On the Interaction of the Elementary Particles. II.

§1. Quantization of the U-field in the absence ~~presence~~ of the electromagnetic field. ~~in the absence~~ in vacuum

Wave equations for the potentials $U, \tilde{U}$ can be derived from the Lagrangian

$$\overline{L}_U = \iiint L_U \, dv ,$$

where

$$4\pi L_U = +\frac{1}{c^2}\frac{\partial \tilde{U}}{\partial t}\frac{\partial U}{\partial t} - \operatorname{grad}\tilde{U}\operatorname{grad}U - \lambda^2 \tilde{U} U .$$

The canonical conjugates to $U, \tilde{U}$ become

$$U^\dagger = \frac{\delta \overline{L}_U}{\delta \frac{\partial U}{\partial t}} = +\frac{1}{4\pi}\frac{1}{c^2}\frac{\partial \tilde{U}}{\partial t}$$

and

$$\tilde{U}^\dagger = +\frac{1}{4\pi c^2}\frac{\partial U}{\partial t}$$

respectively.

Hence,

$$4\pi L_U = +\frac{c^2}{(4\pi)^2}\tilde{U}^\dagger U^\dagger - (\operatorname{grad}\tilde{U}\operatorname{grad}U) - \lambda^2 \tilde{U} U$$

Vertauschungsrel.

$$U^\dagger(\vec{r},t)\,U(\vec{r}\,',t) - U(\vec{r}\,',t)\,U^\dagger(\vec{r},t) = -i\hbar\,\delta(\vec{r},\vec{r}\,')$$

Hamiltonian for the U-field

$$\overline{H}_U = \iiint H_U \, dv$$

$$H_U = U^\dagger\frac{\partial U}{\partial t} + \tilde{U}^\dagger\frac{\partial \tilde{U}}{\partial t} - L_U$$

$$= +\frac{c^2}{4\pi}\tilde{U}^\dagger U^\dagger + \frac{1}{4\pi}(\operatorname{grad}\tilde{U}\operatorname{grad}U) + \frac{\lambda^2}{4\pi}\tilde{U} U$$

$$\frac{1}{4\pi}\left(\frac{\hbar}{m_U c}\right)^2$$

$$\frac{1}{4\pi}\left(\frac{m_U c}{\hbar}\right)^2 \quad u = \left(\frac{m_U}{4\pi}\right)^{\frac{1}{2}}\frac{1}{\hbar}$$

Non-relativistic Hamiltonian for ~~the~~ heavy particle

$$\overline{H}_M = \iiint \tilde{\Psi}\left[\frac{\vec{p}^{\,2}}{2M} + \frac{g}{2}\left\{\tilde{U}(\tau_1 - i\tau_2) + U(\tau_1 + i\tau_2)\right\} + \frac{D}{2}\tau_3\right]\Psi \, dv$$

Hamiltonian for the system consisted of the U-field and the heavy particle

$$\overline{H} = H_U + H_M .$$

§2. Quantization in the presence of the electromagnetic field

The momentum operator $\vec{p}$ should be replaced by $\vec{p} - \frac{e}{c}\vec{A}$ if the particle has the charge $+e$.

In like manner

$$-i\hbar \,\mathrm{grad}\, U \quad , \quad -i\hbar \,\mathrm{grad}\, \tilde{U}$$

should be replaced by

$$-i\hbar \,\mathrm{grad}\, U - \frac{e}{c}\vec{A}U \qquad -i\hbar \,\mathrm{grad}\, \tilde{U} + \frac{e}{c}\vec{A}\tilde{U}$$

as respectively, as they have the charges $+e$ and $-e$ respectively. ~~Thus~~ Further,

$$i\hbar\frac{\partial U}{\partial t} \quad , \quad i\hbar\frac{\partial \tilde{U}}{\partial t}$$

should be replaced by

$$i\hbar\frac{\partial U}{\partial t} - eVU \quad , \quad i\hbar\frac{\partial U}{\partial t} + eVU$$

respectively.

Thus,

$$L_U = -\frac{1}{c^2}\left(\frac{\partial \tilde{U}}{\partial t} + \frac{ie}{\hbar}V\tilde{U}\right)\left(\frac{\partial U}{\partial t} - \frac{ie}{\hbar}VU\right)$$
$$+\left(\mathrm{grad}\,\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)\left(\mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U\right) + \lambda^2\tilde{U}U$$

$$H_U = \frac{ie}{\hbar c^2}\frac{\partial \tilde{U}}{\partial t}VU - \frac{ie}{\hbar c^2}\tilde{U}V\frac{\partial U}{\partial t} - \frac{e^2}{\hbar^2 c^2}\tilde{U}V^2U$$
$$-\left(\mathrm{grad}\,\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)\left(\mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U\right) - \lambda^2\tilde{U}U$$

$$H_M = \tilde{\Psi}\left\{\left(\frac{1}{2M}\vec{p} - \frac{e}{c}\frac{1-\tau_3}{2}\vec{A}\right)^2 + e\tilde{V}\left(\frac{1-\tau_3}{2}\right)\right.$$
$$\left. + 4\pi g\frac{\tau_1 - i\tau_2}{2}\tilde{U} + 4\pi g\frac{\tau_1 + i\tau_2}{2}U\right\}\Psi$$

Finally the Hamiltonian for the electromagnetic field is

$$\overline{H}_S = \iiint H_S\, dv$$

$$H_S = \frac{1}{8\pi}\left\{V\Delta V - \vec{A}\Delta\vec{A} + V\frac{\partial^2 V}{\partial t^2} - \vec{A}\frac{\partial^2\vec{A}}{\partial t^2}\right\}$$

Lagrangian　　　　　　　　　　　　　　Jan 11, 1937.

~~Hamilton 形式~~

U-field の quantization ; Lagrangian は

$$L'_U = -\tilde{U}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \lambda^2\right)U \quad \text{or} \quad L_U = -\frac{1}{c^2}\frac{\partial \tilde{U}}{\partial t}\frac{\partial U}{\partial t} + (\operatorname{grad}\tilde{U}\operatorname{grad}U) + \lambda^2 \tilde{U}U$$

$$\overline{L}_U = \int \cdots \int L_U \, dv\, dt$$

$$\cancel{L_M = \frac{\vec{p}_4^{\,2}}{2M}} -$$

$$U^+ = \frac{\partial \overline{L}_U}{\partial \frac{\partial U}{\partial t}} = \frac{-1}{c^2}\frac{\partial \tilde{U}}{\partial t}.$$

$$\tilde{U}^+ = \frac{-1}{c^2}\frac{\partial U}{\partial t}.$$

$$L_U = \tilde{U}^+U^+ + (\operatorname{grad}\tilde{U}\operatorname{grad}U) + \lambda^2\tilde{U}U$$

$$\tilde{U}^+U' - U'U^+ = -i\hbar\,\delta(\vec{r}\,\vec{r}')$$

$$\overline{L}_M = \int\cdots\int \left\{ -\frac{1}{2M}(\tilde{\Psi}\, p^2\Psi) + i\hbar\tilde{\Psi}\frac{\partial\Psi}{\partial t} - 4\pi g\, \tilde{\Psi}\frac{(\tau_1 - i\tau_2)\tilde{U}}{2}\Psi - 4\pi g\, \frac{\tilde{\Psi}(\tau_1 + i\tau_2)U\Psi}{2} \right\} dv\,dt$$

$$\delta(\overline{L}_U + \overline{L}_M) = \int\cdots\int \Big[\delta\tilde{\Psi}\Big(\frac{p^2}{2M} + i\hbar\frac{\partial}{\partial t} - 4\pi g\, \frac{\tau_1 - i\tau_2}{2}\tilde{U} - 4\pi g\, \frac{\tau_1 + i\tau_2}{2}U\Big)\Psi$$

$$+ \tilde{\Psi}\Big(\frac{p^2}{2M} + \cdots\cdots\Big)\delta\Psi - \delta\tilde{U}\Big\{\Big(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \lambda^2\Big)U$$

$$+ 4\pi g\, \tilde{\Psi}\frac{\tau_1 - i\tau_2}{2}\Psi\Big\} - \tilde{U}\Big\{\Big(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \lambda^2\Big)$$

$$+ 4\pi g\, \tilde{\Psi}\frac{\tau_1 + i\tau_2}{2}\Psi\Big\}\delta U \Big] dv\,dt = 0.$$

Electromagnetic field $V, \vec{A}$ が与えられたとき charge が $+e$ なら

$$p \to p - \frac{e}{c}\vec{A}$$

とおき、又 U の charge が $+e$ なら

$$-i\hbar\,\mathrm{grad}\,U \to -i\hbar\,\mathrm{grad}\,U - \frac{e}{c}\vec{A}U$$

$$\text{or}\quad \mathrm{grad}\,U \to \mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U.$$

とおき、更に

$$i\hbar\frac{\partial U}{\partial t} \to i\hbar\frac{\partial U}{\partial t} - eVU \quad \text{or} \quad \frac{\partial U}{\partial t} \to \frac{\partial U}{\partial t} - \frac{ie}{\hbar}VU$$

とおけばよい。すると

$$L_U = -\frac{1}{c^2}\left(\frac{\partial \tilde{U}}{\partial t} + \frac{ie}{\hbar}V\tilde{U}\right)\left(\frac{\partial U}{\partial t} - \frac{ie}{\hbar}VU\right) + \left(\mathrm{grad}\,\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)$$
$$\times\left(\mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U\right) + \lambda^2\tilde{U}U$$

$$L_M = -\frac{1}{2M}\tilde{\Psi}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2\Psi + i\hbar\tilde{\Psi}\frac{\partial\Psi}{\partial t} - eV\tilde{\Psi}\Psi$$
$$-4\pi g\,\tilde{\Psi}\frac{(\tau_1 - i\tau_2)}{2}\tilde{U}\Psi - 4\pi g\,\tilde{\Psi}\frac{(\tau_1 + i\tau_2)}{2}U\Psi$$

$$\delta\iint(L_U + L_M)\,dt = \iint\Big[\cdots \quad + \delta V\{$$

$$L_S = \frac{1}{8\pi}\left\{V\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V + \vec{A}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}\right\}$$

charge and Current Density.

$$\rho = +e\tilde{\Psi}\cdots\Psi + \frac{ie}{\hbar c^2}\tilde{U}\frac{\partial U}{\partial t} - \frac{ie}{\hbar c^2}\frac{\partial\tilde{U}}{\partial t}U$$
$$= +e\tilde{\Psi}\cdots\Psi + \frac{ie}{\hbar}\tilde{U}\dot{\tilde{U}}^{\dagger} - \frac{ie}{\hbar}U^{\dagger}U$$
$$+\cdots$$

$$\vec{I} = \cdots$$

~~U-quanta~~ Conservation Law.

Hamiltonian.

$$H_U = U^{+}\frac{\partial U}{\partial t} + \tilde{U}^{+}\frac{\partial \tilde{U}}{\partial t} - L_U$$

$$= +\frac{ie}{\hbar c^2}\frac{\partial \tilde{U}}{\partial t} V U - \frac{ie}{\hbar c^2}\tilde{U}\frac{\partial U}{\partial t} - \frac{e^2}{\hbar^2 c^2} V^2 \tilde{U} U$$

$$+ \left(\operatorname{grad}\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)\left(\operatorname{grad} U - \frac{ie}{\hbar c}\vec{A} U\right) + \lambda^2 \tilde{U} U.$$

$$H_M = \frac{1}{2M}\tilde{\Psi}\left(\vec{p} - \frac{e}{c}\left(\frac{1+\tau_3}{2}\right)\vec{A}\right)^2 \Psi + e\tilde{\Psi} V \left(\frac{1+\tau_3}{2}\right)\Psi$$

$$+ 4\pi g\, \tilde{\Psi}\frac{\tau_1 - i\tau_2}{2}\tilde{U}\Psi + 4\pi g\, \tilde{\Psi}\frac{\tau_1 + i\tau_2}{2} U \Psi$$

$$H_S = \frac{1}{8\pi}\left\{ V\Delta V - \vec{A}\Delta\vec{A} \right\}$$

$$+ \frac{1}{8\pi}\left\{ V\frac{\partial^2 V}{\partial t^2} - \vec{A}\frac{\partial^2 \vec{A}}{\partial t^2} \right\}.$$

Lagrangian

$$L_U = -\frac{1}{c^2}\frac{\partial \tilde{U}}{\partial t}\frac{\partial U}{\partial t} + (\mathrm{grad}\,\tilde{U}\ \mathrm{grad}\,U) + \lambda^2 \tilde{U} U$$

$$L_M = i\hbar \tilde{\Psi}\frac{\partial \Psi}{\partial t} - \frac{1}{2M}\tilde{\Psi} p^2 \Psi - 4\pi g\, \tilde{\Psi}\frac{(\tau_1 - i\tau_2)\tilde{U}}{2}\Psi - 4\pi g\, \tilde{\Psi}\frac{(\tau_1 + i\tau_2)U}{2}\Psi$$

$$\bar{I} = \iiint (L_U + L_M)\, dv$$

$$\frac{1-\tau_3}{2}$$

$$L_U = -\frac{1}{c^2}\left(\frac{\partial \tilde{U}}{\partial t} + \frac{ie}{\hbar}V\tilde{U}\right)\left(\frac{\partial U}{\partial t} - \frac{ie}{\hbar}VU\right) + \left(\mathrm{grad}\,\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)\left(\mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U\right) + \lambda^2\tilde{U}U$$

$$L_M = i\hbar\tilde{\Psi}\frac{\partial \Psi}{\partial t} - e\tilde{\Psi}\frac{(1-\tau_3)}{2}V\Psi - \frac{1}{2M}\tilde{\Psi}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2\Psi - 4\pi g\,\tilde{\Psi}\frac{(\tau_1 - i\tau_2)\tilde{U}}{2}\Psi - 4\pi g\,\tilde{\Psi}\frac{(\tau_1 + i\tau_2)U}{2}\Psi$$

$$L_S = \frac{1}{8\pi}\left(\cancel{H^2 - E^2}\right)\left(E^2 - H^2\right)$$

$$\bar{I} = \iiint (L_U + L_M + L_S)\, dv$$

Hamiltonian.

$$U^\dagger = \frac{\delta \bar{I}}{\delta \frac{\partial U}{\partial t}} = -\frac{1}{c^2}\left(\frac{\partial \tilde{U}}{\partial t} + \frac{ie}{\hbar}V\tilde{U}\right)$$

$$\tilde{U}^\dagger = \frac{\delta \bar{I}}{\delta \frac{\partial \tilde{U}}{\partial t}} = -\frac{1}{c^2}\left(\frac{\partial U}{\partial t} - \frac{ie}{\hbar}VU\right)$$

$$H_L = U^\dagger\frac{\partial U}{\partial t} + \tilde{U}^\dagger\frac{\partial \tilde{U}}{\partial t} - L_U$$

$$\cancel{= U^\dagger \frac{ie}{\hbar}VU + \tilde{U}^\dagger}$$

$$= U^\dagger\left(-c^2\tilde{U}^\dagger + \frac{ie}{\hbar}VU\right) + \tilde{U}^\dagger\left(-c^2 U^\dagger - \frac{ie}{\hbar}V\tilde{U}\right)$$

$$+ c^2 U^\dagger\tilde{U}^\dagger - \left(\mathrm{grad}\,\tilde{U} + \frac{ie}{\hbar c}\vec{A}\tilde{U}\right)\left(\mathrm{grad}\,U - \frac{ie}{\hbar c}\vec{A}U\right)$$

$$- \lambda^2\tilde{U}U$$

$$H_M = e\tilde{\Psi}\frac{(1-\tau_3)V}{2}\Psi + \frac{1}{2M}\tilde{\Psi}\left(\vec{p} - \frac{e}{c}\frac{(1-\tau_3)}{2}\vec{A}\right)^2\Psi$$
$$+ 4\pi g\,\tilde{\Psi}\frac{(\tau_1 - i\tau_2)\tilde{U}}{2}\Psi + 4\pi g\,\tilde{\Psi}\frac{(\tau_1 + i\tau_2)U}{2}\Psi$$

$$H_s = \frac{1}{8\pi}(E^2 + H^2).$$

$$\overline{H} = \iiint (H_U + H_M + H_s)\,dv.$$

~~Field Equations~~

charge density: $\rho = e\tilde{\Psi}\frac{(1-\tau_3)}{2}\Psi + \frac{ie}{\hbar}U^+U - \frac{ie}{\hbar}\tilde{U}^+\tilde{U}$