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3. Spin Angular Momentum

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# Note on Dirac's Generalized Wave Equations

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Abstract

A few elementary results of Dirac's relativistic theory of the particle with the spin larger than $\frac{1}{2}$ were deduced. Namely, the expressions for the velocity, the current density, the spin angular momentum and the electric and magnetic moments, which had not been given explicitly in Dirac's paper, ~~was~~ were obtained. It is noticeable that the spin and the magnetic moment are not proportional to each other in general. The invariance of the wave equations under reflection (Spiegelung) was also discussed.

## § 1. Introduction

It will not be altogether trivial to deduce a few properties of the particle with the spin larger than $\frac{1}{2}$ satisfying the generalized relativisttic wave equations recently proposed by Dirac,[1] although these equations are not immediately applicable to elementary particles known at present, the atomic nuclei other than the proton and the neutron, being considered usually as composite particles.

1) Proc. Roy. Soc. (A) **155**, 447, 1936.

以下の式は通常のhに / をつけた活字を使ふこと

The new wave equations for the particle with the mass m in energy unit and with the charge e can be written in the form

$$\left.\begin{aligned}(p_t + k^{-1}\vec{\alpha}\vec{p})\psi_A - \frac{1}{2}\frac{m}{\sqrt{k\ell}}(u^1v^1 + u^2v^2)\psi_B = 0\\ (p_t - \ell^{-1}\vec{\beta}\vec{p})\psi_B - \frac{1}{2}\frac{m}{\sqrt{k\ell}}(u_1v_1 + u_2v_2)\psi_A = 0\end{aligned}\right\} \quad (1)$$

where

$$p_t = i\hbar\frac{\partial}{\partial t} - eA_0$$

$$\vec{p} = -i\hbar\,\mathrm{grad} - e\vec{A}$$

and $\vec{\alpha}$ and $\vec{\beta}$ are two spin vectors, which commute with each other, each of their magnitudes k and $\ell$ being an integer or a half integer. Besides these equations, ~~the part~~ the wave functions $\psi_A$ and $\psi_B$ should satisfy several supplementary conditions.

$u^1$ and $u^2$ form a spinor with $(2k+1)\times 2\ell$ rows and $2k\times 2\ell$ columns, while $v_1$ and $v_2$ form one with $2k\times 2\ell$ rows and $(2k+1)\times 2\ell$ columns.[1] They are connected with $\vec{\alpha}$ by the relations

$$\left.\begin{aligned}\alpha_z - k &= -u^1v_1 \qquad & \alpha_x - i\alpha_y &= -u^1v_2\\ \alpha_x + i\alpha_y &= -u^2v_1 \qquad & -\alpha_z - k &= -u^2v_2,\end{aligned}\right\} \quad (2)$$

so that $\vec{\alpha}$ ($\alpha_3$) and k in (2) are matrices with $(2k+1)\times 2\ell$ rows and columns. Similarly, $u_1$ and $u_2$ form a dotted spinor with $2k\times(2\ell+1)$ rows and $2k\times 2\ell$ columns, while $v^1$ and $v^2$ form one with $2k\times 2\ell$ rows and $2k\times(2\ell+1)$ columns. They are connected with $\vec{\beta}$ by the relations

1) Complete account of the spinor calculus can be found in Laporte and Uhlenbck, Phys. Rev. 37, 1380, 1931 or in van der Waerden, Die gruppentheor. Methode in der Quantenmech., Berlin, 1932.

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$$\beta_z - l = -u_i v^i \qquad \beta_x - i\beta_y = -u_i v^2$$
$$\beta_x + i\beta_y = -u_2 v^i \qquad -\beta_z - l = -u_2 v^2 \quad \Big\} \quad (3)$$

so that $\beta$'s and $l$ in (3) are matrices with $2k \times (2l+1)$ rows and columns. Thus the wave functions $\psi_A$ and $\psi_B$ have $(2k+1) \times 2l$ and $2k \times (2l+1)$ components respectively.

From the wave equations (1), the Hamiltonian of the particle becomes

$$H = \begin{pmatrix} -k^{-1}\vec{\alpha}\vec{p} + eA_0 & \frac{1}{2}\frac{m}{\sqrt{kl}}(u^i v^i + u^2 v^2) \\ \frac{1}{2}\frac{m}{\sqrt{kl}}(u_i v_1 + u_2 v_2) & l^{-1}\vec{\beta}\vec{p} + eA_0 \end{pmatrix} \quad (4)$$

§ 2. Current Density and Velocity

The equations of (1), are multiplied by $\psi_A^*$ and $\psi_B^*$ respectively on the left and are summed up together with ~~the expressions~~ those which are complex conjugate to them. We have, then, the equation

$$\frac{\partial}{\partial t}(\psi_A^* \psi_A + \psi_B^* \psi_B) + \mathrm{div}(-\psi_A^* k^{-1}\vec{\alpha}\psi_A + \psi_B^* l^{-1}\vec{\beta}\psi_B) = 0, \quad (5)$$

which becomes the equation of continuity, if we define the density and the current density of the particle in the state $(\psi_A, \psi_B)$ by

$$\left.\begin{aligned} \rho &= \psi_A^* \psi_A + \psi_B^* \psi_B \\ \vec{i} &= -\psi_A^* k^{-1}\vec{\alpha}\psi_A + \psi_B^* l^{-1}\vec{\beta}\psi_B \end{aligned}\right\} \quad (6)$$

Next, by inserting (4) in the equation of motion

$$i\hbar\dot{\vec{r}} = \vec{r}H - H\vec{r},$$

we obtain the expression of the velocity of the particle

$$\dot{\vec{r}} = \begin{pmatrix} -k^{-1}\vec{\alpha} & 0 \\ 0 & l^{-1}\vec{\beta} \end{pmatrix}, \qquad (7)$$

which is intimately connected with the current density as it should be.

## §3. Spin Angular Momentum

In the central field with the potential

$$A_0(\vec{r}) = A_0(r) \qquad \vec{A}(\vec{r}) = 0,$$

the components of the orbital angular momentum

$$\vec{m} = \vec{r} \times \vec{p}$$

are not constants of motion, since

$$i\hbar\dot{\vec{m}} = \vec{m}H - H\vec{m} \neq 0 .$$

Now we can show that each of the vectors ~~are the spin~~ defined by

$$\left.\begin{aligned} \vec{\alpha}' &= \sum_{\mu,\nu=1,2} \frac{u_\mu v_\nu \vec{\alpha} u^\nu v^\mu}{2(k+1)(2l+1)} \\ \vec{\beta}' &= \sum_{\mu,\nu=1,2} \frac{u^\nu v_\mu \vec{\beta} u^\mu v_\nu}{2(l+1)(2k+1)} \end{aligned}\right\} \qquad (8)$$

~~are the spin vectors, each of which~~ satisfies the usual commutation relations for the angular momentum and the relation

$$\left.\begin{aligned} \alpha_x'^2 + \alpha_y'^2 + \alpha_z'^2 &= (k-\tfrac{1}{2})(k+\tfrac{1}{2}) \\ \text{or} \quad \beta_x'^2 + \beta_y'^2 + \beta_z'^2 &= (l-\tfrac{1}{2})(l+\tfrac{1}{2}), \end{aligned}\right\} \qquad (9)$$

where the $\alpha$'s are matrices with $2k \times (2l+1)$ rows and columns, while $\beta$'s are those with $(2k+1) \times 2l$ rows and columns.

Further we can show that the sum

$$\vec{M} = \vec{m} + \vec{s}$$

commutes with the Hamiltonian (4), where $\vec{s}$ is a vector defined by

$$\vec{s} = \hbar \begin{pmatrix} \vec{\alpha} + \vec{\beta}' & 0 \\ 0 & \vec{\beta} + \vec{\alpha}' \end{pmatrix}, \tag{10}$$

so that $\vec{s}$ can be considered as the spin angular momentum of the particle, the total angular momentum $\vec{M}$ being the sum of the orbital and the spin angular momentum.

As the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\alpha}'$ and $\vec{\beta}'$ have fixed magnitudes $k$, $l$, $k-\frac{1}{2}$ and $l-\frac{1}{2}$ respectively, the magnitude of $\vec{\alpha}+\vec{\beta}'$ becomes either of

$$k+l-\tfrac{1}{2},\ k+l-\tfrac{3}{2},\ \cdots\cdots,\ |k-l+\tfrac{1}{2}|,$$

while that of $\vec{\alpha}'+\vec{\beta}$ becomes either of

$$k+l-\tfrac{1}{2},\ k+l-\tfrac{3}{2},\ \cdots\cdots,\ |k-l-\tfrac{1}{2}|.$$

Thus, in general, the spin ~~magnitudes of the spin for~~ ~~spin values for two~~ has different magnitudes for different states ~~are different from each other~~. For the most simple case $k=l=\frac{1}{2}$, $\vec{\alpha}'$ and $\vec{\beta}'$ are reduced to zero identically, so that

$$\vec{s} = \hbar \begin{pmatrix} \vec{\alpha} & 0 \\ 0 & \vec{\beta} \end{pmatrix}$$

and the spin value is always $\frac{1}{2}$ as it should be.

It should be noticed that the expression (10) has the following group-theoretical meaning. Namely, $\vec{\alpha}+\vec{\beta}'$ is the vector of the simultaneous infinitesimal rotations of the spin spaces of dimensions $2k+1$ and $2l$ respectively, whereas $\vec{\alpha}'+\vec{\beta}$ is that of the spaces of dimensions $2k$ and $2l+1$ respectively.

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## § 4. Magnetic Moment

By the usual procedure, one can obtain from the linear wave equations (1) the second order equations

$$\left.\begin{aligned}(p_t^2-p_x^2-p_y^2-p_z^2)\psi_A+e\hbar k^{-1}(\vec{\alpha}\vec{H}+i\vec{\alpha}\vec{E})\psi_A-m^2\psi_A=0\\(p_t^2-p_x^2-p_y^2-p_z^2)\psi_B+e\hbar l^{-1}(\vec{\beta}\vec{H}-i\vec{\beta}\vec{E})\psi_B-m^2\psi_B=0\end{aligned}\right\}\quad(11)$$

so that the electric and the magnetic moments of the particle can be defined by

$$i\frac{e\hbar}{2m}\begin{pmatrix}k^{-1}\vec{\alpha} & 0\\ 0 & -l^{-1}\vec{\beta}\end{pmatrix}\qquad(12)$$

and

$$\frac{e\hbar}{2m}\begin{pmatrix}k^{-1}\vec{\alpha} & 0\\ 0 & l^{-1}\vec{\beta}\end{pmatrix}\qquad(13)$$

respectively.

It is rather surprising that the spin (10) and the magnetic moment (13) are not proportional to each other except for $k=l=\frac{1}{2}$.

## § 5. Invariance of the Wave Equations under Reflection

According to Dirac,[1] the wave equations (1) together with the supplementary conditions can be written alternatively in the form

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1) l. c. p.459.

$$\left.\begin{aligned} p^{\dot\alpha\kappa} A^{\dot\beta\dot\gamma\cdots}_{\kappa\lambda\mu\cdots} &= m\sqrt{\frac{k}{l}}\, B^{\dot\alpha\dot\beta\dot\gamma\cdots}_{\lambda\mu\cdots} \\ p_{\dot\alpha\kappa} B^{\dot\alpha\dot\beta\dot\gamma\cdots}_{\lambda\mu\cdots} &= m\sqrt{\frac{l}{k}}\, A^{\dot\beta\dot\gamma\cdots}_{\kappa\lambda\mu\cdots} \end{aligned}\right\} \quad (14)$$

where $A^{\dot\beta\dot\gamma\cdots}_{\kappa\lambda\mu\cdots}$ is a spinor with 2k undotted suffixes downstairs and $(2l-1)$ dotted suffixes upstairs and $B^{\dot\alpha\dot\beta\dot\gamma\cdots}_{\lambda\mu\cdots}$ with (2k-1) undotted suffixes downstairs and $2l$ dotted suffixes upstairs, both being symmetrical between all the dotted and between all the undotted suffixes.

The equations (14) are, of course, invariant under proper Lorentz transformations, but ~~are not~~ are so under the reflection (Spiegelung) with respect to the origin, which transforms the spinor with undotted suffixes downstairs into that with dotted suffixes upstairs and vice versa, only when $k=l$ and the relations

$$\left.\begin{aligned} A^{\dot\beta\dot\gamma\cdots}_{\kappa\lambda\mu\cdots} &= B^{\dot\beta\dot\gamma\cdots}_{\kappa\lambda\mu\cdots} \\ A^{\dot\alpha\dot\beta\dot\gamma\cdots}_{\lambda\mu\cdots} &= B^{\dot\alpha\dot\beta\dot\gamma\cdots}_{\lambda\mu\cdots} \end{aligned}\right\} \quad (15)$$

are assumed for all values of $\alpha, \beta, \gamma, \cdots$ and $\kappa, \lambda, \mu \cdots$.

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