Quantum ...

$U(xyzt)\,U^+(x'y'z't') - U^+(x'y'z't')\,\times U(xyzt)$

$= \left(\frac{1}{2\pi}\right)^3 \frac{i\hbar}{2} \iiint \Big[\frac{1}{k_0} \Big(e^{-ick_0(t-t')+ik(r-r')}$

$+ e^{ick_0(t-t')-ik(r-r')} \Big] dk_x dk_y dk_z$

$\iiint e^{ikr} dk_x dk_y dk_z \,(\cdots)$

$= 2\pi \int_0^\infty k\,dk \frac{e^{ikr}-e^{-ikr}}{iR} (\cdots)$

$= \left(\frac{1}{2\pi}\right)^2 \frac{\hbar}{2R} \int_0^\infty \frac{k\,dk}{\sqrt{k^2+k_0^2}} \Big\{ e^{-i(ck_0T-kR)} - e^{-i(ck_0T+kR)}$

$+ e^{i(k_0T+kR)} - e^{-i(ck_0T-kR)} \Big\}$

$= \left(\frac{1}{2\pi}\right)^2 \frac{\hbar}{2R} \int_{-\infty}^\infty \frac{k\,dk}{\sqrt{k^2+k_0^2}} \Big\{ e^{-i(ck_0T-kR)} + e^{i(ck_0T+kR)} \Big\}$

$= \left(\frac{1}{2\pi}\right)^2 \frac{\hbar}{2R} \int_{-\infty}^\infty \lambda \sinh\varphi \cdot \Big\{ e^{-i\lambda s\cosh(\varphi-\chi)} + e^{i\lambda s\cosh(\varphi+\chi)} \Big\} d\varphi$

$\varphi-\chi\to\varphi$　　$\varphi+\chi\to-\varphi$

$$H_0^{(1)}(x) = -\left(\frac{2i}{\pi}\right)\int_0^\infty e^{xi\cosh\varphi}\,d\varphi$$

$$H_1^{(1)}(x) = -\left(\frac{2}{\pi}\right)\int_0^\infty \cosh\varphi\; e^{xi\cosh\varphi}\,d\varphi$$

$c|T| > R$

$$\left.\begin{aligned} k &= \lambda \sinh\varphi \\ k_0 &= \lambda \cosh\varphi \end{aligned}\right\}$$

$$\left.\begin{aligned} R &= S \sinh\chi \\ cT &= S \cosh\chi \end{aligned}\right\} \qquad S^2 = c^2T^2 - R^2$$

$$ck_0T - k_0R = \lambda S \cosh(\varphi - \chi)$$

$$\cancel{ck}k_0T + kR = \lambda S \cosh(\varphi + \chi)$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{\cancel{\hbar}\hbar\lambda}{2R}\int_{-\infty}^{\infty}\underbrace{\{\sinh(\varphi+\chi) + \sinh(\chi-\varphi)\}}_{2\sinh\chi\cosh\varphi} e^{-i\lambda S\cosh\varphi}\,d\varphi$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{\hbar\lambda}{R}\sinh\chi\left(\frac{-\pi}{2}\right)\{H_1^{(1)}(-\lambda S) + H_1^{(2)}(-\lambda S)\}$$

$$= \frac{-\cancel{\hbar}\lambda\sinh\chi}{8\pi R} J_1(-\lambda S) = \frac{\hbar\lambda}{8\pi R} J_1(\lambda S)\sinh\chi$$

$$= \frac{\hbar\lambda}{8\pi S} J_1(\lambda S)$$

$c|T| < R$,

$$\left.\begin{aligned} k &= \lambda \sinh\varphi \\ k_0 &= \lambda \cosh\varphi \end{aligned}\right\}$$

$$\left.\begin{aligned} R &= s\cosh\chi \\ cT &= s\sinh\chi \end{aligned}\right\} \quad s^2 = R^2 - c^2T^2$$

$$ck_0T - kR = -\lambda s \sinh(\chi - \varphi)$$

$$ck_0T + kR = \lambda s \sinh(\chi + \varphi)$$

$$= \left(\frac{1}{2\pi}\right)\frac{\lambda}{2R}\int_{-\infty}^{+\infty}\Big\{\sinh\varphi\Big\{e^{-i\lambda s\sinh(\chi-\varphi)} + e^{i\lambda s\sinh(\chi+\varphi)}\Big\}d\varphi$$

$-\varphi$ $\quad \sinh \int_{+\infty}^{-\infty} \cdots d(-\varphi)$

$$= \int_{-\infty}^{+\infty} \sinh\varphi\Big\{-e^{-i\lambda s\sinh(\chi+\varphi)} + e^{i\lambda s\sinh(\chi+\varphi)}\Big\}d\varphi$$

$$= \left(\frac{1}{2\pi}\right)\frac{\lambda}{2R}\int_{-\infty}^{+\infty}\Big\{\sinh(\varphi+\chi) + \sinh(\varphi-\chi)\Big\}e^{i\lambda s\sinh\varphi}\,d\varphi$$

$$\underbrace{\sinh(\varphi+\chi) + \sinh(\varphi-\chi)}_{2\sinh\varphi\cosh\chi}$$

~~$\sinh(\chi-\varphi)\ 2i\sin(\lambda s\sinh\varphi)$~~

$$\lim_{x\to 0} \frac{\sin Nx}{Nx} = N$$

$$\lim_{x\to 0} \cdot \frac{\sin Nx}{Nx} \quad N$$

$$\left(\frac{1}{2\pi}\right)^2 \frac{\pi}{2R} \int_0^\infty \frac{k\,dk}{k_0} \left\{ e^{-i(ck_0T-kR)} - e^{-i(ck_0T+kR)} \right.$$

$$\left. + e^{i(ck_0T+kR)} - e^{i(ck_0T-kR)} \right\}$$

$$\doteqdot \left(\frac{1}{2\pi}\right)^2 \frac{\pi}{2R} \int_{-\infty}^{\infty} \frac{k\,dk}{k_0} \left\{ e^{-i(ck_0T-kR)} - e^{i(ck_0T-kR)} \right\}$$

$$= -\left(\frac{1}{2\pi}\right)^2 \frac{i\pi}{R} \int_{-\infty}^{\infty} \frac{k\,dk}{k_0} \sin(ck_0T-kR)$$

$$\left.\frac{\cos(ck_0T-kR)}{\cancel{ck_0}\,R}\right|_{-\infty}^{+\infty}$$

$c|T| > R$

$$= -\left(\frac{1}{2\pi}\right)^2 \frac{i\pi\lambda}{R} \int_{-\infty}^{\infty} \cosh\varphi \cdot \sin\{\lambda s \cosh(\varphi - \chi)\}\, d\varphi$$

$$= -\left(\frac{1}{2\pi}\right)^2 \frac{i\pi\lambda}{R} \int_{-\infty}^{\infty} (\cosh\varphi \cdot \cosh\chi + \sinh\varphi \cdot \sinh\chi) \cdot \sin(\lambda s \cosh\varphi)\, d\varphi$$

$$= -\left(\frac{1}{2\pi}\right)^2 \frac{i\pi\lambda}{R} \cosh\chi \cdot 2 \cdot \frac{\pi}{2} \cdot \frac{\partial}{\partial(\lambda s)} Y_0(\lambda s)$$

$$= \frac{1}{4\pi} \frac{i\pi\lambda}{R} \cosh\chi \cdot Y_1(\lambda s)$$

$s \to 0$, $Y_1(\lambda s) \to \dfrac{-2}{\pi \cdot \lambda s}$

$\dfrac{cT}{R} \to \pm 1$ $\quad \cosh\chi \doteqdot \dfrac{cT}{s}$

$\to \pm \dfrac{i\pi}{4\pi^2 s^2}$

C(T) < R

$$= +\left(\frac{1}{2\pi}\right)^2 \frac{i\hbar\lambda}{R} \int_{-\infty}^{\infty} \sinh\varphi \, \sin(\lambda s \sinh(\varphi - \chi))\, d\varphi$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{i\hbar\lambda}{R} \cdot \int_{-\infty}^{\infty} (\sinh\varphi \cdot \cosh\chi + \cosh\varphi \cdot \sinh\chi\}$$

$$\times \underline{\sin(\lambda s \cdot \sinh}\, \varphi)\, d\varphi$$

$$\sinh\varphi = -i \sin i\varphi$$