~~The~~ Theory of Disintegration of the Nucleus by Neutron Impact

On the Collision of the Heavy Particle with the Nucleus. I.

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E21 090 P05

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General Considerations

Introduction

~~The elastic scattering of the heavy particle~~

Among ~~the~~ various ~~cases~~ ~~varieties~~ processes ~~caused by~~ of the collision of ~~process~~ the heavy particle with the nucleus, there are two cases, in which the exact expressions of ~~the~~ cross sections can be ~~calculated~~ obtained ~~exactly~~ ~~by using the~~ by the ~~ordinary~~ exact ~~well known~~ method ~~theory of~~ ~~the~~ quantum mechanics. They are, namely,

i) the elastic scattering

and

ii) the capture of the heavy particle by the nucleus with the emission of the γ-ray. ~~with the the emission of~~

Other processes ~~cases~~ ~~are~~ ~~such as the disintegration~~ ~~more complicated~~, as they can ~~not be reduced to~~ ~~have~~ been considered to be ~~too~~ so complicated ~~that~~ as to ~~make~~ they can not be reduced to one body problem. such as the ~~For example~~ neutron produced, the disintegration of the nucleus ~~with the~~ ~~the produced by the α-particle~~ with the emission of the ~~neutron~~ proton or the α-particle

~~We~~ Now, according to the present theory of ~~the nuclear~~ Heisenberg, Fermi and others, the atomic nucleus ~~is~~ consists of the neutrons and the protons and the radioactive β-disintegration ~~are~~ is ~~considered to be~~ considered to be accompanied by the ~~change~~ transformation of a neutron in the nucleus into a proton. On ~~such~~ this point of view, ~~a neutron collides with~~ it is possible

(1) The meeting of ~~the~~ Osaka branch.

and ~~the~~
that ~~the~~ neutron ~~colliding~~ with the nucleus ~~of the~~
~~atomic number Z changes into the proton,~~
~~which goes away~~ with the atomic number
collide each other
that the neutron colliding with the nucleus
of atomic number Z changes itself into the
proton, ~~the nu~~ the atomic number of the nucleus
being reduced to Z − 1 simultaneously. ~~This~~
~~process is nothing but~~ In other ~~word~~ the disintegration of
the nucleus by the neutron with the emission
of the proton can ~~thus~~ be considered as a sort of
scattering of the heavy particle ~~process~~ caused by the ~~resultant~~ ~~exchange~~
interaction ~~force~~ between the neutron and the proton
in the nucleus. colliding Thus anyone of
~~of exchange type~~ ~~resulting~~ due to ~~from~~ the exchange

Similarly the disintegration of the nucleus
by the proton with the emission of the neutron
can also be considered as ~~the~~ a sort of
scattering caused by the exchange interaction between
the heavy particle and the nucleus.
~~To deal~~ In order to deal with these cases mathematically,
we ought to ~~assume~~ consider ~~first the that~~ the
neutron and the proton ~~are~~ to be two states of the heavy
particle and ~~to need~~ to assume ~~that~~ the following types
of forces between the heavy particle and the
nucleus. They are, namely,
i) ~~the~~ ordinary ~~force~~ short range force
between the heavy particle and the nucleus,
ii) Coulomb force between the proton
and the nucleus.

iii) the force which causes the transition of the
heavy particle from ~~the pro~~ the neutron to the
proton state at the same time with the reduction
of the atomic number of the nucleus by one,
iv) the force which causes the inverse transition
of the former at the same time with the increase
of the ~~atomic number~~ latter by one.
Now ~~as~~ we consider ~~a system of the~~ a heavy particle
and ~~the~~ a nucleus. Let us denote ~~the~~ the state of ~~consisting~~ the system,
in which the former is ~~the~~ neutron and the latter
has the atomic number Z, by 1 and the
state, in which the former is the proton and the
latter has the atomic number Z − 1, by 2.
The eigenfunction of the system thus has two
components, which ~~will~~ may be written as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Further, the interaction energy has the form

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

satisfying the condition $J_{12} = J_{21}^*$
~~with the satisfying the condition~~
in which J_{11} ~~and~~ & J_{22} correspond to the ordinary forces
~~of type~~ i) ~~and~~ & ii), whereas J_{12} and J_{21}
correspond to the exchange forces iv) and iii)
respectively.
In general ~~the~~ J's will depend on the coordinates, the momenta
and the spins of the whole particles constituting
the system. ~~As~~ We want, however, to deal
with the problem as simply and as phenomenologically
as possible, so that ~~they~~ it will be permitted to
assume J's to be function of the ~~relative~~ ~~coordinate~~
distance r 距

between the states 1 and 2 and ~~call~~ denote them by m and M resp.

If ~~the~~ we neglect small ~~differences~~ changes of masses of the heavy particles and the nucleus respectively due to the transition between the heavy particle and the centre of mass of the nucleus.

~~If~~ The kinetic energy of the system has the ordinary form

$$\frac{1}{2m}\vec{p}'^2 + \frac{1}{2M}\vec{P}'^2,$$

where $\vec{p}'$, $\vec{P}'$ are the momentum vectors of the heavy particle and the nucleus respectively, ~~where~~ and ~~the~~ m and M are their masses respectively.

~~If~~ we denote ~~These masses~~ the ~~proper~~ mass of the neutron, the proton, the nucleus of atomic number Z and that of atomic number Z-1 by m_n, m_p, M_Z, M_{Z-1} respectively, the differences $\Delta m = m_n - m_p$, ~~and~~ $\Delta M = M_Z - M_{Z-1}$ ~~are small and the di~~ are small, so that the kinetic energy of the system can be written approximately as

$$\frac{1}{2m'}\vec{p}'^2 + \frac{1}{2M'}\vec{P}'^2,$$

where $\vec{p}'$ and $\vec{P}'$ are the momentum vectors of the heavy particle and the nucleus respectively and

$$m' = \frac{m_n + m_p}{2} \qquad M' = \frac{M_Z + M_{Z-1}}{2},$$

whereas the proper energy of the ~~states~~ system ~~1 and~~ 2 can be written as

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} = \begin{pmatrix} m_n c^2 + M_Z c^2 & 0 \\ 0 & m_p c^2 + M_{Z-1} c^2 \end{pmatrix}.$$

As only the difference of the energy of two states is important, we write instead as

$$\begin{pmatrix} \cancel{D}0 & 0 \\ 0 & -\cancel{D} \end{pmatrix},$$

where $D = (\Delta m + \Delta M)c^2$.

Hence, the Hamiltonian of the system becomes

$$H = \frac{1}{2m'}\vec{p}'^2 + \frac{1}{2M'}\vec{P}'^2 + \begin{pmatrix} J_{11}(r_{12}) & J_{12}(r_{12}) \\ J_{21}(r_{12}) & J_{22}(r_{12}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix},$$

~~If we denote~~ which can be transformed into the form

$$H = \frac{1}{2\cancel{\mu}m}\vec{p}^2 + \frac{1}{2M}\vec{P}^2 + \begin{pmatrix} J_{11}(r) & J_{12}(r) \\ J_{21}(r) & J_{22}(r) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix},$$

where $\vec{p}$ and $\vec{P}$ are the relative momentum and the ~~total~~ momentum of the centre of mass respectively ~~and the~~ m and M are the reduced mass $\left(\frac{m'+M'}{m'M'}\right)$ and the total mass ~~is~~ $m'+M'$ respectively.

~~Thus the~~ In the Schrödinger equation formed from this Hamiltonian, the eigenfunction can be separated at once into the relative coordinates $\vec{r}$ and the coordinates of the centre of mass $\vec{R}$. Thus $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ can be written

in the form $u''(\vec{R})\begin{pmatrix} u_1(\vec{R}) \\ u_2(\vec{R}) \end{pmatrix}$,
and the energy E' of the system can be decomposed into the kinetic energy ~~of~~ "E''" of the centre of mass and the energy E' of the relative motion. The ~~eigenfunctions~~ u'', u_1 and u_2 satisfy the differential equations

$$\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)u''+E''u''=0$$

$$\frac{\hbar^2}{2m}\Delta u_1 + \cancel{\frac{2m}{\hbar^2}} + (E-J_{11})u_1 - J_{12}u_2 = 0$$

$$\frac{\hbar^2}{2m}\Delta u_2 + (E+D-J_{22})u_2 - J_{21}u_1 = 0.$$

As J's are assumed to be functions of r only, u_1 and u_2 can take the forms

$$u_1 = \frac{f_1(r)}{r}Y_{lm}(\theta,\varphi)$$

$$u_2 = \frac{f_2(r)}{r}Y_{lm}(\theta,\varphi),$$

where Y_{lm} is the spherical harmonics, so that f_1 and f_2 should satisfy the simultaneous equations

$$(\ )\quad \begin{cases} \dfrac{d^2 f_1}{dr^2} + \left\{\kappa^2(E-J_{11}) - \dfrac{l(l+1)}{r^2}\right\}f_1 - \kappa^2 J_{12} f_2 = 0 \\ \dfrac{d^2 f_2}{dr^2} + \left\{\kappa^2(E+D-J_{22}) - \dfrac{l(l+1)}{r^2}\right\}f_2 - \kappa^2 J_{21} f_1 = 0, \end{cases}$$

where $\kappa^2 = \dfrac{2m}{\hbar^2}$.
Thus the ~~whole~~ problem of the interaction of the heavy particle with the nucleus ~~are~~ is reduced to ~~the pro~~ that of solving the simultaneous equations ( ) in special conditions.

§2. The Nuclear Disintegration by the Neutron with the Emission of the Proton

Now we want to apply the general ~~form~~ method to the ~~pr~~ calculation of the cross section of the nuclear disintegration by the neutron with the emission of the proton.
~~For simplicity, we take~~ As the simplest form of the interaction potential, we take

$$J_{12} = J_{21} = -J$$

for $r \leq a$, ~~where~~ and

$$J_{12} = J_{21} = 0$$

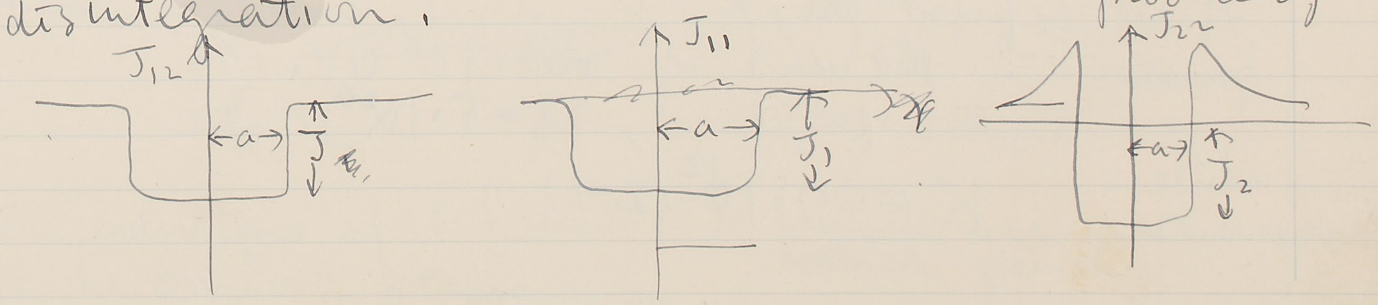
for $r > a$, where J is a ~~real positive~~ positive ~~real~~ quantity and a is the nuclear radius.
Similarly,

$$J_{11} = -J_1 \quad ~~and~~ \quad J_{22} = -J_2$$

for $r < a$, and

$$J_{11} = 0 \quad \text{and} \quad J_{22} = \frac{(Z-1)e^2}{r}$$

for $r > a$, where J_1, J_2 are positive ~~real~~ and $Z-1$ is the atomic number of the nucleus ~~after~~ produced by disintegration.



~~If the int~~ ~~$J \gg J_1, J_2$, The interaction~~

J_1 and J_2 are large or small compared with J according as the interaction is mainly due to the ordinary ~~force~~ or exchange force.

~~The equati~~ simultaneous equations ( ) becomes, now

If we put $\rho = \ldots$ the

$$\left\{\frac{d^2}{d\rho^2} + E_0 + J_{10} - \frac{l(l+1)}{\rho^2}\right\} f_1 + J_0 f_2 = 0$$

for $r<a$, ~~and~~

$$\left\{\frac{d^2}{d\rho^2} + E_0 - D_0 + J_{20} - \frac{l(l+1)}{\rho^2}\right\} f_2 + J_0 f_1 = 0 \qquad (\quad)$$

for $r<a$ and

$$\left\{\frac{d^2}{d\rho^2} + E_0 - \frac{l(l+1)}{\rho^2}\right\} f_1 = 0$$

$$\left\{\frac{d^2}{d\rho^2} + E_0 - D_0 - \frac{K_0}{\rho} - \frac{l(l+1)}{r^2}\right\} f_2 = 0 \qquad (\quad)$$

for $r>a$, where $\rho = \frac{r}{r_0}$

$E_0 = \frac{2m}{\hbar^2}\kappa^2 E$, $\quad D_0 = \frac{2m\kappa^2}{\hbar^2} D$

$J_0 = \frac{2m}{\hbar^2}\kappa^2 J$, $\quad J_{10} = \kappa^2 J_1$ $\quad J_{20} = \kappa^2 J_2$

$K_0 = \kappa^2 (Z-1) e^2$.

If we ~~express~~ take $r_0 = 1.2\cdot 10^{-13}$ cm and express ~~the~~ E, D, J, J_1 and J_2, ~~they the factor~~ in the unit of ~~milli~~ 10^6 eV,

$E_0 = 5\cdot 10^{24} E$, $\quad D_0 = 5\cdot 10^{24} D$ etc,

while $\quad K_0 = 0.7\cdot 10^{2}(Z-1)$

$\frac{2m}{\hbar^2}\left(\frac{\hbar}{mc}\right)^2 \quad \frac{2mc^2 E_0}{2}$

The solutions for

i) ~~$r \geq 0$~~ $r<a$:

~~The solution of the first~~ simultaneous equations If we take ~~you~~

$$h_1 = f_1 + \lambda_1 f_2 \qquad h_2 = f_1 + \lambda_2 f_2$$

instead of f_1 and f_2, ~~in~~ the equations ( ) ~~for r<a~~ ~~the~~ for $r<a$ ~~can be~~ are separated into the form

$$\frac{d^2 h_1}{dr^2} + \left(k_1^2 + \frac{l(l+1)}{r^2}\right) h_1 = 0$$

$$\frac{d^2 h_2}{dr^2} + \left(-k_2^2 - \frac{l(l+1)}{r^2}\right) h_2 = 0, \qquad (\quad)$$

where $\lambda_1 = -\frac{D_0 + J_{10} - J_{20}}{2J_0} + \sqrt{(\quad)^2 + 1} > 0$

$\lambda_2 = -\frac{D_0 + J_{10} - J_{20}}{2J_0} - \sqrt{\quad ,,} < 0$

$k_1^2 = E_0 + J_{10} + \lambda_1 J_0$

$= \sqrt{1+\varepsilon^2}\, J_0 - \frac{D_0 - J_{10} - J_{20}}{2} + E_0$

$k_2^2 = \sqrt{1+\varepsilon^2}\, J_0 + \quad ,, \quad - E_0$

$\varepsilon = \frac{D_0 + J_{10} - J_{20}}{2J_0}$

$\lambda_1 = -\varepsilon \pm \sqrt{1+\varepsilon^2} > 0$

$\lambda_2 = -\varepsilon - \sqrt{1-\varepsilon^2} < 0$

$k_1^2 = \cancel{(\sqrt{1+\varepsilon^2}\, J_0 - \varepsilon)}\lambda_1 J_0 + E_0$

$k_2^2 = -\lambda_2 J_0 + E_0$

The solutions of ( ) and ( ), ~~which are continuous~~ which vanish at the origin, are

heavy quantum }
light quantum

$$\left.\begin{aligned} h_1 &= A_1\sqrt{k_1 r}\; J_{l+\frac{1}{2}}(k_1 r) \\ \text{and}\quad h_2 &= A_2\sqrt{ik_2 r}\; J_{l+\frac{1}{2}}(ik_2 r) \end{aligned}\right\} \qquad (\quad)$$

respectively, ~~thus~~ the functions f_1, f_2 being determined ~~at the same time~~ as their linear combinations at the same time.

ii) $r > a$. The solutions for $r > a$.

The solution of the first ~~equations~~ of ( ) for $r > a$ has the ~~are~~ general form the first of

$$f_1 = \sqrt{kr}\; J_{l+\frac{1}{2}}(kr)$$

$$\{B_1 H^{(1)}_{l+\frac{1}{2}}(kr) + B_2 H^{(2)}_{l+\frac{1}{2}}(kr)\},$$

whereas the solution of the second is ~~the~~ Schrödinger's nothing but ~~equat~~ function of for the Coulomb field.

In order to simplify the calculation, however, we ~~want~~ take for the latter the approximate solution ~~frequently used, which~~ of W. K. B type. ~~Namely,~~ Thus, for $r \gg r_0$, where r_0 is the positive ~~solution of~~

$$F(r) = -(E_0 - D_0) + \frac{K_0}{r} + \frac{l(l+1)}{r^2} = 0,$$

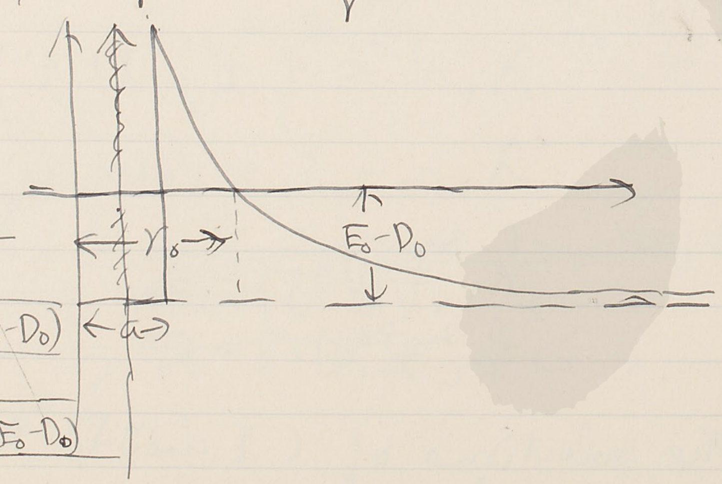
i.e.

$$r_0 = \frac{b}{2}\quad r_0 =$$

$$r_0 = \frac{K_0 + \sqrt{K_0 + 4l(l+}}{2(E_0 - D_0)}$$

$$r_0 = \frac{\cancel{2K_0} + \sqrt{K_0^2 + 4l(l+1)(E_0 - D_0)}}{2l(l+1)}$$

$$r_0 = \frac{K_0 + \sqrt{K_0^2 + 4l(l+1)(E_0 - D_0)}}{2(E_0 - D_0)}$$



we have ~~the~~ an approximate expression ~~a~~
~~we have the following a~~

$$f_2(r) \cong \{F(r)\}^{-\frac{1}{4}}$$

If the energy $E - D$ of the ~~particle~~ is positive ~~larger than D~~ mass defect, but far smaller than the ~~core~~ maximum of the potential $\frac{2e^2}{a}$, i.e. $\frac{K_0}{a} \gg E_0 - D_0 > 0$, $K_0 \gg (E_0 - D_0)a$, i.e. $D < E < D + \frac{2e^2}{a}$ ~~$E_0 > D_0$ or $E_0 - D_0 > 0$~~ the equation

$$F(r) = -\frac{1}{4}(E_0 - D_0) + \frac{K_0}{r} + \frac{l(l+1)}{r^2} = 0$$

has a positive solution

$$r_0 = \frac{K_0 + \sqrt{K_0^2 + 4l(l+1)(E_0 - D_0)}}{2(E_0 - D_0)}$$

which is large compared with a.

In this case, ~~we~~ the ~~an~~ approximation solution of f_2 ~~for $r \gg r_0$ has the form~~ can be written approximately in the form

$$f_2(r) \cong \{F(r)\}^{-\frac{1}{4}}\; e^{\pm i\int_{r_0}^{r}\sqrt{F(r)}}$$

$$\left\{\begin{aligned} &\exp\left(\pm i\int^{r}\sqrt{F(r)}\,dr\right) \\ &C_1 \exp\left(i\int_{r_0}^{r}\sqrt{F(r)}\,dr\right) + C_2\exp\left(-i\int_{r_0}^{r}\sqrt{F(r)}\,dr\right) \end{aligned}\right. \qquad (\quad)$$

for $r \gg r_0$,

~~for $r \gg r_0$,~~ ~~For~~ ~~If~~ ~~For~~ r ~~comparable with~~ nearly equal to r_0, $f_2(r)$ ~~becomes~~ takes the form

$$f_2(r) \cong$$

$$C_1' \cancel{D_1} (r - r_0)^{\frac{1}{2}} e^{i\frac{5}{12}\pi}\, H^{(1)}_{\frac{1}{3}}\left(\frac{2\alpha}{3}\sqrt{F'(r_0)}(r - r_0)^{\frac{3}{2}}\right)$$

$$+ \cancel{D_2}\,C_2'(\cancel{r - r_0})^{\frac{1}{2}} e^{-i\frac{5}{12}\pi}\, H^{(1)}_{\frac{1}{3}}\left(-\frac{2\alpha}{3}(r - r_0)^{\frac{3}{2}}\right)\Big\} \qquad (\quad)$$

for r ~~comparable with~~ ~~nearly equal to r_0~~,

and

$$\{-F(r)\}^{-\frac{1}{4}}\left\{C_1'' \exp\left(\int_{r}^{r_0}\sqrt{-F(r)}\,dr\right) \cancel{-\frac{\pi}{4}}\right. \qquad (\quad)$$

$$\left. + C_2'' \exp\left(-\int_{r}^{r_0}\sqrt{-F(r)}\,dr\right)\right\}$$

for ~~all~~ $r \ll r_0$.

If the energy of the proton is comparable with the maximum potential, ~~the region we have only to consider the region r we need the region~~ ~~$r \ll r_0$ need~~ r_0 ~~become~~ is not small compared with a, so that the region r [illegible] r_0 ~~is excluded.~~ should be excluded.

On the contrary, if ~~$E_0 < D$~~ the neutron energy E is smaller than the mass defect D, the proton can not be emitted from the nucleus, the solution ~~for $r >$~~ a being approximately written in the form

$$f_2 = \{-F(r)\}^{-\frac{1}{4}} \{ C_1 e^{-\int_a^r \sqrt{-F(r)}\,dr} \cancel{+ C_2 e^{-\int_a^r \sqrt{-F(r)}\,dr}} \}$$

for any value of $r > a$, where $\sqrt{-F(r)}$ denotes the positive root of $-F(r) > 0$.

~~§3. Calculation of the case~~

iii) The boundary conditions

$$(6)\begin{cases} \dfrac{d^2u_1}{dr^2} + \left\{\kappa^2(E+V_1) - \dfrac{l(l+1)}{r^2}\right\}u_1 + \kappa^2 J e^{i\theta} u_2 = 0 \\ \dfrac{d^2u_2}{dr^2} + \left\{\kappa^2(E \pm D+V_2) - \dfrac{l(l+1)}{r^2}\right\}u_2 + \kappa^2 J e^{-i\theta} u_1 = 0 \end{cases}$$

$$u_1 = \lambda_{11} f_1 + \lambda_{12} f_2, \qquad u_2 = \lambda_{21} f_1 + \lambda_{22} f_2.$$

$$\lambda_{11} f_1'' + \left[\left\{\kappa^2(E+V_1) - \frac{l(l+1)}{r^2}\right\}\lambda_{11} + \kappa^2 J e^{i\theta}\lambda_{21}\right] f_1$$
$$+ \lambda_{12} f_2'' + \left[\left\{\kappa^2(E+V_1) - \frac{l(l+1)}{r^2}\right\}\lambda_{12} + \kappa^2 J e^{i\theta}\lambda_{22}\right] f_2 = 0$$

$$\lambda_{21} f_1'' + \left[\left\{\kappa^2(E \underbrace{+V_2}_{-D}) - \frac{l(l+1)}{r^2}\right\}\lambda_{21} + \kappa^2 J e^{-i\theta}\lambda_{11}\right] f_1$$
$$+ \lambda_{22} f_2'' + \left[\left\{\kappa^2(E-D+V_2) - \frac{l(l+1)}{r}\right\}\lambda_{22} + \kappa^2 J e^{-i\theta}\lambda_{12}\right] f_2 = 0$$

$$\left.\begin{aligned} & \cancel{\kappa^2} J e^{-i\theta}\lambda_{11}^2 + \{-D+V_2-V_1\}\lambda_{21}\lambda_{11} - \cancel{\kappa^2} J e^{i\theta}\lambda_{21}^2 = 0. \\ & \kappa^2 J e^{-i\theta}\lambda_{12}^2 + \{-D+V_2-V_1\}\lambda_{12}\lambda_{22} - J e^{i\theta}\lambda_{22}^2 = 0. \end{aligned}\right\}$$

$$\lambda_{11} = \frac{V_1 - V_2 + D \pm \sqrt{(V_1-V_2+D)^2+4J^2}}{2Je^{-i\theta}}\lambda_{21}$$

$V_1 - V_2 + D \mp \sqrt{\quad''}$

$$\lambda_{12} = \frac{V_1 - V_2 + D \pm \sqrt{(V_1-V_2+D)^2+4J^2}}{2Je^{-i\theta}}\lambda_{22}$$

$$\lambda_{22} = \frac{-(V_1 - V_2 + D) \mp \sqrt{\quad''}}{2Je^{i\theta}}\lambda_{12}$$

$$u_1 = \frac{1}{2}\{V_1 - V_2 + D + \underbrace{\sqrt{(V_1-V_2+D)^2+4J^2}}_{2K}\} f_1 + Je^{i\theta} f_2$$

$$u_2 = Je^{-i\theta} f_1 - \frac{1}{2}\{V_1 - V_2 + D + \underbrace{\sqrt{(V_1-V_2+D)^2+4J^2}}_{2K}\} f_2$$

$$\cancel{k_1}\ \frac{V_1+V_2-D+\sqrt{(V_1-V_2+D)^2+4J^2}}{V_1+V_2-D-\sqrt{(V_1\pm V_2+D)^2+4J^2}}$$

$$\cancel{f_2''+\left(k_1^2-\frac{l(l+1)}{r^2}\right)f_1=0}$$

$$\frac{d^2u_2}{dr^2}+\left\{\kappa^2(E-D)-\frac{(Z-1)e^2}{r}\right\}u_2=0.$$

$$F=\kappa^2(E-D)-\frac{(Z-1)e^2}{r}-\frac{l(l+1)}{r}\quad -\frac{l(l+1)}{r^2}$$

$$u_2''+F\,u_2=0$$

$$u_2=\{-F\}^{-\frac{1}{4}}\exp\left(\pm i\int_{r_0}^{r}\sqrt{F}\cdot dr\right)$$

$$u_2'=(\pm i)F^{\frac{1}{2}}\sqrt{F}\cdot\exp(\quad{}''\quad)$$

$$u_2''=-F$$

$$u_2=\{-F(r)\}^{-\frac{1}{4}}\exp\left(\pm\int_r^{r_0}\sqrt{-F}\,dr\right)$$

$$u_2'=\{(-F(r))^{\frac{1}{4}}\ \pm(-F(r))^{-\frac{5}{4}}\frac{F'}{4}\}\exp(\pm\ \cdots)$$

$$u_2''=\{(-F(r))^{\frac{3}{4}}\ \pm\cancel{(-F(r))^{-\frac{5}{4}}}\frac{F'}{2}-(-F(r))^{-\frac{9}{4}}\left(\frac{5}{4}\right)\frac{F'^2}{4}$$

$$\pm(-F(r))^{\frac{3}{4}}\cdot\frac{(-F')}{4}\ +(-F(r))^{-\frac{3}{4}}\frac{(-F')}{4}$$

$$\int_{r_0}^{r}\sqrt{-F}\,dr=\kappa^2(E-D)(r-r_0)-(Z-1)e^2(\log r-\log r_0)$$

$$+\ l(l+1)\left(\frac{1}{r}-\frac{1}{r_0}\right)$$

$r\gg r_0$

$$u_2=\cancel{\{-F(r)\}}^{-\frac{1}{4}}\{\kappa^2(E-D)\}^{-\frac{1}{4}}\ (Z-1)e^2.$$

$$\frac{1}{2}\{V_1-V_2+D+\sqrt{(V_1-V_2+D)^2+4J^2}\}f_1''+\left\{\kappa^2(E+V_1)-\frac{l(l+1)}{r^2}\right\}$$

$$\times\frac{1}{2}\{\quad{}''\quad\}f_1+\kappa^2J^2f_1=0\quad\times\{\quad\}f_1$$

$$\frac{1}{2}\{V_1-V_2+D+\sqrt{\quad{}''\quad}\}f_2''+\left\{\kappa^2(E-D+V_2)-\frac{l(l+1)}{r^2}\right\}$$

$$\times\frac{1}{2}\{\quad{}''\quad\}f_2+\kappa^2J^2f_2=0$$

$$u_1=Kf_1+Je^{i\theta}f_2$$

$$u_2=Je^{-i\theta}f_1-Kf_2$$

$$Kf_1''+\left\{\kappa^2(E+V_1)K-\frac{l(l+1)}{r^2}K+\kappa^2J^2\right\}f_1=0$$

$$Kf_2''+\left\{\kappa^2(E-D+V_2)K-\frac{l(l+1)}{r^2}K+\kappa^2J^2\right\}f_2=0$$

$$f_1''+\left\{\kappa^2\left(E+\frac{V_1+V_2-D}{2}+\frac{1}{2}\sqrt{(V_1-V_2+D)^2+4J^2}\right)-\frac{l(l+1)}{r^2}\right\}f_1=0$$

$$\frac{J^2}{K}=\frac{2J^2}{(V_1-V_2+D)+\sqrt{\quad}}=\frac{\sqrt{\quad}-(V_1-V_2+D)}{2}$$

$$f_2''+\left\{\kappa^2\left(E+\frac{V_1+V_2-D}{2}-\frac{1}{2}\sqrt{(V_1-V_2+D)^2+4J^2}\right)-\frac{l(l+1)}{r^2}\right\}f_2=0.$$

$$k_1^2=\kappa^2\left(E+\frac{V_1+V_2-D}{2}+\frac{1}{2}\sqrt{(V_1-V_2+D)^2+4J^2}\right)>\kappa^2(E+V_1)$$

$$k_2^2=\kappa^2\left(-\left(E+\frac{V_1+V_2-D}{2}\right)+\frac{1}{2}\sqrt{\quad{}''\quad}\right)>\kappa^2(D-V_2-E)$$

$$\left.\begin{aligned}f_1&=A_1\sqrt{k_1r}\,J_{l+\frac{1}{2}}(k_1r)\\ f_2&=A_2\sqrt{k_2r}\,J_{l+\frac{1}{2}}(ik_2r)\end{aligned}\right\}$$

$$\sqrt{F} = \sqrt{\kappa^2(E-D) - \frac{(Z-1)e^2}{r} - \frac{l(l+1)}{r^2}}$$

$$= \frac{1}{r}\sqrt{ar^2 - br - c}$$

$$\frac{d}{dr}\left(\frac{\sqrt{ar^2-br-c}}{r}\right)^{\frac{3}{2}} = \frac{3}{2}\frac{\sqrt{(ar^2-br-c)}}{r}\{2ar-b\}$$

$$\frac{3}{2}\sqrt{\quad}^{\frac{3}{2}}\ \frac{1}{r^3}$$

$$= \frac{\sqrt{\quad}}{r}\left\{3ar - \frac{3b}{2} - ar + \frac{b}{r} + \frac{c}{r^2}\right\}$$

$$= \frac{\sqrt{\quad}}{r}\left\{2ar - \frac{b}{2} + \frac{c}{r}\right\}$$

$$\frac{d}{dr}\left(a - \frac{b}{r} - \frac{c}{r^2}\right)^{\frac{3}{2}} = \frac{3}{2}\left(a - \frac{b}{r} - \frac{c}{r^2}\right)^{\frac{1}{2}}\left(\frac{1}{r^2} + \frac{2c}{r^3}\right)$$

$\sqrt{F}$

$$u_2'' + F u_2 = 0$$

$$F = \frac{1}{r^2}(r - r_0)\left(r - \frac{(Z-1)e^2 - \sqrt{(Z-1)^2e^4 + 4l(l+1)(E-D)\kappa^2}}{2\kappa^2(E-D)}\right)$$

$$\cong \frac{1}{r_0^2}(r_0 - \ldots)(r_1 - r_0)$$

$$= \frac{1}{r_0^2}\frac{\sqrt{(Z-1)^2e^4 + 4\kappa^2 l(l+1)(E-D)}}{\kappa^2(E-D)}(r - r_0)$$

$$= \alpha^2(r - r_0).$$

$$u_2'' + \alpha^2 x\, u_2 = 0.$$

$$u_2 = x^{\frac{1}{2}} H^{(1)(2)}_{\frac{1}{3}}\left(\frac{2\alpha}{3}x^{\frac{3}{2}}\right)$$

$$u_2 = (r - r_0)^{\frac{1}{2}}\left\{C_1' H^{(1)}_{\frac{1}{3}}\left(\frac{2\alpha}{3}x^{\frac{3}{2}}\right) + C_2' H^{(2)}_{\frac{1}{3}}(\ ''\ )\right\}$$

$$\underset{x^{\frac{1}{2}}}{''}$$

$x \gtrless 0.$ $\quad x^{\frac{1}{2}} > 0$

$$H^{(1)}_{\frac{1}{3}}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i(x - \frac{5}{12}\pi)}$$

$$H^{(2)}_{\frac{1}{3}}(x) = \sqrt{\frac{2}{\pi}}$$

$$H^{(1)}_{\frac{1}{3}} = J_{\frac{1}{3}}(x) + iY_{\frac{1}{3}}(x)$$

$$H^{(2)}_{\frac{1}{3}} = J_{\frac{1}{3}}(x) - iY_{\frac{1}{3}}(x)$$

$x \ll 1$

$$= \left\{\left(\frac{x}{2}\right)^{\frac{1}{3}} \cdot \frac{2}{\frac{\pi}{3}+3}\, e^{-\frac{i\pi}{3}} + \left(\frac{x}{2}\right)^{-\frac{1}{3}}\right\}\frac{i}{\sin\frac{\pi}{3}}$$

$$= \left\{\left(\frac{x}{2}\right)^{\frac{1}{3}} \cdot e^{\frac{i\pi}{3}} + \left(\frac{x}{2}\right)^{-\frac{1}{3}}\right\}\frac{-i}{\sin\frac{\pi}{3}}$$

$$u_2'' + \alpha^2 x u_2 = 0.$$

$$u_2 = x^\nu \sum_{n=0}^{\infty} a_n x^n$$

$$x^{n+\nu-2}: \quad (\nu+n)(\nu+n-1)a_n + \alpha^2 a_{n-3} = 0.$$

$$a_n = \frac{-\alpha^2}{(\nu+n)(\nu+n-1)} a_{n-3}.$$

$n=0.$ $\quad \nu(\nu-1)=0 \quad \nu = 0$ or $1.$

$$\nu=0: \quad u_2 = \sum_{n=0,1,\dots} \frac{\alpha^{2n} x^{3n} (-1)^n}{3n\cdot(3n-1)\cdot(3n-3)\cdots 3\cdot 2}$$

$$= \sum_{n=0,1,\dots}^{\infty} \frac{(-1)^n \cdot x^{3n} \left(\frac{\alpha}{3}\right)^{2n}}{n! \prod_{m=1}^{n}(m-\frac{1}{3})}$$

$$\nu=1: \quad = x \sum_{n=0,1,\dots}^{\infty} \frac{(-1)^n x^{3n} \left(\frac{\alpha}{3}\right)^{2n}}{n! \prod_{m=1}^{n}(m+\frac{1}{3})}.$$

$|x| \to \infty.$

$$\frac{10 \cdot 2\times10^{-19} \cdot 6\times10^{-27}}{10^{48} \cdot 27\times10^{30}}$$

$$\frac{\pi^2 e^2 h\nu}{m^2 c^3 V a^2} \cdot \chi \frac{1}{4\pi a^2} \qquad -25 \qquad = \frac{12\times10^{-26}}{10^{2}\cdot 70}$$

$$= \quad \qquad = 5\times10^{-28}$$

$$10^{-40} \cdot \frac{v}{V} \cdot \frac{1}{4\pi a^2} = \frac{5\times10^{-28}}{15\times10^{-25}} \quad \frac{5\times20^{20}}{10^5}$$

5×10^{-25}

$$= 5\times10^{2}\times10^{2}$$

$5\times$

If J is the order of 10^{-25} eV.

k_1, k_2 are the order of mag 10 in the unit above

used. ~~so that~~ Thus

In this case

the factor Λ