On the Theory of Internal Pair Production.[1]

By Hideki Yukawa and Shoichi Sakata

Abstract

The probability of internal pair production by radiationless transition of the radioactive nucleus was calculated and its ratio to the probability of emission of electrons from K levels by the same nuclear transition was compared with the experiment in the case of RaC'.

1) This paper was read before the meeting of Osaka branch on July 6, 1935.

On the other ~~n/~~ hand, according to Alichanow, total

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§ 1. Introduction and Summary

The theory of the production of the pair of an electron and a positron by the internal conversion of the γ-ray with the energy larger than 2mc , was developped recently by Jaeger and Hulme , as a natural extension of the theory of ordinary internal conversion of the γ-ray with the emission of the β-ray.

They compared their result with the experiment of Alichanow and Kosodaew , who measured the number of positrons emitted from a Ra(B+C) source. According to he theory the second and the third maxima of the experimental distribution curve, at about 0.7×10^6 eV and 10^6 eV, were to be due to the γ-rays of energies 1.7×10^6 eV and 2.2×10^6 eV, as the coefficient of internal pair production ~~were~~ was maximum at the upper limit of the energy of the positron.

The exceptional case, where a pair of an electron and a~~ao~~ positron is emitted by taking up the energy liberated by a nuclear transition between two S levels, was not treated by the former authors, because of lack of the coresponding γ-ray.

~~1) This paper was read before the meeting of Osaka branch on July 6, 1935.~~

2) J.C.Jaeger & H.R.Hulme, Proc.Roy.Soc. **148**, 708, 1935.

3) A.I.Alichanow & M.S.Kosodaew, Zeits.f.Phys. **90**, 249, 1934.

4) See Fig. 1 below, which is the same with Fig. 15 of Alichanow & Kosodaew, loc.cit.

Now, if the maximum of the distribution curve of the positron produced by such a process lies also at the upper limit of the energy, the first hump of the curve of Fig. 1, at about 0.4×10^6 eV, can be attributed to the missing γ-ray of energy 1.4×10^6 eV.

Hence it seems worth while to investigate this case in detail and to determine whether such a process is sufficient to account for the large hump of the curve or not.

The nuclear transition between two levels of energy difference ΔW can be considered in general as a perturbing field on the electron varying with the frequency $\frac{\Delta W}{h}$. This perturbing field extends only over the region of nuclear dimension in the case of S-S transition, so that the probability of transition of the electron is large only when the eigenfunction of the initial and the final states are not small in the neighborhood of the nucleus. Accordingly, we have only to consider the case, where the total angular momenta of both of the electron and the positron take the smallest values.

The distribution function of the positron calculated in this manner has the maximum in the neighborhood of the upper limit of the energy, which agrees with the first maximum of the curve of Fig. 1.

To find the absolute value of the probability of the pair production we have to know the nuclear ~~tran~~ structure in detail. To avoid this this difficulty, we considered only the ratio of the probability of the pair production to that of emission of K electrons caused by the same nuclear transitions, as they have the common factor relating to the nuclear transition, if the vector potential of the perturbing is negligible compared with the scalar potential. The result obtained shows that the ratio is so small that only a small part of the hump in the exp~~p~~erimental curve can be attributed to the positrons emitted by the above process.

[If the vector potential is also taken into account, the ratio depends on the special forms of scalar and vector potentials and in the favorable case it becomes large enough to be in accord with the experiment.] We can not, however, have much confidence ~~of~~ in our results anyhow, as the application of the present theory of the electron up to the immediate neighborhood of the nucleus may happen to be totally wrong. Hence we are not certain whether other processes such as the emission of the positron alone from the nucleus or the internal conversion of the β-ray are necessary for the complete explanation of the experimental result or not.

A brief account of the calculation will be given in the following sections.

§ 2. Probability of Internal Pair Production

Now we consider the general case, where an electron moving in the field of the nucleus with the charge Ze is perturbed by a nuclear transition between two levels of energy difference ΔW.

The perturbation can be expressed by scalar and vector potentials of of the form $A_0(\vec{r})\exp(-2\pi i\nu t)$ and $\vec{A}(\vec{r})\exp(-2\pi i\nu t)$, where $\nu = \frac{\Delta W}{h}$.
If we assume that only one particle with the effective charge Qe and the reduced mass M in the nucleus falls from a state $\varphi(\vec{r})$ to ~~x~~ a

state $\varphi_0(\vec{r})$, the energy difference being ΔW, the potentials can be expressed in general by

$$A_0(\vec{r}) = Qe\int \frac{\exp(iq|\vec{r}-\vec{r}_1|)}{|\vec{r}-\vec{r}_1|}\varphi_0^*(\vec{r}_1)\varphi(\vec{r}_1)\,d\vec{r}_1, \qquad (1)$$

$$\vec{A}(r) = \frac{Qeh}{4\pi mc}\int \frac{\exp(iq|\vec{r}-\vec{r}_1|)}{|\vec{r}-\vec{r}_1|}\{\varphi_0^*(\vec{r}_1)\,\mathrm{grad}_1\,\varphi(\vec{r}_1) - \varphi(\vec{r}_1)\,\mathrm{grad}_1\,\varphi_0^*(\vec{r}_1)\}\,d\vec{r}_1, \quad [5] \qquad (2)$$

where $q = \frac{2\pi\Delta W}{hc}$. The symbol $\rightarrow$ means always a vector quantity and $*$ a conjugate complex quantity.

Now, according to the perturbation theory, the probability per unit time per unit energy range of the transition of the electron from the continuous state of negative energy $-E$ ($E > mc^2$) to the continuous state of positive energy E_+ ($> mc^2$), satisfying the relation $E_- + E_+ = \Delta W$, is given by

$$\rho(E_+) = \frac{4\pi}{h}\sum_{j',u'}\sum_{j,u}\left|\int \psi^*_{E_+,j',u'}\{-eA_0(\vec{r}) + e(\vec{\alpha}\cdot\vec{A}(\vec{r}))\}\psi_{E_-,j,u}(\vec{r})\,d\vec{r}\right|^2 \qquad (3)$$

where $\psi_{E,j,u}$ denotes the solution of Diracs equation

$$\left\{E + \frac{Ze^2}{r} + c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2\right\}\psi = 0$$

with energy E, inner quantum number $j - \frac{1}{2}$, and its z-component $u + \frac{1}{2}$. j is a positive or a negative integer excluding zero and u takes a value between $-j$ and $j-1$. α, β are Diracs matrices with four rows and four columns and ψ^* can be considered as a matrix with one row and four columns, while ψ as that with four rows and one column, the corresponding elements being complex conjugate to each other.

5) H.M.Taylor & N.F.Mott, Proc.Roy.Soc. 138, 665, 1932.

The eigenfunctions are normalized with respect to energy, the definition of which is

$$\int \tilde{\psi}_{E'} \psi_E \, d\vec{r} = \delta(E'-E)$$

This corresponds to the process, in which a positron of energy E and an electron of energy E are emitted simultaneousely from the nucleus.

Hence, ~~the total probability per unit time of the~~ production of the pair is given by

$$P_{pair} = \int_{mc^2}^{\Delta W - mc^2} p(E_1)\, dE_1 \qquad (4)$$

Now, in the case of S-S transition, φ and $\mathcal{G}$ are functions of r only, so that the scalar potential becomes a function of r and the vector potential reduces to the form $\frac{\vec{r}}{r}$ multiplied by a function of r. Thus the perturbing potential eA +e A can be written in a form

$$V_0(r) + \frac{\alpha \vec{r}}{r} V(r)$$

which commutes with j and u.

Hence we obtain the selection rules

$$\Delta j = 0 \qquad , \qquad \Delta u = 0$$

for the transition of the electron.

In this case, moreover, A and A can be reduced to zero outside of the nucleus by a suitable gauge transformation, which does not affect the transition probability in general. Consequently, the probability of the transition of the electron is appreciable only when the eigenfunctions of the initial and the final states are not small in the

In this case, moreover, $\vec{A}$ can be reduced to zero everywhere and A_0 to the form $V_0'(r)$ ~~where~~ which vanishes outside of the nucleus, by a suitable . . .

neighborhood of r=0.

Thus, the expression (3) becomes approximately

The eigenfunctions in this ex pression are well known. Since only the values of the eigenfunctions in the neighborhood of r=0 are important, their radial parts are expanded in powers of r and the first terms with the power $r^{\delta-1}$ are taken as the first approximation, where

Inserting these expressions for the eigenfunction in ( ) and performing the integrations with respect to , we have finally

p(E) =

where

F ( ) is the only factor, which depends on the energy of the positron emitted and has the form

F ( ) =

where

6) Jaeger & Hulme, loc.cit. We have to take k=0 in (2) for j=-1 and k=1 in (2) for j=1, u being the same as in our case.

The forms of the distribution function F (   ) in special cases are shown in Fig, 2 and Fig. 3.

The total probability of pair production ~~is~~ now becomes

In these formulae, R and ~~λ~~ depend on the detailed structure of the nucleus and can not be estimated easily.

If we neglect the vector potential, thus the calculation is formally similar to that of Beck and Sitte on -disintegration. /~~7)~~ The physical interpretations, however, differ in two cases. In the latter case, either the electron or the positron is reabsorbed by the original nucleus, whereas, in our case, both of them escape the nucleus.

§ 3. Probability of Electron Emission from K Levels

Next we consider the emission of K electrons by the same nuclear transitions.

In general, the probability per unit time of the transition of the the electrons from K states of energy mc √ 1- Z to continuous states of energy E (mc), satisfying the condition E =mc √ 1- Z + is given by

7) G.Beck und K.Sitte, Zeits.f.Phys. 86, 105, 1933.

the meaning of W, A (r), A(r) and being the same as in the previous section. are/~~eigen~~ normarized eigenfunctions of two K states.

Now we consoder the same S-S transition of the nucleus as in the previous section. The expression ( ) becomes simply

Taking for and the first terms of the expansion in powers of r, we have finally

where

~~In this case, the probability does not depend on the magnitude of the vector potential.~~

Now the ratio of the probability of pair production by the S-S transition of the nucleus to that of the emission of electrons from K levels by the same transition is given by

8) See, for example, H.R.Hulme, Proc.Roy.Soc. 138, 643, 1932.

§ 4. General Discussions

Before entering into numerical calculations in special cases, it should be noticed that the above formulation is incomplete in two points.

The first point is that the perturbing potentials ~~wre~~ were assumed to have the special forms (1) and (2). Now, we can show more generally that the perturbing potentials are expressed in the forms

F(r)exp(-2 i t),

G(r)exp(-2 i t),

F, G being certain functions of r only, whenever both the initial and the final states of the nucleus have zero spin or, more gen~~/~~erally, have the same spin in the same direction.

To prove this, we consider the fac~~/~~t that the total angular momentum of the system consisting of the nucleus and the electron should be constant throughout the above processes. So, if the spin of the nucleu~~/~~s does not change, the total angular momentum of the electron

M = m +

should be constant also, where m = r p and is the spin vector. Hence, the perturbed Hamiltonian of the electron should commute with M, so that we have

M(eA +e A) - (eA +e A)M = 0,

where A (r)exp(-2 i t), A(r)exp(-2 i t) are perturbing potentials due to the nuclear transition. As the coefficients of 1 (the unit matrix), $\alpha_x = \rho_1\sigma_x$

$\alpha_y = \rho_1\sigma_y$, $\alpha_z = \rho_1\sigma_z$ in the left hand sides of these equations should vanish separately, we have

$$\vec{m}A_0 - A_0\vec{m} = 0 \qquad (\quad)$$

$$\left.\begin{aligned} m_x A_x - A_x m_x &= 0 \\ m_y A_x - A_x m_y &= -\frac{ih}{2\pi} A_z \\ m_z A_x - A_x m_z &= \frac{ih}{2\pi} A_y \end{aligned}\right\} \qquad (\quad)$$

and similar equations for A, A.

The equations ( ) show that A is a function of r only, while the equations show that the relations of commutation between $\vec{m}$ and $\vec{A}$ are the ~~sm~~ same as those between $\vec{m}$ and $\vec{r}$.

Now, by using ( ) and the relations of commutation between $\vec{m}$ and $\vec{r}$, we have

$$m_y(xA_y - yA_x) - (xA_y - yA_x)m_y = \frac{ih}{2\pi}(yA_z - zA_y)$$

On the other hand, by using ( ) and the relations of commutation between $\vec{p}$ and $\vec{r}$, we have

$$p_x(xA_y - yA_x) - (xA_y - yA_x)p_x = x(p_x A_y - A_y p_x) - \frac{ih}{2\pi}A_y - (yp_x A_x - A_x yp_x).$$

From the third equation

$$(xp_y - yp_x)A_x - A_x(xp_y - yp_x) = \frac{ih}{2\pi}A_y,$$

( ), the last two terms together become $xp_y A_x - A_x xp_y$, so that we have

$$p_x(xA_y - yA_x) - (xA_y - yA_x)p_x = x(p_x A_y - A_y p_x - pA_x + A_x p_y). \qquad (\quad)$$

Further, by using the relations

$$(zp_x - xp_z)A_y - A_y(zp_x - xp_z) = 0, \qquad (yp_z - zp_y)A_x - A(yp_z - zp_y) = 0,$$

we can deduce the equation

$$p_z(xA_y - yA_x) - (xA_y - yA_x)p_z = z(pA_y - A\cancel{p}_y p_x - p_y A_x + A_x p_y). \qquad (\quad)$$

Combining (   ) and (   ), we have

$$(zp - xp)(xA - yA) - (xA - yA)(zp - xp) = 0.$$

Comparing this with (   ), we have finally

$$yA - zA = 0.$$

Similarly, we can further deduce the relations

$$zA - xA = 0, \qquad xA - yA = 0,$$

and

$$m(rA) - (rA)m = 0.$$

These equations together show that A has the form $\frac{\vec r}{r}\, a(r)$

Thus the proof is ended.

We can show, moreover, that the perturbing field is zero outside of the nucleus in this general case. Firstly, the magnetic field H vanishes everywhere, since

$$\vec H = \text{curl } \vec A = 0.$$

Secondly, from the field equations

$$\frac{1}{c}\frac{\partial \vec E}{\partial t} = \text{curl } \vec H - \frac{4\pi}{c}\vec I,$$

in which $\vec H = 0$ and both the electric field $\vec E$ and the current density $\vec I$ have the time factor $\exp(-2\pi i\nu t)$, we have the relation

$$\vec E = \left(\frac{2}{i\nu}\right)\vec I,$$

which shows that the electric field vanishes outside of the nucleus, where the current density due to the nuclear transition vanishes.

Hence, in this case, we can ~~we can~~ reduce the ~~vector~~ scalar potential~~s themselves~~ to zero ~~everywhere~~ outside of the nucleus by a suitable gauge transformation.

~~and the scalar~~ and ~~the scal~~

Hence, in this case, we can reduce the vector potentials themselves to zero everywhere and the scalar potential to the form ~~F(~~ F(r) exp(−2πiνt), ~~where~~ F(r) ~~is~~ being a function of r, which vanishes outside of the nucleus.

The second point is that the solutions of Diracs equation in Coulomb field of the nucleus were taken as the eigenfunctions of the unperturbed states of the electron. As the field is no more of Coulomb type in the neighborhood of the nucleus, we should modify the forms of the eigenfunctions in this region. Thus, each eigenfunction in the previous sections should be multiplyed by a certain function, which deviates appreciably from 1 for small value of r comparable with the nuclear radius a.

The exact determination of such functions is very complicated and necessitates special assumptions on the nuclear structure, we should be satisfyed with taking for the values of the eigenfunctions in~~s~~ the nucleus simply the values of the solutions in Coulomb field at r = a,
while their values outside of the nucleus do not concern us here.

Then, in the formulae of the preceeding sections, we have only to change the definition~~s~~ of R_ρ , ~~S_ρ~~ , other points being unaltered. Namely, we have to put

$$R_\rho = a_N^{2(\gamma-1)} \int V_0 \, d\vec{r} ,$$

$$\cancel{S_\rho = a_N^{2(\gamma-1)} \int V \, d\vec{r}} ,$$

instead of the corresponding expression~~s~~ in § 2.

~~§ 4. Numerical Results and Discussions~~

We want now to obtain the numerical values of F ( ) and   in the case of RaC. Inserting the values

Z=84,   =   ,   W=1.4 10 eV=2.8mc ,

we have

Z=0.6,   =0.8,

=3.6,   =3.5,

so that

The distribution function F ( ) in this case is given in Fig. 2. Its maximum ~~of~~ lies near the upper limit of the energy of the positron and coincides approximately with the first maximum of Alichanows curve, ~~as~~ shown in Fig. 1, which represents the energy distribution of positrons emitted from ~~a~~ Ra(B+C) source.

From Fig. 2, we have

so that

11 9) The energy is written in unit of mc =0.5 10 eV, the proper energy 1 being inclusive.

10 8) E.Fermi, Zeits.F.Phys. 88, 161, 1934.

exhaustively.

As shown by the formula (12), depends, in general, on W and Z. It increases rapidly as W increases and the maximum of F ( ) displaces towards = at the same time.

For example, assuming Z=84 and W=4mc , we have for F ( ) a curve as shown in Fig. 3 and for a value

decreases rapidly as Z increases, owing to the factor (Z) in the denominater.

In conclusion the authors wish to express their cordial thanks to Prof. G. Beck for valuable discussions on his visit to Osaka.

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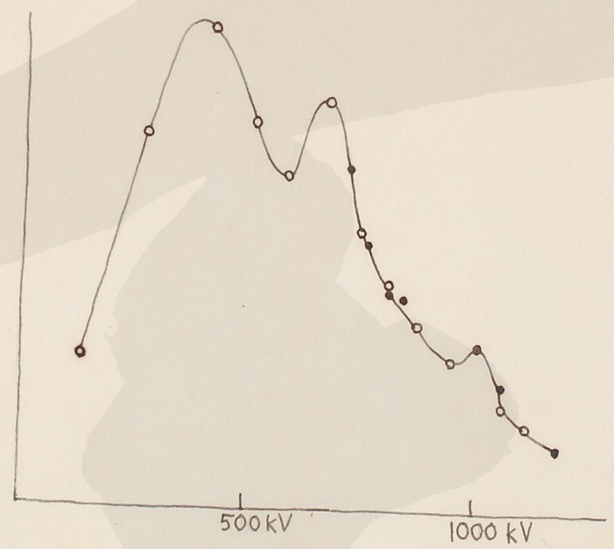

Fig. 1

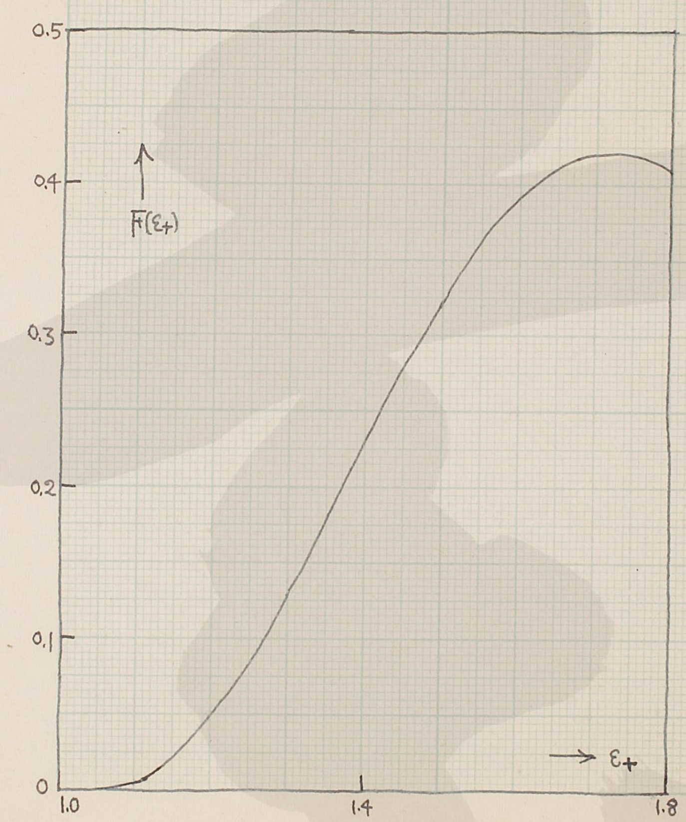

Fig. 2.

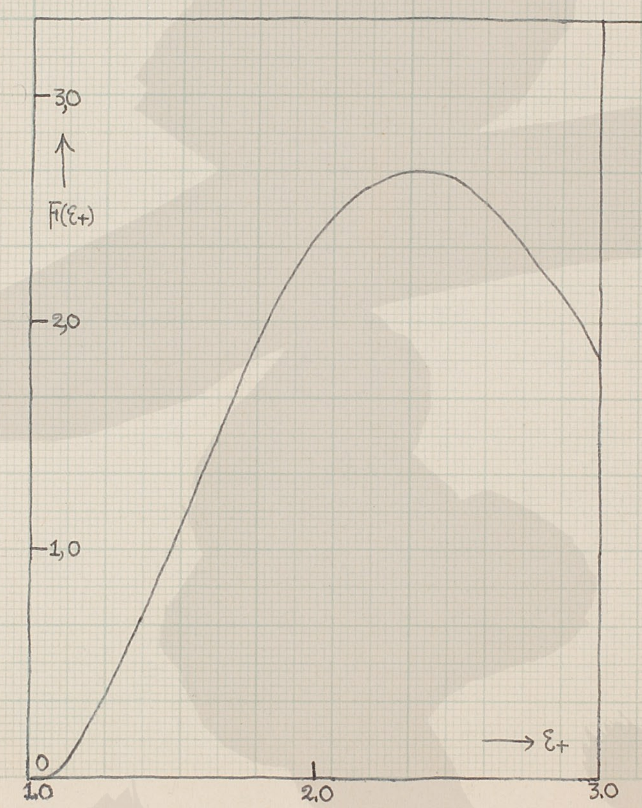

Fig. 3.