Letter to the Editor of the Physical Review

Density Matrix in the Theory of the Positron

In the present theory of the electron and the positron, only one sort of them is considered at first, the existence of the other being deduced as necessary consequence of the theory. One can proceed, however, on the reverse way, accepting the existence of both at the beginning and introducing afterwards theoretically possible relations between them. The mathematical formulation of the latter method will be as follows. The conclusions as results in a slight modification the conclusions as a little more satisfactory than the usual former method follows.

The quantized wave functions $\psi_-(x\,k)$ and $\psi_+(x\,k)$ of the electron and the positron satisfy Dirac's equations

$$\left\{\frac{W \pm eV}{c} + \vec{\alpha}\left(\vec{p} \pm \frac{e}{c}\vec{A}\right) + \beta mc\right\}\psi_{\mp} = 0 \qquad (1)$$

respectively, where x denotes position and time and k takes either of the values 1, 2, 3, 4. If we adopt a representation, in which all matrix elements of α's are real and those of β are pure imaginary, the wave functions ψ_+^* and ψ_-^*, which are complex conjugate to ψ_+ and ψ_- respectively, satisfy the same equations (1) for ψ_- and ψ_+ respectively, so that if the relations

$$\psi_- = \psi_+^* \qquad \psi_+ = \psi_-^* \qquad (2)$$

are assumed at an instant for all points, they will remain to hold good forever. These are obviously mathematical expressions of the equivalence of the anti-electron and the positron on the

one hand and that of the anti-positron and the electron on the other.

Now the density matrix, from which physical quantities such as the resultant charge density

$$-\frac{e}{2}\sum_{k}\{\psi_-^*(xk)\psi_-(xk)-\psi_+^*(xk)\psi_+(xk)\} \qquad (3)$$

etc. can be derived, takes the form

$$R(x\,k,\,x'k') = \frac{1}{2}\{\psi_-^*(x'k')\psi_-(xk)-\psi_+^*(x'k')\psi_+(xk)\}$$

or by (2)

$$= \frac{1}{2}\{\psi_-^*(x'k')\psi_-(xk)-\psi_-(x'k')\psi_-^*(xk)\}, \qquad (4)$$

in contrast to the symmetrical density matrix of Heisenberg[1]

$$\frac{1}{2}\{\psi_-^*(x'k')\psi_-(xk)-\psi_-(xk)\psi_-^*(x'k')\}, \qquad (4)$$

the factor $\frac{1}{2}$ in (3) and (4) being needed on account of the above equivalence. In Dirac's approximation, in which particles are moving in a common field, it reduces to

$$\frac{1}{2}\{\sum_{occ}\psi_n^*(x'k')\psi_n(xk)-\sum_{unocc}\psi_n(x'k')\psi_n^*(xk)\}, \qquad (5)$$

in contrast to Dirac's

$$\frac{1}{2}\{\sum_{occ}\psi_n^*(x'k')\psi_n(xk)-\sum_{unocc}\psi_n^*(x'k')\psi_n(xk)\}, \qquad (5)'$$

---

(1) Heisenberg, Zeits. f. Phys. **90**, 209; (1934) **98**, 714, 1936.
(2) Dirac, Proc. Camb. Phil. Soc. **30**, 150, 1934.

where ψ_n, ψ_n^* denote normalized (unquantized) wave functions of stationary states of the electron.

In the case $V=0$ and $\vec{A}=0$, if we denote the wave function of a positive energy state by ψ_n, its complex conjugate ψ_n^* will be the wave functions of a negative energy state. Hence, if all the negative energy states of the electrons are occupied and all the positive energy states are empty, (5) will be zero for any values of x' k and x' k, each term in the first summation being compensated by the same term in the second, whereas (5)' is not zero and becomes infinite for the limit $t-t'=r-r'=0$! Thus, if there are finite number of unoccupied negative energy states and occupied positive energy states, i.e. finite number of electrons and positrons, (5) will remain to be finite everywhere.

If potentials are not zero, we should take gauge invariant density matrix

$$\frac{1}{2}\left\{\psi_-^*(x'k')\psi_-(xk)\, e^{\frac{ei}{\hbar c}\int_{x'}^{x}(\vec{A}d\vec{s}-cVdt)} - \psi_-(x'k')\psi_-^*(xk)\, e^{-\frac{ei}{\hbar c}\int_{x'}^{x}(\vec{A}d\vec{s}-cVdt)}\right\}, \quad (6)$$

which also reduces to zero in Dirac's approximation for empty space without external field.

For weak external field, the density matrix (6) was found to be infinite for $x=x'$, in general, applying ordinary perturbation theory. Thus, drastic change of the formalism seems to be needed in the presence of the field, so which can not easily be foreseen.

that we must still go far befoóre complete theory of the positron is reached. Further discussions and the application to neutrino problems will be made elsewhere.

Hideki Yukawa

Institute of Physics,
Osaka Imperial University,
April 21, 1936.