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On the Theory of Internal Pair Production

Note ~~on~~ on Pair Production by Radiationless Transition of the Radioactive Nucleus

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§1. Jaeger and Hulme[1] recently calculated the probability of the production of electrons and positrons by the internal conversion of γ-rays as a natural extension of the theory of ordinary internal conversion of γ-rays and compared their result with the experiment of Alichanow and Kosodaew,[2] who measured the number of positrons emitted from ~~the~~ Ra(B+C) source. The exceptional case of S-S transition, corresponding to energy liberation of amount 1.426×10^6 eV for RaC', was not treated by the former authors. If the maximum of the distribution of the positron lies in this case at the upper limit of its energy as in other cases, it coincides with the first maximum of Alichanow's curve.[3]

Hence it seems worth while to investigate this case in detail, and to determine whether ~~it~~ such a process is sufficient to account for the largest hump in Alichanows curve or not. To avoid special assumptions on nuclear structure, we cosidered only the ratio of the probability of the pair production to that of ~~and~~ the emission of K-electrons caused by the same nuclear transitions.

The result of our calculation shows that the ratio is so small that only a small part of the hump can be attributed to ~~such~~ positrons emitted by ~~the above~~ such process.

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1) J.C. Jaeger and H.R. Hulme, Proc. Roy. Soc. **148**, 708, 1935.
2) A. I. Alichanow ~~und~~ M.S. Kosodaew, Zs. f. Phys. **90**, 249, 1934
3) See Fig. 1, which ~~corresponds to~~ is the same with Fig. 15 of Alichanow's ~~paper~~ and Kosodaew, loc. cit.

Now we consider the general case, where a perturbation expressed by the potentials $A_0(\vec{r})\exp(-2\pi i\nu t)$, $\vec{A}(\vec{r})\exp(-2\pi i\nu t)$ scalar and vector of frequency ν acts on the electron moving in the Coulomb field of potential $\frac{e^2 Z}{r}$

§2. Now the nuclear transition, by which the energy of amount $h\nu$ ΔW is liberated can be considered as a periodic perturbation of frequency $\nu = \frac{\Delta W}{h}$ on the electron and can be expressed by ~~potentials~~ scalar and vector potentials $A_0(\vec{r})\exp(-2\pi i\nu t)$, $\vec{A}(\vec{r})\exp(-2\pi i\nu t)$. Hence, according to ordinary perturbation theory, the probability per unit energy range of the transition of the electron from the state of negative energy $E = -E_+ (E_+ > mc^2)$ to the continuous state of positive energy $E' (> mc^2)$, satisfying the relation $E' - E = h\nu$ or $E' + E_+ = h\nu$, is given by

$$p(E_+) = \frac{4\pi^2}{h} \sum_{j',u'} \sum_{j,u} \left| \int \tilde{\Psi}_{E',j',u'}(\vec{r}) \{ e A_0(\vec{r}) + e \vec{\alpha} \vec{A}(\vec{r}) \} \times \Psi_{E,j,u}(\vec{r})\, d\vec{r} \right|^2 \quad (1),$$

where $\Psi_{E,j,u}$ denotes the solution of Dirac's equation

$$\left(E + \frac{e^2 Z}{r} + c\vec{\alpha}\vec{p} + \beta mc^2 \right)\Psi = 0 ,$$

with energy E, total angular momentum $|j| - \frac{1}{2}$ (Dirac's j), and its z-component $u + \frac{1}{2}$. j is positive or negative integer excluding zero and u takes a value between $-|j|$ and $|j| - 1$. $\rightarrow$ means always a vector quantity.

This corresponds to the process, in which a positron of energy E_+ and an electron of energy E' are emitted simultaneously.

The total probability of the production of a pair is given by

$$P_{pair} = \int_{mc^2}^{h\nu - mc^2} p(E_+)\, dE_+ .$$

The calculation in this case is formaly (similar to) the same as that of Beck and Sitte[4] on β-disintegration. The physical interpretation, however, differs in two cases. In the latter case, either the positron or the electron is assumed to be reabsorbed by the original nucleus, whereas in our case both the electron and the positron escape the nucleus.

4) G.Beck und K.Sitte, ZS. f.Phys. 86,105,1933.

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Next the emission of K-electrons from the nucleus is given by

$$P_K = \frac{4\pi^2}{h} \sum_{j'',u''} \sum_{u=0,-1} \left| \int \tilde{\Psi}_{E'',j'',u''}(\vec{r}) \{ eA_0(\vec{r}) + e\vec{\alpha}\vec{A}(\vec{r}) \} \times \psi_{K,u}(\vec{r}) d\vec{r} \right|^2, \quad (2)$$

where $\psi_{K,u}$ denotes the eigenfunction of either of two 1S states of energy $mc^2\sqrt{1-\alpha^2 Z^2}$. $\Psi_{E'',j'',u''}$ denotes that of energy $E''(>mc^2)$ satisfying the condition $E'' = mc^2\sqrt{1-\alpha^2 Z^2} + h\nu$. [4] (in which ~~the common time factor~~ ~~exp(−i2πνt) is omitted~~,

The perturbing potentials $A_0(\vec{r})$, $\vec{A}(\vec{r})$ ~~are common to both cases and~~ can be expressed in general by [5]

$$A_0(\vec{r}) = Qe \int \frac{e^{iq|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} \phi(\vec{r}_1)\phi_0^*(\vec{r}_1)\, d\vec{r}_1$$

$$\vec{A}(\vec{r}) = \frac{Qeh}{4\pi mci} \int \frac{e^{iq|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} \{ \phi_0^*(\vec{r}_1)\,\mathrm{grad}_1\,\phi(\vec{r}_1) - \phi(\vec{r}_1)\,\mathrm{grad}_1\,\phi_0^*(\vec{r}_1) \} d\vec{r}_1,$$

~~in the nucleus and $A_0(\vec{r}) = \vec{A}(\vec{r}) = 0$ outside~~,

These expressions are always applicable provided ~~if we~~ assume that only one particle with the effective charge Qe in the nucleus falls from a state $\phi(r)$ to a state $\phi_0(r)$, the energy of amount $h\nu$ being liberated. q denotes

* In the case of S-S transition, (A_0 and) $\vec{A}$ are zero outside of the nucleus, and moreover $A_0(\vec{r})$ is independent of θ, ϕ and $\vec{A}(\vec{r})$ can be ~~made~~ reduced to zero by a suitable gauge transformation, in the nucleus, so that hereafter we simply write $V(r)$ instead of $eA_0(\vec{r}) + e\vec{A}(\vec{r})$, which is a function of r in the nucleus and zero outside of it.

If we further neglect ~~the~~ the vector potential, the perturbation potential can becomes

~~where the time factor $\exp(-i2\pi\nu t)$ is omitted and~~

~~be written simply as $V(r)$,~~

~~a function V say of r in the nucleus and zero outside of it.~~

where q denotes $\frac{2\pi\nu}{c}$.

5) H. M. Taylor and N. F. Mott, Proc. Roy. Soc. 138, 665, 1932.

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§3. Now ~~the~~ four components of the eigenfunction of the electron in continuous state can be written in the following form:[6)]

$$\psi^{(1)}_{E,j,u}(\vec{r}) = -i\,F_{E,j}(r)\,w^{(1)}_{j,u}(\theta,\phi)$$
$$\psi^{(2)}_{E,j,u}(\vec{r}) = -i\,F_{E,j}(r)\,w^{(2)}_{j,u}(\theta,\phi)$$
$$\psi^{(3)}_{E,j,u}(\vec{r}) = G_{E,j}(r)\,w^{(3)}_{j,u}(\theta,\phi)$$
$$\psi^{4}_{E,j,u}(\vec{r}) = G_{E,j}(r)\,w^{(4)}_{j,u}(\theta,\phi),$$

where

$$\left.\begin{aligned} w^{(1)}_{j,u}(\theta,\phi) &= N_{j,u}\,P^{u}_{-j}\cancel{(\theta,\phi)} \\ w^{(2)}_{j,u}(\theta,\phi) &= N_{j,u}\,P^{u+1}_{-j}\cancel{(\theta,\phi)} \\ w^{(3)}_{j,u}(\theta,\phi) &= N_{j,u}(-j+u)\,P^{u}_{-j-1}\cancel{(\theta,\phi)} \\ w^{(4)}_{j,u}(\theta,\phi) &= N_{j,u}(j+u+1)\,P^{u+1}_{-j-1}\cancel{(\theta,\phi)} \end{aligned}\right\} \text{ or } \left\{\begin{aligned} & N_{j,u}(j+u)\,P^{u}_{j-1}\cancel{(\theta,\phi)} \\ & N_{j,u}(-j+u+1)\,P^{u+1}_{j-1} \\ & N_{j,u}\,P^{u}_{j} \\ & N_{j,u}\,P^{u+1}_{j} \end{aligned}\right.$$

according as $j<0$ or $j>0$, and

$$P^{u}_{j}(\theta,\phi) = \frac{(j-u)!}{2^{j}\,j!}(\sin^2\theta)^{u}\frac{d^{\,j+u}}{(d\cos\theta)^{j+u}}(\cos^2\theta-1)^{j}e^{iu\phi}$$

$$N_{j,u} = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{(|j|-u-1)!\,(|j|+u)!}}$$

~~The~~ As the value of the eigenfunction in the neighborhood of the nucleus is important, $F_{E,j}(r)$ and $G_{E,j}(r)$ are expanded in powers of r and the first terms $f_{E,j}\,r^{\gamma-1}$ and $g_{E,j}\,r^{\gamma-1}$ are taken as the first approximation, where $\gamma = \sqrt{j^2-\alpha^2Z^2}$. For $E > mc^2$

$$f_{E,j} = \mp N\sqrt{2(j+\gamma)(j\varepsilon-\gamma)}$$
$$g_{E,j} = N\sqrt{2(j-\gamma)(j\varepsilon\mp\gamma)}$$

acording as $j \gtrless 0$, where

$$N = \sqrt{\frac{2\pi m}{h^2}}\,(2k)^{\gamma-\frac{1}{2}}\,e^{\frac{\pi b}{2}}\,\frac{|\Gamma(\gamma+ib)|}{\Gamma(2\gamma+1)}, \quad \varepsilon = \frac{E}{mc^2}$$

6) Jaeger and Hulme, loc. cit.

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† In the neighborhood of the nucleus the factor $e^{-\frac{r}{a_0}}$ can be omitted.

$$k = \frac{2\pi mc}{h}\sqrt{\left(\frac{E}{mc^2}\right)^2 - 1}\ , \qquad b = \alpha Z \frac{\frac{E}{mc^2}}{\sqrt{\left(\frac{E}{mc^2}\right)^2 - 1}}\ ,$$

normalization being performed with respect to the energy.

For $E = -E_+ \leq -mc^2$, $\quad f_{-E_+, j} = \overline{g}_{E_+, -j}$
$g_{-E_+, j} = \overline{f}_{E_+, -j}$,

where the signe – upon g ,f means to change Z into –Z.

Next for ~~K-electrons~~ K-state, the components of the eigenfunction are

$$\psi^{(1)}_{K,u} = -iN_0\sqrt{1-\gamma_1}\, r^{\gamma_1 - 1} e^{-\frac{r}{a_0}}\, W^{(1)}_{-1,u}$$
$$\psi^{(2)}_{K,u} = -iN_0\sqrt{1-\gamma_1}\, r^{\gamma_1 - 1} e^{-\frac{r}{a_0}}\, W^{(2)}_{-1,u}$$
$$\psi^{(3)}_{K,u} = N_0\sqrt{1+\gamma_1}\, r^{\gamma_1 - 1} e^{-\frac{r}{a_0}}\, W^{(3)}_{-1,u}$$
$$\psi^{(4)}_{K,u} = N_0\sqrt{1+\gamma_1}\, r^{\gamma_1 - 1} e^{-\frac{r}{a_0}}\, W^{(4)}_{-1,u}\ ,$$

~~where~~

where $\gamma_1 = \sqrt{1-\alpha^2 Z^2}$ $\qquad N_0 = \frac{1}{\sqrt{2\,\Gamma(2\gamma_1+1)}}\left(\frac{2Z}{a_0}\right)^{\gamma_1+\frac{1}{2}}$ $\qquad a_0 = \frac{h^2}{4\pi^2 m e^2 Z}$,

§4. Inserting the above expressions in the equations (1) and (2), we have

$$p(E_+) = \frac{4\pi^2}{h}\sum_{j',u'}\sum_{j,u}\Big\{ f_{E',j'}\, f_{-E_+,j}\, A_{j';u';j,u} + g_{E',j'}\, g_{-E_+,j}\, B_{j';u';j,u}\Big\}^2 \left|\int r^{\sqrt{j^2-\alpha^2Z^2}+\sqrt{j'^2-\alpha^2Z^2}}\, V(r)\, dr\right|^2$$

and

$$P_K = \frac{4\pi^2}{h}\sum_{j'',u''}\sum_{u=0,-1}\Big\{ \underbrace{f_{E'',j''}\, N_0\sqrt{1-\gamma_1}}_{A_{j'';u'';-1,u}} + \underbrace{g_{E'',j''}\, N_0\sqrt{1+\gamma_1}}_{B_{j'';u'';-1,u}}\Big\}^2 R^2$$

$$\left|\int r^{\gamma_1+\sqrt{j''^2-\alpha^2Z^2}}\, V(r)\, dr\right|^2$$

where

$$\cancel{R^{4} = \int r^{2\gamma} V(r)\,dr} \int r^{\sqrt{j^2+\alpha^2 Z^2}+j'}$$

$$A_{j';u';j;u} = \int d\phi \int \sin\theta\, d\theta \,(w^{(1)}_{j,u} w^{*(1)}_{j',u'} + w^{(2)}_{j,u} w^{*(2)}_{j',u'})$$

$$B_{j';u';j;u} = \int d\phi \int \sin\theta\, d\theta \,(w^{(3)}_{j,u} w^{*(3)}_{j',u'} + w^{(4)}_{j,u} w^{*(4)}_{j',u'}),$$

Performing the integration, we have

$$A_{j';u';j;u} = B_{j';u';j;u} = \delta_{j';j}\,\delta_{u';u},$$

from which the selection rules

$$\Delta j = 0 \quad , \quad \Delta u = 0$$

for the transition of the electron are obtained.

We have ~~finally~~ thus

$$p(E_+^n) = \frac{8\pi^2}{h} \sum_j \{ f_{E';j} f_{-E;j} + g_{E';j} g_{-E;j} \}^2 \left| \int r^{2\gamma} V(r)\,dr \right|^2$$

$$\cancel{\cong \frac{4\pi}{h} \sum_j M_j^2 \cdot \left| \int r^{2\gamma} V(r)\,dr \right|^2} \qquad (3)$$

$$P_K = \frac{8\pi^2}{h} \{ f_{E';-1} N_0\sqrt{1-\gamma_1} + g_{E';-1} N_0\sqrt{1+\gamma_1} \}^2 \left| \int r^{2\gamma_1} V(r)\,dr \right|^2$$

$$\cancel{\cong \frac{4\pi}{h} L^2 \cdot \left| \int r^{2\gamma_1} V(r)\,dr \right|^2}, \qquad (4)$$

~~so that the required ratio becomes~~

$$\cancel{\rho = \frac{P_{pair}}{P_K} = }$$

~~Now~~ In the summation ~~in the equation~~ on the right hand side of the equation (3), the terms other than $j = \pm 1$ are very small, so that

$$p(E_+^n) \cong \frac{8\pi^2}{h} \{ (f_{E';-1} f_{-E,-1} + g_{E';-1} g_{-E,-1})^2 + (f_{E';+1} f_{-E,+1} + g_{E';1} g_{-E;1})^2 \} \left| \int r^{2\gamma_1} V(r)\,dr \right|^2$$

~~where~~

and the required ratio becomes

$$\rho = \frac{P_{pair}}{P_K} = \frac{\int_{mc^2}^{h\nu - mc^2} (M_{-1}^2 + M_1^2)\, dE_+}{L^2},$$

~~In the numerator of , M is small except for j = +1, so that~~

approximately, where $M_j^2 = (f_{E';j} f_{-E_+;j} + g_{E';j} g_{-E_+;j})^2$

$$\doteq (f_{E';j}\, \overline{g_{E_+,-j}} + g_{E';j}\, \overline{f_{E_+,-j}})^2$$

$$L^2 = N_0^2 (f_{E'';-1} \sqrt{1-\gamma_1} + g_{E'';-1} \sqrt{1+\gamma_1})^2 .$$

~~N, N₀ can be rewritten in the following manner:~~

The expression for ρ can be further simplifyed and we have finally

$$\rho = \frac{\gamma_1^2}{\pi (\alpha Z)^{2\gamma_1+1} \Gamma(2\gamma_1+1)} \frac{\int_1^{\frac{h\nu}{mc^2}-1} F(\varepsilon_+^*)\, d\varepsilon_+^*}{\Phi(\varepsilon'')(\varepsilon''+\gamma_1)},$$

where ~~$=\sqrt{\ }$, $= \frac{h}{\ }$.~~ If we write further $\varepsilon_+ = \frac{E_+}{mc^2}$

~~we have~~

$$F(\varepsilon_+^*) \equiv \Phi(\varepsilon')\, \Phi(\varepsilon_+)\, (\varepsilon_+ \varepsilon' - \gamma_1^2)$$

$$\Phi(\varepsilon) = e^{\pi b} |\Gamma(\gamma_1 + ib)|^2 \eta^{2\gamma_1 - 1}$$

~~and~~

$$\eta = \sqrt{\varepsilon^2 - 1}$$

$$\eta_+ = \sqrt{\varepsilon_+^2 - 1}.$$

$$\Phi(\varepsilon_+) = e^{-\pi b} |\Gamma(\gamma_1 + ib)|^2 \eta_+^{2\gamma_1 - 1}$$

~~where indicates the energy distribution of emitted positive and negative electrons.~~

Hereafter denotes $\sqrt{1-}$ Z.

$F(\varepsilon_+)$ indicates the energy distribution of the positrons emitted.

§4. General Discussions and Numerical Results

Before entering into numerical calculations in special cases, two points in the above formulation should be noticed.

The first of them is that the solutions of Dirac's equation in Coulomb field of the nucleus were taken as the eigenfunctions of initial and final states of the electron. As in ~~f~~ the neighborhood of the nucleus the field is no more of Coulomb type, we should, in general, multiply each eigenfuntion in Coulomb field by the function of r, which deviates appreciably from 1 in the immediate neighborhood of $r = 0$.

As a first approximation, instead of this, we take for the values of the eigenfunctions in the nucleus those of solutions in Coulomb field at $r = a_N$, a_N being the radius of the nucleus. Then, in the formulae of the preceeding sections, we have only to modify the definitions of R_p, S_p, other points being unaltered.

We have namely

$$R_p = a_N^{2(\gamma-1)} \int V_0 \, d\vec{r} \, ,$$

$$S_p = a_N^{2(\gamma-1)} \int V \, d\vec{r} \, .$$

The second point is that we have assumed for the perturbing potentials the special forms (1) and (2). We can show more generally that the perturbing potential $eA_0 + e\vec{\alpha}\vec{A}$ has a form $V_0(r) + \frac{\vec{\alpha}\vec{r}}{r} V(r)$, whenever the nucleus falls from 同じ 状態 a state of zero spin to another of also zero spin.

These two points will be discussed in detail in a paper of the authors,[9] in which similar calculations in the case of β-disintegration will be performed.

9) This paper will soon appear in this journal.

§4. ~~The~~ Numerical Results and Discussions[8]

~~§3.~~ We want now to obtain the numerial values of $F(\varepsilon_+^*)$ and ρ in the case of RaC'. Inserting the values

$Z = 84 \quad \alpha = 1/137 \quad \Delta W \cong 1.4\cancel{3}\times 10^6\ \mathrm{eV} = 2.8mc^2,$

we have $\alpha Z = 0.6 \quad \gamma_1 = 0.8$, ~~and~~

$$\frac{\gamma_1^2}{\pi(\alpha 2)^{2\gamma_1+1}\,\Gamma(2\gamma_1+1)} = 0.5,$$

$$\Phi(\varepsilon) = \eta^{0.6} e^{0.6\pi\frac{\varepsilon}{\eta}} \left|\Gamma\left(0.8+0.6i\frac{\varepsilon}{\eta}\right)\right|^2$$

$$\cong 4.5 + 1.6\eta$$

$$\Phi(\varepsilon) \cong (4.5+1.6\eta)\, e^{-1.2\pi\frac{\varepsilon}{\eta}}$$

$$\varepsilon'' = 2.8 + 0.8 = 3.6, \quad \eta'' = 3.5$$

$$\Phi(\varepsilon'')(\varepsilon''+\gamma_1) = 44.4,$$

so that

$$\rho = 1.1\times 10^{-2} \int F(\varepsilon_+^*)\, d\varepsilon_+^*$$

$$F(\varepsilon_+^*) = (4.5+1.6\eta')(4.5+1.6\eta_+)(\varepsilon_+\varepsilon' - 0.6)\, e^{-1.2\pi\frac{\varepsilon_+}{\eta_+}}$$

$$\varepsilon' + \varepsilon_+ = 2.8.$$

The distribution function $F(\varepsilon_+^*)$ is given by Fig. ~~1~~ 2. The maximum of the curve lies ~~at~~ near the upper limit of the energy of the positron and it coincides, approximately, with the first maximum of Alichanow's curve.

From Fig. 2 we have

$$\int F(\varepsilon_+^*)\, d\varepsilon_+^* \cong 0.18$$

and

$$\rho \cong 1.1\times 10^{-2}\times 0.18 = 2\times 10^{-3}.$$

On the other hand the number of the electron emitted from the K-levels is known to be about 0.003 per disintegration in this case, so that the number of positrons in question is about

$$0.003\times 2\times 10^{-3} \cong 0.6\times 10^{-5}$$

per disintegration.

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7) E. Fermi, ZS. f. Phys. **88**, 161, 1934.

Now according to Alichanow and Kosodaew, total number of positrons emitted is 0.003 per disintegration. Hence, if the above result of the calculation is correct, the number of positrons emitted by the S-S transition in question amounts only 0.002 of total number of positrons emitted. Thus only very small part of the first ~~maximum~~ hump of the distribution curve of Alichanow can be interpreted as ~~the~~ ~~p~~ due to the process above considered.

In general ~~§5.~~ The ratio ρ increases rapidly and the maximum of ~~the~~ $F(\varepsilon_+^*)$ displaces by and by to $\varepsilon_+^* = \frac{\Delta W}{2}$ as ΔW, i. e. the energy liberated by the nuclear transition, increases.

For example, assuming $Z = 84$ and $h\nu = 4mc^2$, we have for $F(\varepsilon_+^*)$ a curve as shown by Fig. 3 and

$$\rho = 4.0 \times 10^{-2}.$$

In conclusion the authors wish to express his cordial thanks to Prof. G. Beck for valuable discussions on his visit to Osaka.

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§6. We ~~should~~, however, notice that in this case ~~above treated~~ ~~we a~~ the behavior of the electron in the neighborhood of the nucleus is mainly concerned, so that the result of the calculation ~~may~~ happen to be totally wrong.