On the Magnetic moment of the Electron

By C. G. Darwin

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1. In a recent paper‡ the present writer discussed Dirac's new formulation§ of the quantum mechanics of the electron from the point of view of the wave theory. In the course of that work formulae were obtained for the electric density and current associated with any type of electron wave. These formulae link the electron with the aether, so that to a first approximation (that is, excluding such questions as the reaction of its own emitted radiation on the electron) we have a complete expression of the state of affairs, which is suitable for use in the classical electromagnetic theory. But there are certain problems in which it is convenient to attribute part of the magnetic field of a moving electron to an electric convection current and part to its intrinsic magnetisation; for example, this would be so in discussing the Stern-Gerlach effect for free electrons. In the earlier paper a separation of this kind was carried out for ~~an atom~~ an electron in an atom and a short account was given of the corresponding case for an electron moving freely. As the paper was intended to cover a much wider field, these matters were only reviewed rather briefly, and the purpose of the present note is to elaborate them somewhat more ~~briefly~~ fully. The process is one of pure classical theory, for the establishment of current and density is the only appeal that need be made to the quantum theory. We shall exhibit directly the magnetic moment of the electron, working it out in the first instance for slow motions, as it is easy to generalise this ~~way~~ case by relativity principles. It would no doubt be possible to develop the whole result in one step, but the purely classical problem of the fields of a swiftly moving magnet is quite troublesome, even though the principles have been fully mastered, so that it would be an unnecessary and profitless exercise to follow this procedure.

It may be recalled that Dirac's equations || were easily shown to be

‡. 118 654. many formulae in this paper are denoted by a †.

§ 117 610.

|| With his matrix methods, Dirac never actually writes down these equations, but they are extracted from his work and given in (2.2)†

invariant for any space or relativity transformation, * even though their form is quite unsymmetrical. The associated density-current functions naturally have the property of covariance, though also quite unsymmetrical in appearance. We shall here meet other tensors, also quadratic in the ψ's, and having the same unsymmetrical appearance. Their construction is by no means obvious, and the only process seems to be a direct application of all the transformations of the group to each component — a straightforward but laborious method. Before Dirac's equations were found one would have said without hesitation that the correct procedure in such a case would be to reduce the equations themselves to tensor form, for then associated tensors should become obvious. It is not hard || to throw the equations into space ~~time~~ vector form, but if the relativity transformation is to be included, they obstinately refuse to go into any but a very clumsy form and nothing is gained; this being so, it seems not worth ~~wh~~ while to use the space vector ~~form~~ either. I have the hope that this subject may interest some analytical geometer and may tempt him to investigate these curiously unsymmetrical ~~form~~ invariant properties. As we shall see, they appear to be connected with the stereographic projection of a sphere and perhaps with the homographic transformation of a complex variable, subjects with which I am not very familiar.

The purpose of the present note is illustration rather than proof, as it makes no new demand on the quantum theory, but is in effect pure electro-dynamics. It arose out of an investigation given in the immediately following paper, which aims at seeing whether any magnetic polarisation of electrons could arise when they are diffracted, by say by a crystal. The results there obtained made it necessary to consider somewhat more deeply the interpretation of the wave equations and led to the present work.

|| Dirac uses a somewhat similar process, by splitting his matrices into pairs of factors one set of which is a space vector, but the work is still very unsymmetrical.

2. There are two different ways in which magnetic moment can be observed, which may be called the internal and the external, typified respectively by the Stern-Gerlach method and magnetometer. In the internal method we expose the magnet to a magnetic field and find its changed energy, in the external we isolate it and observe the magnetic field at distance. As long as the electromagnetic equations are satisfied, the two methods must be equivalent and may choose whichever is most convenient. There can be no question but that the external method is much simpler, for even if we avoid the troublesome non-uniform field of the Stern-Gerlach experiment, we still have to use something equivalent to the pondermotive force of Lorentz in order to find the energy, whereas the external moment can be calculated from Maxwell's equations directly.

We therefore first consider how in the classical theory magnetic moment would exhibit itself externally. The general electromagnetic eq. may be written

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial E}{\partial t} + 4\pi\frac{j}{c} &= \operatorname{curl} H & \operatorname{div} E &= 4\pi\rho \\ -\frac{1}{c}\frac{\partial H}{\partial t} &= \operatorname{curl} E & \operatorname{div} H &= 0 \end{aligned}\right\} \quad (2.1)$$

where ρ and j are density and current. As long as no single magnetic poles are allowed, these equations can describe any system. In Lorentz's theory the current is attributed to the convection of electricity so that $j = \rho v$, but this must now be changed so as to allow for the magnetic moment. The change is simple and brings the equations to the form they have in the elementary theory of magnetism. Let μ be the magnetisation (density of magnetic moment); then B, the magnetic induction is given by $B = H + 4\pi\mu$ and the last two equations must be replaced by
$$-\frac{1}{c}\frac{\partial B}{\partial t} = \operatorname{curl} E \qquad \operatorname{div} B = 0.$$

Eliminating H from the first equation we have

$$\frac{1}{c}\frac{\partial E}{\partial t} + 4\pi\left(\rho\frac{v}{c} + \operatorname{curl}\mu\right) = \operatorname{curl} B \qquad (2.2)$$

At the distant point of observation we need not distinguish between H and B and so can attribute the effects to a current

$$j = \rho v + c \operatorname{curl}\mu. \qquad (2.3)$$

We shall therefore show that part of j is proportional to the electron's velocity and part is the curl of a certain vector, and from this we shall deduce the magnetisation of the electron.

3. The magnetisation of the electron can only be observed by supposing that the electron wave is confined to a limited region of space. If it is observed by a Stern-Gerlach experiment, slits must be used, or if by a magnetometer, the electron wave must be somewhere away from the instrument. Moreover, if part of the magnetic field is to be attributed to the convection of electricity, we must be able to assign a definite velocity to that convection. All these conditions are fullfilled if we take as a "wave-packet." The accurate form of such a packet can be set down by the use of Fourier integrals, but, unlike the case of Schrödinger's equation, the integrals cannot be worked out, so that the distribution of the wave in space is not made explicit. However, by taking low velocity an approximate solution can be found, and this is all that is needed.

The accurate solution of Dirac's equation in the form of plane waves is

$$\left.\begin{aligned} &\psi_3 = A\cdot S \qquad \psi_4 = B\cdot S \\ &\psi_1 = \frac{-Ar - B(p-iq)}{mc + W/c}\cdot S \qquad \psi_2 = \frac{-A(p+iq) + Br}{mc + W/c}\, S \end{aligned}\right\} \quad (3.1)$$

where $S = \exp i\frac{2\pi}{h}(px + qy + rz - Wt)$

and $W/c = \sqrt{m^2c^2 + p^2 + q^2 + r^2}$.

If the velocity is small, we have $W = mc^2 + (p^2+q^2+r^2)/2m$, and the equations for ψ_3, ψ_4 reduce practically to Schrödinger's. In order to construct a wave packet we replace A by

$$A\left(\frac{\sigma}{h}\sqrt{2\pi}\right)^3 \exp -\tfrac{1}{2}\left(\frac{2\pi\sigma}{h}\right)^2\left[(p-mu)^2 + (q-mv)^2 + (r-mw)^2\right],$$

and taking the new A as a constant we now integrate over all values of p, q, r. As was shown in an earlier paper ‡, the result will be a packet ~~intensity~~ initially round the origin, moving with velocity u, v, w and spreading somewhat as it goes. The initial value, which is all that we need here, is

$$\psi_3 = AP \quad , \quad \psi_4 = BP \qquad (3,2)$$

where $P = \exp -\frac{1}{2\sigma^2}(x^2+y^2+z^2) + i\frac{2\pi}{h}m(ux+vy+wz)$;

To evaluate ψ_1, ψ_2 approximately we may replace W/c in the ~~detern~~ denominator by mc and the integrations can then be done. For our purpose we only want the initial values, and these can be found alternatively by the use of $(4.1)^{\dagger}$. We have

$$(3,3)\left\{\begin{aligned} \psi_1 &= -\frac{1}{2mc}\left\{A\left[mw + \frac{ih}{2\pi\sigma^2}z\right] + B\left[m(u-iv) + \frac{ih}{2\pi\sigma^2}(x-iy)\right]\right\}P \\ \psi_2 &= -\frac{1}{2mc}\left\{A\left[m(u+iv) + \frac{ih}{2\pi\sigma^2}(x+iy)\right] + B\left[mw + \frac{ih}{2\pi\sigma^2}z\right]\right\}P \end{aligned}\right.$$

These four quantities express the instantaneous wave form at time $t = 0$ for a packet moving with small velocity u, v, w.

Next form the density and current by the use of $(3.3)^{\dagger}$ — apart from a constant factor the four expressions are given below in (4.2). Then approximately

$$\rho = -e(AA^* + BB^*)\exp -(x^2+y^2+z^2)/\sigma^2 \qquad (3,4)$$

$$\begin{aligned} j_1 = e\Big\{ &-AA^*\left(u - \frac{h}{2\pi\sigma^2 m}y\right) - BB^*\left(u + \frac{h}{2\pi\sigma^2 m}y\right) \\ &- AB^*\frac{ih}{2\pi\sigma^2 m}z + A^*B\frac{ih}{2\pi\sigma^2 m}z\Big\} \exp -(x^2+y^2+z^2)/\sigma^2 \\ = \rho u &+ \frac{eh}{4\pi m}\left\{(BB^* - AA^*)\frac{\partial}{\partial y} - (iA^*B - iAB^*)\frac{\partial}{\partial z}\right\} \\ &\times \exp -(x^2+y^2+z^2)/\sigma^2 \end{aligned} \qquad (3.5)$$

In the same way ~~f~~ we find

$$j_2 = \rho v + \frac{eh}{4\pi m}\left\{(-A^*B - AB^*)\frac{\partial}{\partial z} - (BB^* - AA^*)\frac{\partial}{\partial x}\right\}\exp -(x^2+y^2+z^2)/\sigma^2$$

$$j_3 = \rho w + \frac{eh}{4\pi m}\left\{(iA^*B - iAB^*)\frac{\partial}{\partial x} - (-AB^* - A^*B)\frac{\partial}{\partial y}\right\}\exp \;''$$

These are of the form required by § 2, and we may say that there is a magnetisation with components

$$\mu_1 = \frac{eh}{4\pi mc}(-A^*B - AB^*)\exp -(x^2+y^2+z^2)/\sigma^2 \quad \Big\}$$

‡ 117 p. 258

$$\left.\begin{aligned} \mu_2 &= \frac{eh}{4\pi mc}(iA^*B - iAB^*)\exp -(x^2+y^2+z^2)/\sigma^2 \\ \mu_3 &= \frac{eh}{4\pi mc}(-AA^* + BB^*) \quad \text{''} \quad \text{''} \end{aligned}\right\} \quad (3.6)$$

To describe a single electron the constants A, B must be so normalised, that

$$(AA^* + BB^*)\iiint \exp -(x^2+y^2+z^2)/\sigma^2 \, dx\,dy\,dz = 1, \quad (3.7)$$

and it follows that the magnetic moment is $eh/4\pi mc$, as it should be. The expressions (3, 6) are, of course, a special case of (4.6)† according to which

$$\left.\begin{aligned} \mu_1 &= -\psi_3^*\psi_4 - \psi_3\psi_4^* \\ \mu_2 &= i\psi_3^*\psi_4 - i\psi_3\psi_4^* \\ \mu_3 &= -\psi_3\psi_3^* + \psi_4\psi_4^* \end{aligned}\right\} \quad (3.8)$$

If we take an infinite plane wave, the direct observation of the moment is not possible, but it is easy to extend the ~~purpose~~ process so as to cover this case. Experimentally we may imagine a screen to intercept all but a small pencil of the wave, which thus becomes amenable to observation; theoretically we can adopt (3.8) as a definition of the magnetisation. In discussing such questions as the diffraction of a stream of electrons, we want to be able to describe how the magnetic moment will be affected by the diffracting system, and it is thus necessary to find how the direction the magnetisation is correlated with A and B. If the direction of magnetisation has colatitude χ and longitude ω with reference to the z-axis, it is easy to show that $-B/A = \cot\frac{1}{2}\chi e^{i\omega}$. Now if we project stereographically ‡ from the positive pole of a sphere on a plane at unit distance, the point χ, ω has co-ordinates $\cot\frac{1}{2}\chi\cos\omega$, $\cot\frac{1}{2}\chi\sin\omega$. Thus the ratio $-B/A$ bears to the direction of magnetisation to same relation as that borne by the point ζ in the complex plane to the direction in a sphere that corresponds by stereographic projection. In any diffraction problem the diffracted wave must bear a linear relation to the incident, that is to say,

‡ The stereographic projection was 1st used in this connection by Jordan. Z. f. Phys. 44 p.292. Though his purpose was rather different, his work covers much of the matter discussed here.

$A' = \alpha A + \beta B$, $B' = \gamma A + \delta B$, and so the new direction of magnetisation, described by $-B'/A'$, is related by a homographic projection to the old.

4. The expressions (3,8) are easily verified to be components of a vector by applying the transformations (2) and (3) of p. 656†, but they have only been worked out for low velocities and need modification for high. At low velocities ψ_1, ψ_2 are negligible, so that we are at liberty to add terms in them to the μ's, and by doing so we can find six expressions forming an antisymmetric tensor of the second rank which obey ~~ex~~ exactly the same rules of transformation for (1), (2) and (3) as do the electric and magnetic forces for any relativity transformation. The process is quite easy though laborious, for in the view of the unsymmetrical appearance of the components, each of the six has to be tested with each transformation making 18 operation in all. In giving the components we use subscripts $x\,y\,z\,t$ for the tensor, to avoid confusion with the numerical subscripts of the ψ's, which have not a vector meaning. Then

$$\left.\begin{aligned} \mu_{xt} &= -\psi_3\psi_4^* - \psi_4\psi_3^* + \psi_1\psi_2^* + \psi_2\psi_1^* \\ \mu_{yt} &= -i\psi_3\psi_4^* + i\psi_4\psi_3^* + i\psi_1\psi_2^* - i\psi_2\psi_1^* \\ \mu_{zt} &= -\psi_3\psi_3^* + \psi_4\psi_4^* + \psi_1\psi_1^* - \psi_2\psi_2^* \\ \mu_{yz} &= i\psi_1\psi_4^* - i\psi_4\psi_1^* + i\psi_2\psi_3^* - i\psi_3\psi_2^* \\ \mu_{zx} &= -\psi_1\psi_4^* - \psi_4\psi_1^* + \psi_2\psi_3^* + \psi_3\psi_2^* \\ \mu_{xy} &= i\psi_1\psi_3^* - i\psi_3\psi_1^* - i\psi_2\psi_4^* + i\psi_4\psi_2^* \end{aligned}\right\} \quad (4.1)$$

These expressions are, of course, all real, because we have not followed the practice of expressing relativity transformations with the use of imaginary time. The first three components are the magnetic moment, as we have seen, and the last three are therefore the electric moment. Before applying the formulae, we may consider, quite inconclusively, some more mathematical questions that they raise.

In addition to the moment tensor, it is possible to construct other invariants which are quadratic in the ψ's and their ~~components~~ conjugates. Thus we have the density-current vector of (3,3)†, which we may repeat here with the omission of a constant factor,

$$\left.\begin{aligned} c\rho &= -\psi_1\psi_1^* - \psi_2\psi_2^* - \psi_3\psi_3^* - \psi_4\psi_4^* \\ j_x &= \psi_1\psi_4^* + \psi_2\psi_3^* + \psi_3\psi_2^* + \psi_4\psi_1^* \\ j_y &= i\psi_1\psi_4^* - i\psi_2\psi_3^* + i\psi_3\psi_2^* - i\psi_4\psi_1^* \\ j_z &= \psi_1\psi_3^* - \psi_2\psi_4^* + \psi_3\psi_1^* - \psi_4\psi_2^* \end{aligned}\right\} \quad (4.2)$$

Moreover we can see that (3,1)† the coefficient of mc must be (and of course is verified to be) invariant

$$I = -\psi_1\psi_1^* - \psi_2\psi_2^* + \psi_3\psi_3^* + \psi_4\psi_4^* \qquad (4.3)$$

and another invariant can be formed

$$J = i\psi_1\psi_3^* + i\psi_2\psi_4^* - i\psi_3\psi_1^* - i\psi_4\psi_2^*. \qquad (4.4)$$

These are not all independent, for the following relations may be verified;—

$$c^2\rho^2 - j_x^2 - j_y^2 - j_z^2 = I^2 + J^2 \qquad (4.5)$$

$$\mu_{xt}^2 + \mu_{yt}^2 + \mu_{zt}^2 - \mu_{yz}^2 - \mu_{zx}^2 - \mu_{xy}^2 = I^2 - J^2. \qquad (4.6)$$

The relation (4.5) is a familiar one in relativity theory and (4.6) was used by Frenkel‡ in his paper on the spinning electron. I do not know if this is a complete list of quadratic covariants, but it seems probable, and indeed it is rather surprising, that there should even be as many as this.

It would be a problem of some mathematical interest to make a systematic method of dealing with these formulae. I should certainly have expected that the tensor calculus would prove convenient, and it may be so, for I can only say that a good many trials have been unsuccessful. Dirac himself has pointed out that the eq are not final and may need modification later, but that hardly affects the question; for whether right or wrong they conform to relativity and yet seems most unwilling to obey the tensor discipline which has succeeded with ease in ruling all other phenomena.

It may be noted in this connection that a similar set of invariants can be developed for light waves. If the electromagnetic equations for free space are combined by introducing the complex quantities

$$\left.\begin{aligned} \psi_1 &= -iH_z, & \psi_2 &= H_y - iH_x \\ \psi_3 &= E_z, & \psi_4 &= E_x + iE_y \end{aligned}\right\} \quad (4.7)$$

we get four equations which are identical with Dirac's, when m and e are equated to zero. The only difference is that here E_z is real, whereas for the electron ψ_3 is necessarily complex; though at present the distinction is important, it will probably disappear when we understand the meaning of negative energy. The quantities (4.7) must obey the same rules of transformation as Dirac's, but, of course, with other axes the ψ's no longer stand for the same function the forces. The associated invariants must also exist, though they do not look like invariants when expressed in terms of the forces, because, if this is done, two different rules of transformation are being mixed together. Allowing for this peculiarity (4.2) represents the energy and momentum, and (4.1) depends on the polarisation, vanishing, for example, for circularly polarised light. The stereographic ~~projection~~ relation between the "magnetic moment" and the elliptic polarisation of the wave is that given by Jordan. If we turn over to the language of particles, we can describe the relation of light to electricity by saying that a lightquantum is simply an electron without charge or mass.

5. We now consider the free motion of an electron moving at any speed, and shall first take the motion in the z-direction. If the particle-velocity is $c \tanh\beta$, the momentum is $mc\sinh\beta$ and energy $mc^2\cosh\beta$. We then have

$$\left.\begin{aligned} \psi_1 &= -A\sinh\tfrac{1}{2}\beta\cdot S & \psi_2 &= B\sinh\tfrac{1}{2}\beta\cdot S \\ \psi_3 &= A\cosh\tfrac{1}{2}\beta\cdot S & \psi_4 &= B\cosh\tfrac{1}{2}\beta\cdot S \end{aligned}\right\} \quad (5.1)$$

‡ Z. Phys. vol 37 p. 243 (1926)

where $S = \exp i\frac{2\pi}{h}mc(z\sinh\beta - ct\cosh\beta)$. We can reduce the electron to rest by applying the transformation (1) p. 656† and the solution becomes

$$\psi_1 = \psi_2 = 0, \quad \psi_3 = A\cdot S_0, \quad \psi_4 = B\cdot S_0, \quad \text{where } S_0 = \exp - i\frac{2\pi}{h}mc^2t$$

Thus in (5.1) the ratio $-B/A$ determines the direction of the magnetisation for the system of coordinates in which the electron is at rest. In this system we may write

$$\mu_x^0 = -A^*B - AB^*, \quad \mu_y^0 = iA^*B - iAB^*, \quad \mu_z^0 = -AA^* + BB^* \qquad (5.2)$$

These quantities, which may be called the "null moments", are really more important than the actual moments; for example, in an unpolarised beam the null moments, as we shall see, are condensed towards the equator of the motion.

Now substitute (5.1) in (4.1) and we have

$$\left.\begin{array}{lll} \mu_{xt} = \mu_x^0\cosh\beta, & \mu_{yt} = \mu_y^0\cosh\beta, & \mu_{zt} = \mu_z^0 \\ \mu_{yz} = \mu_y^0\sinh\beta, & \mu_{zx} = -\mu_x^0\sinh\beta, & \mu_{xy} = 0 \end{array}\right\} \qquad (5.3)$$

Thus at high speeds the transverse component of magnetisation increases, while the longitudinal is unchanged, and so the moment tends towards the equatorial plane. The electric moment is transverse to the motion and can be expressed as vector product of the magnetisation and the velocity (not the velocity $\frac{dx}{dt}$, but that usually denote $\frac{dx}{ds}$).

We have treated of the motion along the z-direction first because we had available the transformation which would reduce the electron to rest. But we can show that the same result applies for any other direction of motion. In (3.1) take $p = mc\sinh\beta\cdot l_x$, $q = mc\sinh\beta\cdot l_y$, $r = mc\sinh\beta\cdot l_z$, $W = mc^2\cosh\beta$ and substitute in (4.1). As there is no point in normalising an infinite wave, we divide out all the expressions by a common factor ‡ and have

$$\mu_{xt} = \mu_x^0\cosh\beta - l_x(l\mu^0)(\cosh\beta - 1), \qquad (5.4)$$

etc., and

$$\mu_{yz} = -(l_y\mu_z^0 - l_z\mu_y^0)\sinh\beta, \qquad (5.5)$$

etc. Here μ_x^0, etc., are defined by (5.2) and ~~two~~ as we now have vector formulae we can see by comparison with (5.3) that they are the null

‡ If it should be desired to normalise, the appropriate quantity is $AA^* + BB^*$, which is the invariant I when the same factor is taken ~~off~~ out.

moments. Thus the method of stereographic projection gives for all plane waves just what is usually required, the direction of magnetisation for the system of coordinates in which the associated particle-velocity is zero.

The ~~velocity~~ vector product relation of the electric to magnetic moment was introduced by Frenkel — for the particle electron; but there is a difference in the way it arises. In Frenkel's work the relation was taken as a kinematical constraint, necessary in order that the electron should have no electric moment when at rest. This constraint made his system non-holomorphic, whereas we have a purely holo~~morphic~~nomic system. To make a very elementary analogy, the motion of a rolling body is non-holo~~morp~~nomic, but happens to become holonomic for a cylinder; Dirac has shown that Frenkel's rolling body really was a cylinder. In a more recent paper Frenkel ‡ has attempted to extend his kinematical condition to the wave theory, but with less success, because he applied it to the amplitude ψ, whereas we have seen that it is quadratic in the ψ's. It is, in fact, a dynamical, not an undulatory relation.

The electron wave can be specified by its electric moment just as well as by its magnetic, and this ~~is~~ has the convenience that the direction is always transverse. At first sight it would appear that one less element is needed, whereas the magnetic is fixed by the total intensity of the wave. The only defect in using the electric moment is that though it fixes the magn~~etic moment~~ itude of the longitudinal magnetic component, it leaves it arbitrary whether it is ~~polarised light, which can be~~ forwards or backwards. We may liken this to the specification of elliptically polarised light, which can be described by means of the position of the major axis and the ratio of the axes, but still requires the statement of the sense in which the ellipse is traversed.§

In conclusion, we should consider the character of an unpolarised beam of electrons. When two beams of elliptically polarised light of the same

‡ 47 p. 786
§ Jordan: loc. cit.

frequency are superposed, the resultant is again polarised light, and in the same way two superposed ~~, the resultant~~ polarised electron waves compound into a third polarised wave. To represent an unpolarised wave we have to proceed in the same way as ~~for the~~ light. There (though it is perhaps physically rather meaningless) it is the case that rigorously monochromatic light must have a ~~f~~ definite polarisation, and unpolarised light can only arise by occasional chance changes of phase, which imply a departure from monochromatism. So when the electron wave has its wave-length given precisely, the magnetic moment must be in a definite direction, and an unpolarised wave can only be represented by supposing occasional changes of phase. It is a curious, though unpractical, result that if the velocity of an electron is absolutely precise, it must have a definite direction of magnetisation.

Summary.

Starting from the wave equations for an electron and the associated electric density and current, it is shown how the electromagnetic fields of a moving electron can be attributed partly to the convection of electricity and partly to an intrinsic magnetisation. A geometrical construction shows the relation between the wave constants and the magnetisation. The formulae, first worked out for slow motion, are easily generalised by relativity for high speeds, and in this case there are electric as well as magnetic moments, and various invariant properties are given. A comparison is made between an electron wave and a light wave, and the resemblance may be loosely expressed by saying that a light-quantum is an electron without charge or mass.

On the Diffraction of the magnetic Electron

1. As soon as it was appreciated that the phenomenon of the "spinning electron" was to be attributed to the existence of simultaneous wave functions,† the question naturally arose as to how far there would be an analogy with the polarisation of light, and in ~~the~~ particular whether in diffraction by a slit or grating or in reflexion from a mirror, electrons ~~and~~ would be selected having a preferential direction of magnetisation. Dynamical conceptions suggest that it would require a magnetic force to produce a change in the polarisation, so that the general indication was rather strongly negative, and it seemed hardly worth while to verify ~~that~~ the fact by direct solution. I believe, however, that recently experiments have been undertaken to search for such an effect (by means of successive diffraction from two ~~et~~ crystals), and in view of this I have examined the question closely. The result, broadly speaking, bears out the first conjecture that no effect could be observed in practice, but in certain cases (of a rather hypothetical kind) an effect might be found and the whole problem is interesting as an example of the wave theory.

The essence of the matter is sufficiently represented by taking a line grating instead of a crystal, and, furthermore, by limiting the electric and magnetic forces of the grating in such a way that there shall only be first order spectra on each side, two reflected and two transmitted. The grating is supposed made out of a periodic distribution of electric and magnetic material. Actual crystals will consists of almost purely electric material, but the magnetic case is as easy to treat and, though unpractical, gives rise to interesting results. It will appear that when a polarised electron wave falls on such a grating, the direction of magnetisation may be changed in the ~~direct~~ diffracted rays. It is true polarisation effect in that the change in direction is independent of the strength of the field in the grating, which only determines the total intensity. In the case of pure electric forces and of some magnetic forces, the change of direction of magnetisation can be represented by a ~~ratio~~ rotation about an axis through a definite angle, both axis and angle depending only on the directions of the incident and diffracted rays and on the

† 116 p. 227 (1927)

speed of the electrons. It follows that a stream of electrons which are initially unpolarised cannot become polarised by diffraction, since the grating merely rotates all the components like a solid body. In cases where there are simultaneous electric and magnetic fields this might not be so, but the calculation becomes rather heavy, and I have only examined its outline sufficiently to see that some polarisation would occur. The matter is quite unpractical, for not only are the electric forces of atoms far stronger than the magnetic, but even if they were equal the electric would be much more effective in controlling the polarisation. I should conjecture also that it would not be practically possible to make a stream of electrons get anywhere near a powerfully magnetised iron crystal; and for all these reasons it would seem fairly safe to say that no polarisation effect will be detected.

2. Passage of electrons through a grating is a special case of a general problem discussed by Dirac † under the title "The Transition to States of Equal Energy." His process is, of course, indisputably correct, but it calls for some comment. Suppose that an incident Schrödinger wave (to avoid the complication of having several wave functions) falls on a grating, which is represented by a weak perturbing electric field of potential V. Let the incident wave be represented by ψ_0 and one of the diffracted waves by ψ_1. Then he shows that the strength will depend on $\iiint V \psi_0 \psi_1^* \, dx\, dy\, dz$. Now ψ_1, like ψ_0, is a solution of the wave equation with V omitted and so represents a wave extending to infinity on both sides of the grating, incident as well as emergent. What we want is only the half of this wave which emerges from the grating, and it is not clear that this is legitimately represented by ψ_1. A second difficulty will be clear to anyone who has ever tried to apply the

† 114 p. 234.

Summary

The Problem is solved of the diffraction of an electron wave by a line-grating exerting periodic electric and magnetic forces; this represents the essential features of diffraction by a crystal. The incident wave is supposed to be magnetised in a definite direction, and it is shown that when the grating exerts only electric forces, the effect is to rotate the direction of magnetisation through a definite angle about an axis perpendicular to the incident and diffracted rays, and no polarisation can be produced by the diffraction. For some magnetic forces a similar rotation occurs, but in general the simultaneous action of electric and magnetic forces may produce a partial polarisation though the case is too remote from experiment to be worth treating in detail.