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all having the spin $\frac{1}{2}$, although these equations seems to be can not be applied to ~~applicable to none~~ any of elementary particles known at present, as long as the neutron and the proton other than atomic nuclei ~~are consider to be~~ are excluded as composite particles. [1]

Note on Dirac's Generalized Wave Equations

By Hideki Yukawa and Shoichi Sakata

(Read Sept. 26, 1936)

Abstract. A few elementary results of the relativistic theory of the particle with the spin larger than $\frac{1}{2}$ ~~simple properties~~ of ~~the particle satisfying~~ the generalized ~~relativistic wave equation of in~~ were ~~investigated~~. Namely The expressions for velocity, current density, spin angular momentum and electric and magnetic moments were found. It is noticable that the spin and the electric or magnetic moment are not proportional to each other in general.

It will be not altogether trivial to deduce a few ~~In this note, the authors want to investigate simple fundamental~~ properties of the particle with the spin larger than $\frac{1}{2}$ satisfying generalized relativistic ~~equations~~ wave equations recently proposed by Dirac[1], although at present there seems to be no ~~there are~~ we can find no elementary particles with the spin larger than $\frac{1}{2}$ are found in nature.

The new wave equations can be written in ~~have~~ the form

$$\boxed{(2k+1)\times 2l} \quad p_t\psi_A + \frac{1}{k}\vec{\alpha}\,\vec{p}\,\psi_A - \frac{1}{2}\frac{m}{\sqrt{kl}}(u^1v^1+u^2v^2)\psi_B = 0,$$
$$\boxed{(2l+1)\times 2k} \quad p_t\psi_B - \frac{1}{l}\vec{\beta}\,\vec{p}\,\psi_B - \frac{1}{2}\frac{m}{\sqrt{kl}}(u_1v_1+u_2v_2)\psi_A = 0, \qquad (1)$$

for the particle with ~~charge~~ mass ~~e and~~ m and l

where

$$p_t = i\hbar\frac{\partial}{\partial t} - eA_0$$
$$\vec{p} = -i\hbar\,\mathrm{grad} - e\vec{A} \qquad \mu = \frac{m}{\sqrt{kl}}$$

[1] Proc. Roy. Soc. (A) 155, 447, 1936.

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or van der Waerden, die gruppentheoret. Methode in der Quantenmech., Berlin, 1932.

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§2. ~~Spin Angular Momentum~~ Velocity and Current Density

and $\vec{\alpha} = \cancel{k\vec{\alpha}}$ and $\vec{\beta} = \cancel{l\vec{\beta}}$ are two spin vectors ~~with absolute value~~ magnitudes k and l ~~first~~ ~~comm~~ which commute with each other, each their magnitudes k and l ~~are~~ being a integers or a half integer. ~~$m = \sqrt{kl}\,\mu$~~ μ is a number satisfying

$$\mu = \frac{m}{\sqrt{kl}}, \qquad (\cancel{2})$$

where m is the rest mass of the particle.

~~u^1, u^2, v^1_1, v^2_2 etc a u's and v's are spinors satisfying~~ $(u_1$ and $u_2)$ ~~and (v_1, v_2) are~~ form a spinor with $(2k+1)\times 2l$ rows and $(2k)\times 2l$ columns, while v_1 and v_2 form one with $2k\times 2l$ rows and $(2k+1)\times 2l$ columns.[1] They are connected with $\vec{\alpha}$ and $\vec{\beta}$ by the relations

$$\alpha_z - k = -u^1 v_1, \qquad \alpha_x - i\alpha_y = -u^1 v_2 \quad \text{by (2)}$$
$$\alpha_x + i\alpha_y = -u^2 v_1, \qquad -\alpha_z - k = -u^2 v_2,$$

so that ~~α's and β's are~~ α's are ~~$\vec{\alpha}$ axis~~ α's and k are matrices with $(2k+1)\times 2l$ rows and columns. Similarly $u_i\, u_i$ form a dotted spinor with $2k\times(2k+1)$ rows and columns, while $v_i\, v_i$ form one with $2k+2l$ $2k+2l$ rows and $2k\times(2l+1)$ columns. They are connected with $\vec{\beta}$ by the relations

1) ~~Full~~ ~~Complete~~ account description of spinor notations can be found in Laporte and Uhlenbeck, Phys. Rev. 37, 1380, 1931 †

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~~Spin Angular Momentum~~

§ 3. ~~Equa.~~ Charge and Current

$$\rho_z - l = -u_i v^i \qquad \rho_x - i\rho_y = -u_i v^{\dot{i}}$$
$$\rho_x + i\rho_y = -u_{\dot{i}} v^i \qquad -\rho_z - l = -u_{\dot{i}} v^{\dot{i}} \quad (3)$$

Thus, the wave functions ψ_A and ψ_B have $(2k+1)\times 2l$ and $(2l+1)$ ~~2kx~~ components respectively. The Hamiltonian of the particle ~~has the form~~ becomes

$$H = \begin{pmatrix} -k^{-1}\vec{\alpha}\vec{p} + eA_0 & \frac{1}{2}\mu(u^i v^i + u^{\dot{i}} v^{\dot{i}}) \\ \frac{1}{2}\mu(u_i v_i + u_{\dot{i}} v_{\dot{i}}) & l^{-1}\vec{\beta}\vec{p} + eA_0 \end{pmatrix} \quad (4)$$

§ 2. Current density and Velocity.

~~Equation of~~ Continuity

~~We multiply the~~ ~~each of~~ ~~(1) by ψ_A^* and ψ_B^* respectively on the left and, the equations and added~~

~~We add The~~ equations of (1), after having been multiplied by ψ_A^* and ψ_B^* on the left respectively, ~~and~~ are added together with ~~similar~~ ~~the equation~~ expressions ~~equations~~ complex conjugate to them. We have, then, ~~the equation of continuity~~

$$\frac{\partial}{\partial t}(\psi_A^*\psi_A + \psi_B^*\psi_B) \cancel{- k^{-1}\,\mathrm{div}\,(\psi_A^*\vec{\alpha}\psi_A)}$$
$$\cancel{- l^{-1}\,\mathrm{div}\,(\psi_B^*}\ + \mathrm{div}\,(\psi_A^*\underset{k^{-1}\vec{\alpha}}{\vec{\alpha}}\psi_A + \psi_B^*\underset{l^{-1}\vec{\beta}}{\vec{\beta}}\psi_B) = 0 \quad (5)$$

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, which becomes the equation of continuity, if we define the density and the current density of the particle by

$$\left.\begin{aligned} \rho &= \psi_A^* \psi_A + \psi_B^* \psi_B \\ \vec{i} &= -\psi_A^* k^{-1} \vec{\alpha} \psi_A + \psi_B^* l^{-1} \vec{\beta} \psi_B \end{aligned}\right\} \quad (6)$$

Next, by inserting ~~from~~ (14) in the equation of motion

$$i\hbar \dot{\vec{r}} = \vec{r} H - H \vec{r}$$

we obtain the expression of the velocity of the particle ~~the velocity becomes~~

$$\dot{\vec{r}} = \begin{pmatrix} -k^{-1}\vec{\alpha} & 0 \\ 0 & l^{-1}\vec{\beta} \end{pmatrix}. \quad (7)$$

§3. Spin Angular Momentum

In the central field ~~given by~~ with the potential

$$A_0(\vec{r}) = A_0(r) \qquad \vec{A} = 0,$$

the components of the orbital angular momentum

$$\vec{m} = \vec{r} \times \cancel{(-i\hbar\,\mathrm{grad})}\, \vec{p}$$

are ~~is~~ not ~~a~~ constants of motion, since

$$i\hbar \dot{\vec{m}} = \vec{m} H - H \vec{m} \neq 0.$$

$$\vec{\alpha}' = \sum_{\mu,\nu=1,2} u_{\mu}^{\mu} u^{\nu} \vec{\alpha} v_{\nu} u^{\mu}$$

$$\vec{\alpha}' = \sum_{\mu,\nu=1,2} \frac{u_{\mu} v_{\nu} \vec{\alpha} u^{\nu} v^{\mu}}{2(k+1)(2l+1)}$$

$$\vec{\beta}' = \sum_{\mu,\nu=1,2} \frac{u^{\nu} v_{\mu} \vec{\beta} u^{\mu} v_{\nu}}{\cancel{(2k+1)}\ 2(l+1)(2k+1)}$$

$$\cancel{u_1 v^1 u_2 v^i} = (2k+1) v_2 u^1$$

$$=$$

$$v_1 u^1 v_2 u^1 = v_2 u^1 v_1 u^1 + v_2 u^1$$

$$v_2 u^1 v_2 u^2 = v_2 u^2 v_2 u^1 + v_2 u^1$$

$$\vec{s} = \begin{pmatrix} \vec{\alpha} + \vec{\beta}' & 0 \\ 0 & \vec{\beta} + \vec{\alpha}' \end{pmatrix} \qquad \vec{s}^2 = \begin{pmatrix} k(k+1) + (l-\tfrac{1}{2})(l+\tfrac{1}{2}) + \vec{\alpha}\vec{\beta}' + \vec{\beta}'\vec{\alpha} & 0 \\ 0 & (k-\tfrac{1}{2})(k+\tfrac{1}{2}) + l(l+1) + \vec{\alpha}'\vec{\beta} + \vec{\beta}\vec{\alpha}' \end{pmatrix}$$

$$\alpha_x = -\frac{u^1 v_2 + u^2 v_1}{2} \qquad \beta'_x = -\frac{v^1 u_2 + v^2 u_1}{2}$$

$$u^1 v_2 \quad v^i u_2$$

$$-v_2 u^1 u_2 v^i$$

~~for H given~~ Now if we introduce two vectors defined by

$$\left.\begin{aligned} \alpha_x' &= -\frac{v_2u^1+v_1u^2}{2} \\ \alpha_y' &= \frac{v_2u^1-v_1u^2}{2i} \\ \alpha_z' &= \frac{v_2u^2-v_1u^1}{2} \end{aligned}\right\} (8) \qquad \left.\begin{aligned} \beta_x' &= \frac{-v^1u_2+v^2u_1}{2} \\ \beta_y' &= \frac{v^2u_1-v^1u_2}{2i} \\ \beta_z' &= \frac{v^2u_2-v^1u_1}{2} \end{aligned}\right\} (8)$$

~~with $2k\times(2l+1)$ rows and columns and $(2k+1)\times 2l$ rows and columns respectively,~~ $\vec{\alpha}'$ and $\vec{\beta}'$ ~~satisfy the commutation relations similar to~~ form two
each of which satisfy the usual commutation relations for the angular momentum and the relations

$$\left.\begin{aligned} \alpha_x'^2+\alpha_y'^2+\alpha_z'^2 &= (k-\tfrac{1}{2})(k+\tfrac{1}{2}), \\ \beta_x'^2+\beta_y'^2+\beta_z'^2 &= (l-\tfrac{1}{2})(l+\tfrac{1}{2}). \end{aligned}\right\} (9)$$

~~Namely,~~ ~~If we assume~~ $\vec{\alpha}'$ and $\vec{\beta}'$ can be considered as matrices ~~having~~ with $2k\times(2l+1)$ rows and columns and $(2k+1)\times 2l$ rows and columns respectively.

Now, if we introduce the ~~matrices~~ vector ~~spin angular momentum~~

$$\vec{s} = \hbar\begin{pmatrix} \vec{\alpha}'+\vec{\beta}' & 0 \\ 0 & \vec{\beta}'+\vec{\alpha}' \end{pmatrix}, \quad (10)$$

we can show that ~~the~~ vectors are the spin vectors

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~~the sum~~

~~$\vec{M} = \vec{m} + \vec{s}$~~

~~commut can be found to commute with H the Hamiltonian~~

~~A~~ simple calculations shows that the sum

$$\vec{M} = \vec{m} + \vec{s}$$

commutes with the Hamiltonian (4), so that ~~$\vec{s}$ can be~~ considered as ~~defined~~ the spin angular momentum of the particle. ~~can be~~ As $\vec{\alpha}, \vec{\beta}, \vec{\alpha}'$ and $\vec{\beta}'$ have fixed magnitudes k, l, $k-\frac{1}{2}$ ~~$l-\frac{1}{2}$~~ and $l-\frac{1}{2}$ respectively, ~~each~~ $\vec{\alpha} + \vec{\beta}'$ ~~and $\vec{\beta} + \vec{\alpha}'$~~ ~~to take~~ either ~~the absolute values of~~ becomes either ~~the~~ of the set

$$k+l-\frac{1}{2},\ k+l-\frac{3}{2},\ \ldots\ |k-l+\tfrac{1}{2}|,$$

while that of ~~those of~~ $\vec{\alpha}' + \vec{\beta}$ becomes either of

$$k+l-\frac{1}{2},\ k+l-\frac{3}{2},\ \ldots,\ |k-l-\tfrac{1}{2}|,$$

Thus in general the magnitudes of the spin angular momentum ~~a~~ for different states ~~are different, in general,~~ can be different from each other.

For ~~special~~ the most simple case $k = l = \frac{1}{2}$, $\vec{\alpha}'$ and $\vec{\beta}'$ ~~=0~~ are reduces to zero identically, so that

$$\vec{s} = \begin{pmatrix} \vec{\alpha} & 0 \\ 0 & \vec{\beta} \end{pmatrix}$$

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corresponding to the ~~ordinary~~ case of Dirac's electron, and the magnitude of the spin ~~is~~ becomes $\frac{1}{2}$ always, which

§ 4. Magnetic Moment

~~The magnetic moment~~ From the linear wave equations (1), one can obtain, by the ~~usual~~ usual procedure, the second order equations

$$\left.\begin{aligned} &(p_t^2 - p_x^2 - p_y^2 - p_z^2)\psi_A + e\hbar k^{-1}(\vec{\alpha}\vec{H} + i\vec{\alpha}\vec{E})\psi_A - m^2\psi_A = 0, \\ &(\qquad // \qquad)\psi_B + e\hbar l^{-1}(\vec{\beta}\vec{H} - i\vec{\beta}\vec{E})\psi_B - m^2\psi_B = 0, \end{aligned}\right\} \quad (11)$$

so that one ~~one may~~ may consider the particle to have the electric and magnetic ~~moments~~ moments

$$i\frac{\hbar e}{2m}\begin{pmatrix} k^{-1}\vec{\alpha} & 0 \\ 0 & -l^{-1}\vec{\beta} \end{pmatrix} \quad (12)$$

and

$$\frac{\hbar e}{2m}\begin{pmatrix} k^{-1}\vec{\alpha} & 0 \\ 0 & l^{-1}\vec{\beta} \end{pmatrix} \quad (13)$$

respectively.

It is rather surprising that the spin (10) and electric or magnetic moment (12) or (13) are not proportional to each other except the case $k = l = \frac{1}{2}$.

the light velocity is taken as unity.

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It should be noticed that
~~One can~~ The expression ( ) for the ~~angular~~ spin has
~~has~~ the ~~simple~~ ~~following meaning~~ ~~can, also be deduced from~~ ~~the group~~ ~~theory~~ the meaning in the theory of the rotation group.
Namely, $\vec{\alpha}+\vec{\rho}$ is the ~~infinitesimal~~ operator vector
of ~~infinitesimal rotation~~ the simultaneous infinitesimal
rotations of the spin spaces of dimensions ~~2k and~~
$2k+1$ and $2l$ respectively, whereas $\vec{\alpha}'+\vec{\rho}$ is
that of ~~the~~ the ~~space~~ spin spaces of dimensions $2k$
and $2l+1$ respectively.

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§5. Invariance of the Wave Equations under Reflection

The wave equations (1) with the supplementary conditions are invariant ~~un~~ together, ~~of course,~~ of course, under proper Lorentz transformations, but are not so in general under reflection with respect to a point. as long as ~~since ψ_A and the dott~~ (Spiegelung) ~~undotted and the~~ dotted quantities since ~~ψ_A and ψ_B have~~ the numbers of ~~co~~ components of ψ_A and ψ_B are ~~different~~.

Now, according to Dirac[1], the whole equations ~~for ψ_A and ψ_B~~ can be ~~a~~ written alternatively ~~into the form~~

$$\left.\begin{array}{l} p^{\dot{\mu}\nu}\psi^{(i)}_{\nu} = m\psi^{(i)\dot{\mu}} \\ p_{\dot{\mu}\nu}\psi^{(i)\dot{\mu}} = m\psi^{(i)}_{\nu}, \end{array}\right\} \quad \begin{array}{l} \mu, \nu = 1, 2 \\ i = 1, 2, \ldots 4kl \end{array} \qquad (14)$$

where the ~~new~~ ~~The~~ ~~wave~~ functions $\psi^{(i)}_{\nu}$ ~~ψ_2 and~~ $\psi^{(i)\dot{\mu}}$ ~~ψ^1, ψ^2~~ ~~and~~ ~~$\psi_{\dot{\mu}\lambda}$~~, with each ~~$2k\,2kl$~~ components, ~~where~~ are connected with ψ_A and ψ_B by the relations

$$\left.\begin{array}{l} v^{(i\kappa)}_{\nu}\psi^{(\kappa)}_{A} = k^{\frac{1}{2}}\psi^{(i)}_{\nu} \\ v^{(i\lambda)\dot{\mu}}\psi^{(\lambda)}_{B} = l^{\frac{1}{2}}\psi^{(i)\dot{\mu}}, \end{array}\right\} \qquad (15)$$

where $\kappa = 1, 2, \cdots (2k+1)\times 2l$, $\lambda = 1, 2, \cdots 2k\times(2l+1)$.

The equations (14) are invariant under reflection ~~general Lorentz transformation~~ if it ~~$\psi^{(i)}_{\nu}$ and $\psi^{(i)\dot{\mu}}$~~ ($\psi^{(i)}_{1}$ and $\psi^{(i)}_{2}$) ~~and $\psi^{(i)\dot{1}}$ and $\psi^{(i)\dot{2}}$~~ transform into $\psi^{(i)\dot{1}}$ ... $\psi^{(i)\dot{2}}$ and vice versa.

1) l.c. p.457.

This, however, will be ~~a~~ an ~~other~~ additional restriction to the wave equations ψ_A, ψ_B, ~~obtained from the~~ ψ_ν, which can be derived ~~from~~ by the relations from $\psi_\nu^{(i)}$, $\psi^{(i)\mu}$

$$\cancel{u^{(\kappa)(i)\nu}\psi_\nu^{(i)} = k\,\psi_A^{(\kappa)}}$$

$$\left.\begin{aligned} 2k^{\frac{1}{2}}\psi_A^{(\kappa)} &= u^{(\kappa)(i)\nu}\,\psi_\nu^{(i)} \\ 2l^{\frac{1}{2}}\psi_B^{(\lambda)} &= u_\mu^{(\lambda)(i)}\,\psi^{(i)\mu}. \end{aligned}\right\} \qquad (16)$$

At anyrate,

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§5. Invariance of the Wave Equations under Reflection.

According to Dirac[1] ~~the~~ wave equations (1) together with the supplementary conditions can be ~~re~~written alternatively in the form

$$\left.\begin{aligned} p^{\dot{\mu}\varkappa} A^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots} &= m\sqrt{\tfrac{k}{l}}\, B^{\dot{\mu}\dot{\beta}\dot{\gamma}\dots}_{\lambda\mu\dots} \\ p_{\dot{\alpha}\varkappa} B^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dots}_{\lambda\mu\dots} &= m\sqrt{\tfrac{l}{k}}\, A^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots}, \end{aligned}\right\} \quad (14)$$

where $A^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots}$ is a spinor with $2k$ undotted suffixes downstairs and $(2l-1)$ dotted suffixes upstairs and $B^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dots}_{\lambda\mu\dots}$ with $(2k-1)$ undotted suffixes downstairs and $2l$ dotted ones upstairs, both ~~spinors~~ being symmetrical between all the undotted and between all the dotted suffixes.

The equations (14) are ~~invariant~~, of course, ~~Lorentz~~ invariant under ~~the~~ proper Lorentz transformations, but are so under ~~the~~ the reflection with respect to the origin, (Spiegelung) except for only when $k = l$, ~~when A, B~~ the relations

$$\cancel{A^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots} = B}, \quad \left\{ \begin{aligned} A^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dots}_{\lambda\mu\dots} &= B \\ A^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots} &= B^{\dot{\beta}\dot{\gamma}\dots}_{\varkappa\lambda\mu\dots} \end{aligned}\right.$$

are assumed ~~for~~ for all values of $\alpha, \beta, \gamma, \dots$ and $\varkappa, \lambda, \mu, \dots$

1) l.c. p. 459.