相対論とHilbert空間　March 2, 1955 (1)

第9回 素粒子基礎理論討論会

特殊相対論と量子力学の融合という問題を、特殊相対論とHilbert空間との関係という観点から取り上げてる。

先ず第二量子化をやる前の相対論的場の理論を規格的 ~~[illegible]~~ 道筋を、Lorentz群の表現という立場から考察してる；

$$x'_\mu = a_{\mu\nu} x_\nu \qquad x_0 = ct$$

$$a_{\mu\nu} : \text{real}$$

$$\bar{x}'_\mu = \bar{x}_\nu \bar{a}_{\nu\mu} \qquad \left.\begin{array}{l} \bar{a}_{\nu\mu} = a_{\mu\nu} \\ \bar{x}_\mu = x_\mu \end{array}\right\}$$

$$\bar{x}'_\mu g_{\mu\nu} x'_\nu = \bar{x}_\mu g_{\mu\nu} x_\nu \rightarrow \boxed{\bar{a} g a = g}$$

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$x'_\mu = g_{\mu\nu} x_\nu \rightarrow \left.\begin{array}{l} x'_1 = x_1, \; x'_2 = x_2 \\ x'_3 = x_3, \; x'_0 = t x_0 \end{array}\right\}$$

(time reversal) (or space reflection)

$$\psi'_\alpha = A_{\alpha\beta}\psi_\beta \qquad \bar{\psi}'_{\cancel{\alpha\beta}} = \bar{\psi}_\delta \bar{A}_{\delta\gamma}$$

(time reversal) (or space reflection): $\psi'_\alpha = G_{\alpha\beta}\psi_\beta$

$$\bar{\psi}'_\alpha G_{\alpha\beta} \psi'_\beta = \bar{\psi}_\gamma \bar{A}_{\gamma\alpha} G_{\alpha\beta} A_{\beta\delta} \psi_\delta$$

$$= \bar{\psi}_\gamma G_{\gamma\delta} \psi_\delta$$

$$\bar{A}_{\gamma\alpha} G_{\alpha\beta} A_{\beta\delta} = G_{\gamma\delta}.$$

or $\boxed{\bar{A} G A = G}$

有限次元の表現としては、よく知られているように

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(i) vector $A_{\mu\nu} \to a_{\mu\nu}$ $\quad G_{\mu\nu} \to g_{\mu\nu}$

(ii) spinor $G_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \gamma_4$

$$\bar{\psi}'\gamma_4\psi' = \bar{\psi}\gamma_4\psi$$

($(-p_0 + \alpha p + \beta m)\psi = 0$

$(-\beta p_0 + \beta\alpha p + m)\psi = 0$

$-i\beta\alpha = -i\rho_3\rho_1\sigma = \rho_2\sigma = \gamma$

$\gamma_1\gamma_2\gamma_3 = \rho_2\sigma_x\,\rho_2\sigma_y\,\rho_2\sigma_z$

$-i$ $\quad = i\rho_2 \quad \to \quad x_1' = +x_1, \dots \quad x_0' = -x_0$

$i\gamma_4 \quad \to \quad x_1' = -x_1, \dots \quad x_0' = x_0$ }

~~従って~~上の考察で time reversal の代りに space reflection を取ってもよいから、そうすると $\bar{\psi}\gamma_4\psi$ が invariant になる)

$$\bar{A}\gamma_4 A = \gamma_4$$

との例に取れば、有限次元の表現では、

$$\bar{A} G A = G$$

(この関係を満たすような (有限次元の) 行列を作ることが問題になっていた。但し G は 対角行列ではなく、しかも $G^2 = 1$ であるから、(or ±1) G の固有値は 1 及び −1 (違った一般には i, $-i$ をもとめた) で、~~for~~ +1 だけではない。

そこで次元数をふやした極限として無限次元の表現を考えると、それは unitary ではなく、$\bar{\psi} G \psi$ を不変

というような表現である.

無限次元の non-unitary な表現として, 四次元の orthogonal normalised function の complete set を basis とする表現について考えて見る

$$\varphi(x_\mu) = \sum_n c_n \chi_n(x_\mu) \qquad x'_\mu = \sum_\nu a_{\mu\nu} x_\nu$$

$$\downarrow$$

$$\varphi(x'_\mu) = \varphi'(x_\mu)$$

$$= \sum_n c'_n \chi_n(x_\mu)$$

$$\chi_n(x_\mu)$$

$$\downarrow$$

$$\chi_n(x'_\mu) = \chi'_n(x_\mu)$$

$$= \chi_{n'}(x_\mu) A_{n'n}$$

$$c_{n'} \chi_{n'}(x_\mu)$$

$$\downarrow$$

$$c_{n'} A_{nn'} \chi_n(x_\mu) = c'_n \chi_n$$

$$\boxed{c'_n = \sum_{n'} A_{nn'} c_{n'}} \longleftarrow x'_\mu = \sum_\nu a_{\mu\nu} x_\nu$$

space reflection

$x_1 \to x'_1 = -x_1$: $\chi_n(x_\mu) \to \chi_n(x'_\mu) = \chi_n(-x_1, x_2, x_3, x_0)$

$\chi_n(x'_\mu) \to \pm \chi_n(x_\mu)$

according as χ_n is even or odd with respect to x_1

etc.

thus, $c'_n = \varepsilon_{n_1} c_n$

$\varepsilon_{n_1} = \pm 1$ according as χ_n is even or odd

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with respect to x_1, etc

A transformation which leaves $x_\mu g_{\mu\nu} x_\nu$ invariant ~~would~~ should leave

(1) $\sum_n c_n^* \varepsilon_{n_1 n_2 n_3} c_n$ (space reflection)

(1)' or $\sum_n c_n^* \varepsilon_{n_0} c_n$ (time reversal)

invariant.

Thus, the quantity (1) or (1)' should have a physical meaning which is ~~invariant~~ ~~wit~~ independent of the choice of the coordinate system. This ~~Nega~~ formally introduction of negative energy by Dirac*, ~~is~~ very similar to ~~thus very naturally understood in this way~~, although Dirac and Pauli considered it in connection with ~~four-d~~ three dimensional interpretation of probability.

In our case, we can define further invariant quantities from

$$\varphi(x_\mu) = \sum_n c_n \chi_n(x_\mu)$$

$$\psi(x_\mu) = \sum_n d_n \chi_n(x_\mu)$$

(2) $\sum_n d_n^* \varepsilon_{n_1 n_2 n_3} c_n$

(2)' $\sum_n d_n^* \varepsilon_{n_0} c_n$

* Dirac, P.R.S. A180, 1 (1942)
Pauli, R.M.P. 15, 175 (1943)

(2) $\omega(z')$ can be regarded as a kind of correlation coeff. between two states φ and ψ.

Suppose that χ_n is even or odd with respect to x_0 accord. as n_0 is even or odd.

$$C_n = C_{n_1 n_2 n_3}\, e^{-\frac{1}{2}n_0\lambda}$$

$$n_0 = 0, 1, 2, \ldots$$

$$\sum_n C_n^* q_{n_0} C_n = \sum_{n_1 n_2 n_3} |C_{n_1 n_2 n_3}|^2 \underbrace{(1 - e^{-\lambda} + e^{-2\lambda} - \ldots)}_{\frac{1}{1+e^{-\lambda}}}$$

If also
$$d_n = d_{n_1 n_2 n_3}\, e^{-\frac{1}{2}n_0\lambda},$$
then
$$\sum_n d_n^* q_{n_0} d_n = \sum_{n_1 n_2 n_3} d^*_{n_1 n_2 n_3} C_{n_1 n_2 n_3} \times \frac{1}{1+e^{-\lambda}}$$

If
$$\sum_{n_0} C_{n_0} \chi_{n_0}(x_0) = \sum_{n_0} \hat{\chi}_{n_0}(y_0)\chi_{n_0}(x_0) = \delta(x_0 - y_0)$$

If
$$\chi_n(x) = \underbrace{(-1)^n \exp(x^2/2)}_{(2^n n! \sqrt{\pi})^{-1/2}} \frac{d^n \exp(-x^2)}{dx^n}$$