa^2\right)}$

Orthogonality condition for ~~wave~~ mode function ψ_α
instead of state vector Ψ_α

$$\psi_\beta^* \quad \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \kappa^2\right)\psi_\alpha = \rho_\alpha$$

where ρ_α depends on ψ_α, ~~as well as~~ if we consider only one type of the field ψ as representing the ~~entity of~~ fundamental entity.

$$-\psi_\alpha \quad \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \kappa^2\right)\psi_\beta^* = \rho_\beta^*$$

$$\frac{\partial}{\partial x_\mu}\left(\psi_\beta^* \frac{\partial \psi_\alpha}{\partial x_\mu} - \psi_\alpha \frac{\partial \psi_\beta^*}{\partial x_\mu}\right) = \psi_\beta^* \rho_\alpha - \psi_\alpha \rho_\beta^*$$

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$$\boxed{\int \cdots \int (\psi_\rho^* P_\alpha - \psi_\alpha \rho_\beta^*)\, d^4x = 0}$$

Reduction of non-linear ~~de~~ c-number field equation to <u>linear</u> ~~of~~ q-number field equation

$$\Psi_\alpha = a_\alpha \psi_\alpha$$

$$a_\alpha : \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \kappa^2 \right) \psi_\alpha = \sum_{n=2}^{\infty} c_n \psi_\alpha^n$$

$$\left(\frac{\partial}{\partial x_\mu \partial x_\mu} - \kappa^2 \right) \Psi_\alpha = \left(\sum_{n=2}^{\infty} c_n \psi_\alpha^{n-1} \right) \Psi_\alpha$$

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa \right) \psi = \sum_{n=1}^{\infty} c_n \psi^n$$

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa \right) \psi^m = \sum_{k=0}^{m-1} \psi^k \cdot \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa \right) \psi \cdot \psi^{m-k-1}$$

$$= \sum_{k=0}^{m-1} \psi^k \sum_{n=1}^{\infty} c_n \psi^n \cdot \psi^{m-k-1}$$

$$(\quad \cdot\cdot \quad) \sum_{m=1}^{\infty} b_m \psi^m = \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \sum_{n=1}^{\infty} b_m c_n \psi^k \psi^n \psi^{m-k-1}$$

$$= \sum m\, b_m c_n \psi^{m+n-1}$$

arbitrary operators in the Hilbert space

$$\Psi = \sum_\alpha c_\alpha \Phi_\alpha$$

$$\rho = \sum \rho_{\alpha\beta} a_\alpha^* a_\beta$$

Identity $\quad 1 = \sum a_\alpha^* a_\alpha$

What is the law of construction ρ or $\rho_{\alpha\beta}$ which corresponds to a physically important quantity?

$$a_0 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \qquad a_0^* = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$$a_0 = a_0^*$$

$$a_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad a_1^* = \begin{pmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

etc.

$$a_2 = \begin{pmatrix} 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad a_2^* = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$a_0^* a_0 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix} = a_0 \qquad a_0^* a_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = a_1$$

$$a_1^* a_1 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} \qquad a_1^* a_0 = \begin{pmatrix} 0 & 0 & \cdots \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = a_1^*$$

$$a_2^* a_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \qquad a_1^* a_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$a_2^* a_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Properties of the operator

$$\rho = \sum \rho_{\alpha\beta} a_\alpha^* a_\beta$$

$$\rho_{\alpha\beta} a_\alpha^* a_\beta \Psi_\gamma = \rho_{\alpha\beta} \delta_{\beta\gamma} \Psi_\alpha = \rho_{\alpha\gamma} \Psi_\alpha$$

$$\rho \Psi = \rho_{\alpha\beta} a_\alpha^* a_\beta \sum c_\gamma \Psi_\gamma = \rho_{\alpha\gamma} c_\gamma \Psi_\alpha$$

$$= \rho_{\alpha\beta} c_\beta \Psi_\alpha$$

Thus ρ is ~~an~~ a transformation in Ψ-space which changes $\Psi = (c_\alpha)$ into $\rho\Psi = (\sum_\beta \rho_{\alpha\beta} c_\beta)$

According to ~~In~~ the customary method of second quantization, Lagrangian or Hamiltonian of a system, which is a certain space integral of the bilinear form ~~quantity~~ (for a free field) of field quantities, is replaced by the same integral of ~~a~~ the field operators. In our case, the ~~corresponding~~ problem is to find the correct form for the bilinear operator

$$\rho = \sum \rho_{\alpha\beta} a_\alpha^* a_\beta$$

from which physical laws could be derived just as the field equations were derived from the Lagrangian or Hamiltonian in the customary theory. $\rho_{\alpha\beta}$ should be related to the solutions (mode functions) ψ_α, ψ_β of a certain non-linear field equations. The case α or $\beta = 0$ is exceptional, because the corresponding ψ_α, ~~ψ~~ or ψ_β is not well-defined. $\rho_{\alpha\beta}$ should depend on the properties of the states α and β, such as spin, mass, parity etc. of each state.

Let us suppose that

(i) the law which determines the state $\beta \equiv \Psi$ is

$$[a_\alpha, \rho]\Psi = 0$$
$$[a_\alpha^*, \rho]\Psi = 0$$

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Correspondence to linear field theory

$$\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \varkappa^2\right) u_k = 0$$

$$u = \sum_k q_k u_k$$

$$[q_k, q_{k'}] = \delta(k + k')$$

u is the superposition of only one particle solutions u_k while the totality of (φ_α) of the solutions φ_α contains not only one particle-like solutions, but also two, three etc particle-like solutions. ~~Corresponding to the~~ Now the question arises: ~~Foest~~ Did the possible modes increase so very much due to the introduction of the non-linear term in the field equation?

Take a very simple example of particle dynamics.

$$2\frac{dq}{dt}: \quad \frac{d^2q}{dt^2} = -\omega^2 q + \lambda q^3$$

$$\frac{d}{dt}\left(\frac{dq}{dt}\right)^2 = -\frac{d}{dt}\left(\omega^2 q^2 - \frac{\lambda}{2} q^4\right)$$

$$\left(\frac{dq}{dt}\right)^2 = C - \omega^2 q^2 + \frac{\lambda}{2} q^4$$

$$= (a q^2 \pm b)^2$$

$$a^2 = C \qquad 2ab = \omega^2 \qquad b^2 = \frac{\lambda}{2}$$

$$C\lambda = \omega^4 \qquad C = \omega^4/\lambda$$

$$\frac{dq}{dt} = \pm(a q^2 - b)$$

$$\frac{dq}{dt} = i\omega q - \lambda q^2$$

$$\frac{1}{i\omega q - \lambda q^2}\, dq = dt$$

$$\frac{dq}{\cancel{i\omega q(q-\ )}\; q(i\omega - \lambda q)} = \left(\frac{1}{q} + \frac{\lambda}{i\omega - \lambda q}\right)\frac{dq}{i\omega} = dt$$

$$\frac{1}{i\omega}\left(\log q - \log\left(q - \frac{\lambda}{i\omega}\right)\right) = t + c$$

$$\log\left(q \Big/ q - \frac{\lambda}{i\omega}\right) = i\omega t + c'$$

$$\frac{q}{q - \cancel{\frac{\lambda}{i\omega}}\,\frac{i\omega}{\lambda}} = C \exp i\omega t$$

$$(1 - C \exp i\omega t)\, q = - \cancel{\frac{\lambda}{i\omega}}\,\frac{i\omega}{\lambda}\, C \exp(i\omega t)$$

$$q = \frac{\frac{i\omega}{\lambda}\cancel{\frac{\lambda}{i\omega}}\, C \exp(i\omega t)}{C \exp i\omega t - 1}$$

$$= \frac{\cancel{\frac{\lambda}{i\omega}}\;\; \frac{i\omega}{\lambda}}{1 - C^{-1}\exp(-i\omega t)}$$

$$|C^{-1}| < 1 \quad \text{or} \quad |C| > 1$$

$$q = \cancel{\frac{\lambda}{i\omega}}\,\frac{i\omega}{\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot C^{-n} \cdot e^{-i\omega t\,[n]}$$

$C \to \infty$

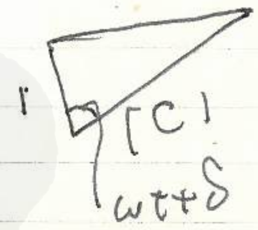
$$q = \frac{i\omega}{\lambda}\,(1 - C^{-1} e^{-i\omega t})$$

$|C| \ll 1$:

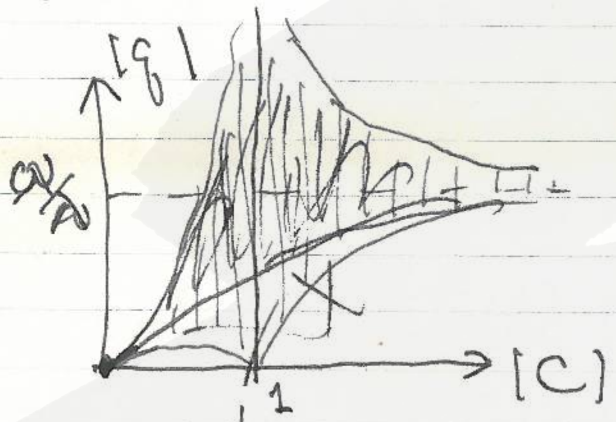
$$q = -\frac{i\omega}{\lambda} C \exp i\omega t \sum_{n=0}^{\infty} \frac{1}{n!} C^{n} e^{in\omega t}$$

$C \to 0$

$$q = -\frac{i\omega}{\lambda} C \exp i\omega t$$



$$\frac{C \exp i\omega t - 1}{\tilde{C} \exp(-i\omega t) - 1} \quad |C|^2 + 1$$



$$|q| = \frac{\lambda}{\omega} \cdots \frac{1}{|1 - C^{-1}\exp(-i\omega t)|}$$

$$C = |C| \exp i\delta$$

$$|C \exp i\omega t - 1| = \big|\,|C| \exp i(\omega t + \delta) - 1\,\big|$$

$$= \sqrt{|C|^2 + 1 - 2|C| \cos(\omega t + \delta)}$$

$$|q| = \frac{\omega}{\lambda}|C| \Big/ \sqrt{|C|^2 + 1 - 2|C| \cos(\omega t + \delta)}$$

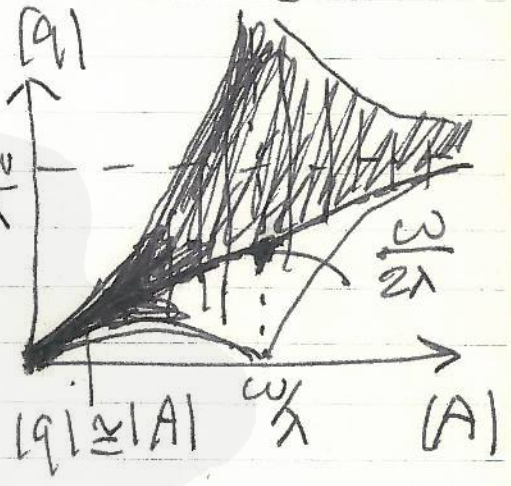
$$= \frac{\omega}{\lambda} \Big/ \sqrt{1 + \frac{1}{|C|^2} - \frac{2\cos(\omega t + \delta)}{|C|}}$$

$\lambda = 0.$ $\cancel{q = A \exp i\omega t}$

$$q = +\frac{i\omega}{\lambda} C \exp i\omega t$$

$$= A \exp i\omega t$$

$$C = \frac{\lambda}{i\omega} A$$



λ=0 の場合には C=0 の場合しか現われてこない。しかしその代り A が任意の値をとり得る。
λが非常に小さい極限では A を量子化しても C は量子化されない。反対に C を量子化すると C≠0 なる固有値に対する A の固有値は∞になってしまう。

$$\frac{i\omega}{\lambda} C = A \qquad A = |A| \cdot \exp i\varepsilon$$

$$q = \frac{A \exp(i\omega t)}{\frac{\lambda}{i\omega} A \exp(i\omega t) - 1}$$

$$q = \frac{|A| \exp i(\omega t + \varepsilon)}{\frac{\lambda}{\omega} |A| \exp i(\omega t + \varepsilon - \frac{\pi}{2}) - 1}$$

$$|q| = \frac{|A|}{\sqrt{\left(\frac{\lambda}{\omega}|A|\right)^2 + 1 - 2\frac{\lambda}{\omega}|A| \cos(\omega t + \varepsilon - \frac{\pi}{2})}}$$

$$\frac{|A|}{\frac{\lambda}{\omega}|A| + 1}$$

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linear theory $(\lambda \ll \omega)$

$q = A \exp i\omega t$

$|A| = 1$: one particle s,

$|A| = \sqrt{2}$: two particle s,

$\vdots$

$|A| = \sqrt{n}$: n-particle s.

non-linear theory

$$q = \frac{A \exp i\omega t}{\frac{\lambda}{i\omega} A \exp(i\omega t) - 1}$$

$|A| = 1$

$|A| = \sqrt{2}$

$\vdots$

$|A| = \sqrt{n}$.

$\frac{\lambda}{\omega} \ll 1$

$q = (A) e^{i\omega t}$

$$q = -Ae^{i\omega t}\left(1 + \frac{\lambda}{i\omega} A e^{i\omega t}\right) = -Ae^{i\omega t} - \frac{\lambda}{i\omega} A^2 e^{2i\omega t}$$

$q = (A) e^{2i\omega t}$

$$\begin{pmatrix} e^{i\omega t} \\ e^{2i\omega t} \\ e^{3i\omega t} \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} e^{-2i\omega t} \\ e^{-i\omega t} \\ 0 \\ e^{i\omega t} \\ e^{2i\omega t} \\ \vdots \end{pmatrix}$$

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$$[q, p] = i\hbar$$

$$H = \frac{1}{2}(q^2 + p^2) - \frac{1}{2}$$

$$H\varphi_n = n\varphi_n$$

$$\left(\psi(q') = \sum_n c_n \varphi_n\right)$$

$$\Psi = \sum_n c_n \Psi_n$$

n: particle number

$$p = \sum_{n',n''} p_{n'n''} a^*_{n'} a_{n''}$$

$$q = \sum_{n'n''} q_{n'n''} a^*_{n'} a_{n''}$$