Angular Momenta in the Theory of Hyperfine Structure

The theory of the hyperfine structure of one electron spectra has recently been treated by means of the relativistic equation of Dirac. The writer wishes to show that in this case the angular momenta of electron alone are no more constant, but the total angular momenta including the nuclear spin angular momenta ~~still~~ conserve. The proof is very simple.

According to the ordinary notation, the Hamiltonian is written in the form

$$H = -eV(r) + c\alpha\left(p \mp \frac{e}{c}A\right) + \alpha_4 mc,$$

where

$$A = \frac{[\mu \times r]}{r^3}$$

represents the field caused by the nuclear magnetic moment μ. The components μ_x, μ_y, μ_z of μ ~~which commu~~ are assumed ~~to take eigenvalues~~

~~$\mu,\ \mu\frac{K-1}{K},\ \mu\frac{K-2}{K},\ \ldots\ldots,\ -\mu\frac{K-1}{K},\ -\mu$~~

~~and~~ to satisfy the relations

$$\mu_y\mu_z - \mu_z\mu_y = i\frac{\mu}{K}\mu_x \qquad \text{etc.}$$

$$\text{or} \quad [\mu \times \mu] = i\frac{\mu}{K}\mu,$$

but, commute with other variables, and so take eigenvalues

$$\mu,\quad \mu\frac{K-1}{K},\quad \mu\frac{K-2}{K},\quad \ldots\ldots\quad -\mu\frac{K-1}{K},\quad -\mu.$$

α and α_4 can also be ~~rex~~pressed by Dirac's matrices ρ, σ, as $\alpha = \rho_1\sigma$, $\alpha_4 = \rho_3$.

In this case, the angular momenta of the electron

$$[r \times p] + \frac{h}{4\pi}\sigma$$

commute with the terms of the Hamiltonian, except the nuclear spin term, ~~which does not commute,~~ for which the relation

$$\left\{[r\times p] + \frac{h}{4\pi}\sigma\right\}\left\{-e\rho_1\frac{\sigma[\mu\times r]}{r^3}\right\} - \left\{-e\rho_1\frac{\sigma[\mu\times r]}{r^3}\right\}\left\{[r\times p] + \frac{h}{4\pi}\sigma\right\}$$

$$= -e\rho_1\frac{h}{2\pi i}\frac{\{[r\times[\sigma\times\mu]] + [\sigma\times[\mu\times r]]\}}{r^3}$$

is ~~established~~ deduced ~~through~~ ~~by means of~~ from the following ~~calculation~~ ~~relations~~ ~~formulae~~ relations formulae,

$$\sigma[r\times p](r[\sigma\times\mu]) - (r[\sigma\times\mu])[r\times p] = \frac{h}{2\pi i}[r\times[\sigma\times\mu]]$$

or

$$\{yp_z - zp_y\}\{x(\sigma_y\mu_z - \sigma_z\mu_y) + y(\sigma_z\mu_x - \sigma_x\mu_z) + z(\sigma_x\mu_y - \sigma_y\mu_x)\}$$
$$-\{x(\sigma_y\mu_z - \sigma_z\mu_y) + y(\sigma_z\mu_x - \sigma_x\mu_z) + z(\sigma_x\mu_y - \sigma_y\mu_x)\}\{yp_z - zp_y\}$$
$$= \frac{h}{2\pi i}\{y(\sigma_x\mu_y - \sigma_y\mu_x) - z(\sigma_z\mu_x - \sigma_x\mu_z)\} = \frac{h}{2\pi i}[r\times[\sigma\times\mu]]_x$$

by using the ~~relations~~ formulae $yp_z - p_zy = 0$ etc $p_x x - xp_x = \frac{h}{2\pi i}$ etc

and

$$\frac{h}{4\pi}\sigma(r[\sigma\times\mu]) - (r[\sigma\times\mu])\frac{h}{4\pi}\sigma = \frac{h}{2\pi i}[\sigma\times[\mu\times r]]$$

or

$$\frac{h}{4\pi}\sigma_x\{\sigma_x(\mu_y z - \mu_z y) + \sigma_y(\mu_z x - \mu_x z) + \sigma_z(\mu_x y - \mu_y x)\}$$
$$-\{\sigma_x(\mu_y z - \mu_z y) + \sigma_y(\mu_z x - \mu_x z) + \sigma_z(\mu_x y - \mu_y x)\}\frac{h}{4\pi}\sigma_x$$

$$= \frac{h}{2\pi i}\{\sigma_y(\mu_x y - \mu_y x) - \sigma_z(\mu_z x - \mu_x z)\} = \frac{h}{2\pi i}[\boldsymbol{\sigma}\times[\boldsymbol{\mu}\times\boldsymbol{r}]]_x$$

by ~~using~~ the formulae ~~formula~~ relations

$$\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z \quad \text{etc.}$$

Accordingly, and $[\boldsymbol{r}\times\boldsymbol{p}] + \frac{h}{4\pi}\boldsymbol{\sigma}$ is not the constant of motion ~~and~~.

$$(1)\quad \left(\left\{[\boldsymbol{r}\times\boldsymbol{p}] + \frac{h}{4\pi}\boldsymbol{\sigma}\right\}H - H\left\{[\boldsymbol{r}\times\boldsymbol{p}] + \frac{h}{4\pi}\boldsymbol{\sigma}\right\}\right) = -e\rho_1 \frac{h}{2\pi i}\frac{\{[\boldsymbol{r}\times[\boldsymbol{\sigma}\times\boldsymbol{\mu}]] + [\boldsymbol{\sigma}\times[\boldsymbol{\mu}\times\boldsymbol{r}]]\}}{r^3} \neq 0$$

Now let us consider the quantity

$$\frac{h}{2\pi}\boldsymbol{\mu},$$

which also commute with the terms of the Hamiltonian other than the nuclear spin term, for which

$$\frac{h}{2\pi}\boldsymbol{\mu}\left\{-e\rho_1\frac{\boldsymbol{\sigma}[\boldsymbol{\mu}\times\boldsymbol{r}]}{r^3}\right\} - \left\{-e\rho_1\frac{\boldsymbol{\sigma}[\boldsymbol{\mu}\times\boldsymbol{r}]}{r^3}\right\}\frac{h}{2\pi}\boldsymbol{\mu} = -e\rho_1\frac{h}{2\pi i}\frac{\mu}{K}\frac{[\boldsymbol{\mu}\times[\boldsymbol{r}\times\boldsymbol{\sigma}]]}{r^3}$$

is established by means of the formula

$$\frac{h}{2\pi}\mu_x\{\mu_x(y\sigma_z - z\sigma_y) + \mu_y(z\sigma_x - x\sigma_z) + \mu_z(x\sigma_y - y\sigma_x)\}$$

$$- \{\mu_x(y\sigma_z - z\sigma_y) + \mu_y(z\sigma_x - x\sigma_z) + \mu_z(x\sigma_y - y\sigma_x)\}\frac{h}{2\pi}\mu_x$$

$$= \frac{h}{2\pi}(-i)\frac{\mu}{K}\{\mu_y[\boldsymbol{r}\times\boldsymbol{\sigma}]_z - \mu_z[\boldsymbol{r}\times\boldsymbol{\sigma}]_y\}$$

$$= \frac{h}{2\pi i}\frac{\mu}{K}[\boldsymbol{\mu}\times[\boldsymbol{r}\times\boldsymbol{\sigma}]]_x,$$

so that

$$(2)\quad \frac{h}{2\pi}\frac{K}{\mu}\boldsymbol{\mu}H - H\frac{h}{2\pi}\frac{K}{\mu}\boldsymbol{\mu} = -e\rho_1\frac{h}{2\pi i}\frac{[\boldsymbol{\mu}\times[\boldsymbol{r}\times\boldsymbol{\sigma}]]}{r^3}.$$

From these two formulae (1) and (2), using the identities

$$[\boldsymbol{r}\times[\boldsymbol{\sigma}\times\boldsymbol{\mu}]] + [\boldsymbol{\sigma}\times[\boldsymbol{\mu}\times\boldsymbol{r}]] + [\boldsymbol{\mu}\times[\boldsymbol{r}\times\boldsymbol{\sigma}]]$$

$$= \boldsymbol{\sigma}(\boldsymbol{r}\boldsymbol{\mu}) - \boldsymbol{\mu}(\boldsymbol{r}\boldsymbol{\sigma}) + \boldsymbol{\mu}(\boldsymbol{\sigma}\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\sigma}\boldsymbol{\mu}) + \boldsymbol{r}(\boldsymbol{\mu}\boldsymbol{\sigma}) - \boldsymbol{\sigma}(\boldsymbol{\mu}\boldsymbol{r}) = 0$$

the required formula

$$\cancel{\boldsymbol{m}}\left([\boldsymbol{r}\times\boldsymbol{p}] + \frac{h}{4\pi}\boldsymbol{\sigma} + \frac{h}{2\pi}\frac{K}{\mu}\boldsymbol{\mu}\right)H - H\left(\boldsymbol{m} + \frac{h}{4\pi}\boldsymbol{\sigma} + \frac{h}{2\pi}\frac{K}{\mu}\boldsymbol{\mu}\right) = 0$$

is deduced. So the sum of the angular momentum of the electron and the nuclear spin angular momentum are constants of motion. Of course, the orbital angular momentum of ~~the~~ a nucleus does not appear in this case as it is fixed in the centre, but still the spin angular momenta ~~takes~~ play their rôle in the expression of the total angular momenta.