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Harish-Chandra
Infinite irreducible representations of the Lorentz group (1)
(expinor)

(P. R. S. A189 (1947), 372)
Dirac, P. R. S. A183 (1945) 284 ~~...~~

infinitesimal L. T.

$$x_k' = x_k + \varepsilon_k{}^l x_l \quad \left\{ = x_k + g_{km}\varepsilon^{ml} x_l \right.$$

$$\varepsilon_{kl} = -\varepsilon_{lk}$$

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$$

$$g_{kl} = 0 \quad \text{for } k \neq l$$

representation of this transf.:

$$1 + \frac{1}{2}\varepsilon_{kl} I^{kl}$$

I^{kl} : antisym. operators

$$[I^{kl}, I^{mn}] = -g^{km}I^{ln} + g^{lm}I^{kn} + g^{kn}I^{lm} - g^{ln}I^{km}$$

$$\because \quad x_k' = (\delta_{kl} + \cancel{g_{km}\varepsilon^{ml}}\ \varepsilon_k{}^l) x_l$$

$$\cancel{= \delta_{kl} +} \; g\, x_m \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_m}$$

$$= \delta_{kl} + \varepsilon_k{}^l \quad x_l \frac{\partial}{\partial x_k}$$

$$= \delta_{kl} + g_{km}\varepsilon^{ml} x_l \frac{\partial}{\partial x_k}$$

$$= \delta_{kl} \mp g_{km}\varepsilon^{lm} x_l \frac{\partial}{\partial x_k}$$

$$= \delta_{kl} - g_{kl}\varepsilon^{ml} x_m \frac{\partial}{\partial x_k}$$

$$\varepsilon_k{}^l = g^{km}\varepsilon_{km} = -g^{ml}\varepsilon_{mk} = -\varepsilon^l{}_k$$

$$x_k' = (\delta_{km} g_{ml} + \varepsilon_{kl}) x^l$$

$$= (\delta_{kl} x_l + \tfrac{1}{4}\varepsilon_{kl} x^l \frac{\partial}{\partial x^m}) x^m$$

$x_4 = i x_0$

$$x_\mu \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_\mu} = -i x_\mu \frac{\partial}{\partial x_0} - i x_0 \frac{\partial}{\partial x_\mu}$$

$$= -i \left(x_\mu \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_\mu} \right)$$

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$K_1 = iI^{23}$, $K_2 = iI^{31}$, $K_3 = iI^{12}$
$L_1 = iI^{10}$, $L_2 = iI^{20}$, $L_3 = iI^{30}$

$$\left.\begin{aligned} [K_1, K_2] &= iK_3 \\ [K_1, L_2] &= iL_3 = [L_1, K_2] \\ [L_1, L_2] &= -iK_3 \end{aligned}\right\}$$

spatial rotation group:
δ_3 : direct sum of ~~or~~ finite, unitary and ~~repr~~ irreducible representations.

$$(m|K^+|m-1) = (m-1|K^-|m) = \sqrt{(k+m)(k-m+1)}$$

$$K^+ = K_1 + iK_2, \quad K^- = K_1 - iK_2$$

$$K^2 = k(k+1)$$

$\mathbf{L}$: vector under δ_3

$$(k|\mathbf{L}|k') = 0 \quad \text{unless } k' = k \text{ or } k\pm 1$$

$$(k, m|L^+|k-1, m-1) = -p^-(k)\sqrt{(k+m)(k+m-1)}$$

etc.

$$(km|L_3|k-1, m) = p^-(k)\sqrt{(k+m)(k-m)}$$

etc

Two invariants

$$J^2 = -\tfrac{1}{2} I^{kl} I_{kl}$$

$$I = -\tfrac{1}{2}\varepsilon^{klmn} I_{kl} I_{mn} \qquad \varepsilon^{0123} = 1$$

$$[J^2, I^{kl}] = [I, I^{kl}] = 0$$

$$I = \mathbf{K}\mathbf{L}, \quad J^2 = \mathbf{K}^2 - \mathbf{L}^2$$

$$k(k+1)p(k) = I$$

$$J^2 = k(k+1)(1 - p(k)) - \cdots$$

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It is shown that corresponding to every pair of complex numbers $\varkappa$, $\varkappa^*$ for which $2(\varkappa-\varkappa^*)$ is real and integral, there exists, in general, one irreducible representation $D_{\varkappa,\varkappa^*}$ of the Lorentz group.

However, if $4\varkappa$, $4\varkappa^*$ are both real and integral there are two representations $\mathfrak{D}^+_{\varkappa,\varkappa^*}$ and $\mathfrak{D}^-_{\varkappa,\varkappa^*}$ associated to the pair $(\varkappa, \varkappa^*)$.

All these repres. are infinite except $\mathfrak{D}^-_{\varkappa,\varkappa^*}$ which is finite if $2\varkappa$, $2\varkappa^*$ are both integral.

For suitable values of $(\varkappa, \varkappa^*)$, $\mathfrak{D}_{\varkappa,\varkappa^*}$ or $\mathfrak{D}^+_{\varkappa,\varkappa}$ is unitary. Namely, $\mathfrak{D}_{\varkappa,\varkappa^*}$ is unitary only if

either $\quad \varkappa = -\frac{1}{2} + i\nu + \frac{1}{2}n \quad \varkappa^* = -\frac{1}{2} + i\nu - \frac{1}{2}n$

or $\quad \varkappa = \varkappa^* = -\frac{1}{2} + \nu \quad |\nu| \leq 2$

ν being arbitrary real number.

However, if $\varkappa$, $\varkappa^*$ are both of the form $\frac{1}{4}n(n \neq -2)$, $\mathfrak{D}^-_{\varkappa,\varkappa^*}$ is unitary only if $\frac{1}{2} \geq |\varkappa + \frac{1}{2}| = |\varkappa^* + \frac{1}{2}|$, ~~while~~ $\mathfrak{D}^+_{\varkappa\varkappa^*}$ is unitary whenever $|\varkappa + \frac{1}{2}| = |\varkappa^* + \frac{1}{2}|$

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