Note on the Theory of ~~the~~ Multiplicative Showers

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$$f_-(l,y) = \sum_{n=1}^{\infty} f_n(l,y)$$

$$f_n(l,y) = \frac{(2\alpha\log 2)^n}{2} e^{-\alpha l} \int_0^l dl' \int_0^y dy'\, e^{\alpha l'}$$

$$\times \frac{l'^n (l-l')^{n-1} (y-y')^{n-1}}{n!(n-1)!(n-1)!} \int_0^{y'} \frac{e^{-t} t^{l'+n-1}}{\Gamma(l'+n)} dt$$

$$\int_0^y \frac{(y-y')^{n-1}}{(n-1)!} \int_0^{y'} \frac{e^{-t} t^{l'+n-1}}{\Gamma(l'+n)} dt$$

$$= -\frac{(y-y')^n}{n!} \int_0^{y'} \frac{e^{-t} t^{l'+n-1}}{\Gamma(l'+n)} \Bigg|_0^y + \int_0^y \frac{(y-t)^n}{n!} \frac{e^{-t} t^{l'+n-1}}{\Gamma(l'+n)} dt$$

$$f_n(l,y) = \frac{(2\alpha\log 2)^n}{2} e^{-\alpha l} \int_0^l dl' \int_0^y \frac{l'(l-l')^{n-1}}{(n!)^2(n-1)!} e^{\alpha l'}$$

$$\times \frac{(y-t)^n e^{-t} t^{l'+n-1}}{\Gamma(l'+n)} dt$$

$$l, l' \ll 1. \quad \int_0^l dl' \int_0^y \frac{l'(l-l')^{n-1}}{(n!)^2(n-1)!} \frac{}{\Gamma(l'+n)} \qquad C_1 = 0.4228$$

$$\Gamma(l'+n) = (l'+n-1)\cdots(l'+1)\,\Gamma(l'+1)$$

$$l' \ll 1: \Gamma(l'+1) = \sqrt{\frac{\pi l'}{\sin(\pi l')} \frac{1-l'}{1+l'}}\; e^{C_1 l' - C_3 l'^3 \cdots}$$

$l' \ll 1$

$$\Gamma(l'+n) \cong (n-1)!\left(1 + \cancel{\sum_{m=1}^{n-1} m} \; \boxed{\sum_{m=1}^{n-1}\frac{1}{m}}\cdot l'\right)$$

$$\times (1-l')\{1 + C_1 l'\}$$

$$= (n-1)!\left[1 + \left(\sum_{m=1}^{n-1}\frac{1}{m} - 1 + C_1\right)l'\right]$$

$$t^{l'} \cong 1 + l'\log t$$

$$e^{l'\log t} \qquad \frac{t^{l'}}{\Gamma(l'+n)} \cong \left[1 + \underbrace{\left(\log t - \sum_{m=1}^{n-1}\frac{1}{m} + 1 - C_1\right)}_{=K} l'\right]$$

$$f_n(l,y) = \frac{(2\alpha\log 2)^n}{2} e^{-\alpha l}\int_0^y \frac{(y-t)^n e^{-t} t^{n-1}}{(n!)^2 (n-1)!}\,dt$$

$$\times \int_0^l l'(l-l')^{n-1}[1 + K l']\,dl'$$

$$\int_0^l dl' = \cancel{-l'\frac{(l-l')^n}{n}\Big|_0^l + \int_0^l \frac{(l-l')^n}{n}}\,dl'$$

$$+K \quad = \int_0^l l'(l-l')^{n-1}dl' - K l'^2\frac{(l-l')^n}{n}\Big|_0^l$$

$$+ \int_0^l \frac{2K}{n} l'(l-l')^n\,dl'$$

$$= \left(1+\frac{2K}{n}\right) l'\frac{(l-l')^{n+1}}{n+1}\Big|_0^l + \int_0^l\left(1+\frac{2K}{n}\right)\frac{(l-l')^{n+1}}{n+1}\,dl'$$

$$= \left(1+\frac{2K}{n}\right)\frac{\quad}{n+}$$

$$\int_0^l l'(l-l')^{n-1}(1+Kl')\,dl'$$

$$= -\cancel{\frac{l'(l-l')^n}{n}}\Big|_0^l + \int_0^l \frac{(l-l')^n}{n}dl'$$

$$- \cancel{\frac{Kl'^2(l-l')^n}{n}}\Big|_0^l + \int_0^l \frac{2Kl'(l-l')^n}{n}dl'$$

$$= \frac{l^{n+1}}{n(n+1)} + \frac{2K(t)\,l^{n+2}}{n(n+1)(n+2)}$$

$$\cancel{(y-t)}^n e^{-t}$$

$$f_n(l,y) = \frac{(2\alpha\log 2)^n}{2}\frac{e^{-\alpha l}\,l^{n+1}}{(n!)^3(n+1)}\int_0^y (y-t)^n e^{-t} t^{n-1}dt$$

$$\left\{1 + \frac{2K(t)\,l}{n+2}\right\}$$

$$\int_0^y e^{-t}t^{n-1}(y-t)^n dt$$

$$= \int_0 e^{-t}$$

$$= \frac{(2\alpha\log 2)^n}{2}\frac{e^{-\alpha l}\,l^{n+1}}{(n!)^3(n+1)}\int_0^y (y-t)^n e^{-t} t^{n-1}dt\ \cancel{(1+}$$

$$\frac{\partial}{\partial l'}\int_\alpha^{y'} e^{-t}\,t^{l'+n-1}dt =$$

$y \to \infty$.

$$f_n(l,\infty) = \frac{(2\alpha\log 2)^n}{2} e^{-\alpha l} \int_0^l dl' \int_0^y \frac{l'^n (l-l')^{n-1}}{(n!)^2 (n-1)!} \quad e^{\alpha l'} \Gamma(l'+n)$$

$$\times \int_0^\infty \frac{e^{-t} t^{l'+n-1} (y-t)^n}{\cancel{\Gamma(l'+n)}} dt$$

~~n=0.~~

$n=1$:
$$f_1(l,\infty) = \frac{2\alpha\log 2\, e^{-\alpha l}}{\Gamma(l'+1)} \int_0^l l' dl' \int_0^\infty e^{-t} t^{l'} (y-t) dt \quad e^{\alpha l'}$$

$$= \frac{\alpha\log 2\, e^{-\alpha l}}{\Gamma(l'+1)} \int_0^l l' dl' e^{\alpha l'} \{y\Gamma(l'+1) - \Gamma(l'+2)\}$$

$(l'+1)\Gamma(l'+1)$

$$= \alpha\log 2 \cdot e^{-\alpha l} \int_0^l l'(y-l'-1) e^{\alpha l'} dl'$$

$$= \alpha\log 2 \cdot e^{-\alpha l} \left\{ (y-1) \frac{e^{\alpha l}(\alpha l-1)+1}{\alpha^2} \right.$$

$$\left. - \frac{l^2 e^{\alpha l}}{\alpha} + \frac{2}{\alpha}\left(\frac{e^{\alpha l}(\alpha l-1)+1}{\alpha^2} \right) \right\}$$

$$= \log 2 \left\{ (y-1) \frac{(\alpha l-1)+e^{-\alpha l}}{\alpha} - l^2 + \frac{2}{\alpha^2}(\alpha l-1) \right.$$

$-\alpha l + \frac{\alpha^2}{2} l^2 -$

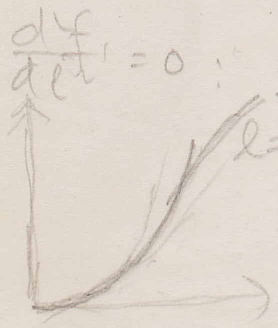
$$\left. + \frac{2}{\alpha^2} e^{-\alpha l} \right\}$$

$$= \log 2 \cdot \left\{ \left(y+\frac{2}{\alpha}-1\right) l - l^2 - \frac{(1-e^{-\alpha l})}{\alpha}\left(y+\frac{2}{\alpha}-1\right) \right\}$$

$$= \log 2 \cdot \left\{ \left(y+\frac{2}{\alpha}-1\right) l - l^2 - \left(l - \frac{\alpha}{2} l^2 + \frac{\alpha^2}{6} l^3\right)\left(y+\frac{2}{\alpha}-1\right) \right\}$$

$2 : 6 : 24$

$\frac{df_1}{dl} = 0$: $2 - 2\alpha l = 0$, $l = \frac{1}{\alpha}$



$$= \log 2 \cdot \left\{ \frac{\alpha}{2}(y-1) l^2 - \frac{\alpha^2}{6}\left(y+\frac{2}{\alpha}-1\right) l^3 \right\}$$

y:大 $=$ $f_1(l,\infty) \cong \left(l^2 - \frac{\alpha}{3} l^3\right) \log 2 \cdot \frac{\alpha}{2} y$

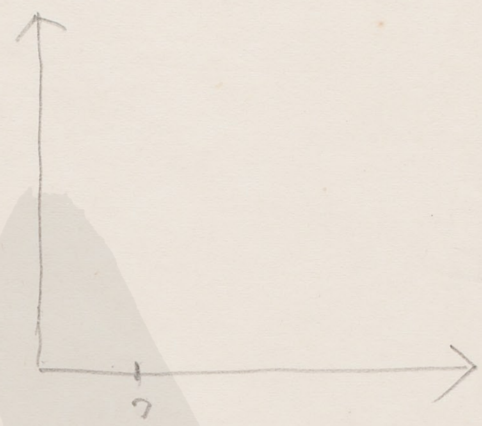
$\alpha \simeq 0.6$

l in unit λ_0 $\left(l = \frac{\lambda}{\lambda_0} \right)$

$\lambda_0 = 0.4$ cm for Pb.
$= 275$ m air.

then $l \cong \frac{3}{\alpha} = 5$ or $\lambda = 2$ cm

で l^3の項がl^2の項と等しくなる.



$l^2 - 0.2 \cdot l^3$

$\lambda^2 - \frac{0.2}{\lambda_0} \lambda^3$

$= \lambda^2 - \frac{\lambda^3}{2}$

$l = \lambda^2 \left(1 - \frac{\lambda}{2}\right)$

$\lambda = 2$

$4 - \frac{\quad}{2}$

$f_1(l, \infty)$

$\lambda = 0.8$ $\quad 0.64 \times 0.6 =$

$$\begin{array}{r} 0.64 \\ 0.6 \\ \hline 0.384 \end{array}$$

$\lambda = 0.6$

$$\begin{array}{r} 0.36 \\ 0.7 \\ \hline 0.252 \end{array}$$

0.9 $\quad$ 0.2

$\frac{\alpha}{\sqrt{2}} \quad \frac{0.6}{\sqrt{2} \cdot 0.4} = \frac{1.5}{\sqrt{2}} \lambda$

$\lambda = 0.4$

$$\begin{array}{r} 0.16 \\ 0.8 \\ \hline 0.128 \end{array}$$

$\lambda = 0.2$

$$\begin{array}{r} 0.04 \\ 0.9 \\ \hline 0.036 \end{array}$$

$$f_n(l,\infty) \cong \frac{(2\alpha \log 2)^n}{2} e^{-\alpha l} y^n \int_0^l \frac{e^{\alpha l'} l'^n (l-l')^{n-1} dl'}{(n!)^2 (n-1)!}$$

$$\int_0^l e^{\alpha l'} l'^n (l-l')^{n-1} dl' = \int_0^l e^{\alpha l'} l'^n \cdot \sum_{m=0}^{n-1} \frac{(n-1)!(-1)^m}{m!(n-1-m)!} l^m l'^{n-1-m}$$

$$= \int_0^l e^{\alpha l'} \sum_{m=0}^{n-1} \frac{(n-1)!(-1)^m l^m}{m!(n-1-m)!} \int_0^l e^{\alpha l'} l'^{2n-1-m} dl' \quad -(-1)^{2n-1-m}$$

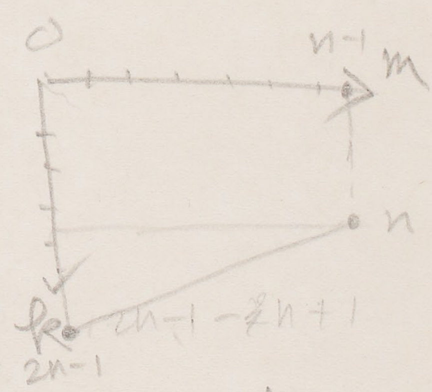
$$= \sum_{m=0}^{n-1} \frac{(n-1)!(-1)^m l^m}{m!(n-1-m)!} \frac{(2n-1-m)!}{\alpha^{2n-1-m+1}} \left\{ e^{\alpha l} \sum \frac{(-\alpha l)^{k}}{(\alpha l)^{2n-1-m}\, (2n-1-m-k)!} \right.$$

$$\left(\because \int_0^l e^{\alpha l'} l'^r dl' = \left\{ \frac{e^{\alpha l}}{\alpha^{r+1}} \sum_{s=0} \frac{r!}{(r-s)!} (\alpha l)^{r-s} - \frac{(-1)^r}{\alpha^{r+1}} r! \right\} \right.$$

$$= \frac{r!}{\alpha^{r+1}} \left\{ (\alpha l)^r e^{\alpha l} \sum_{s=0}^{r} \frac{(-\alpha l)^{-s}}{(r-s)!} - (-1)^r \right\} \Bigg)$$

$$= \sum_{m=0}^{n-1} \frac{(n-1)!(2n-1-m)!}{m!(n-1-m)!} \left\{ \frac{l^{2n-1} e^{\alpha l}}{\alpha} (-1)^m \sum_{k=0}^{2n-1-m} \frac{(-\alpha l)^{-k}}{(2n-1-m-k)!} \right.$$

$$\left. + (-1)^m \frac{l^m}{\alpha^{2n-m}} \right\}$$



$$= \sum \frac{(n-1)!(2n-1-m)!(-1)^m l^{2n-1}}{m!(n-1-m)!}$$

$$= \frac{(n-1)!\, l^{2n-1} e^{\alpha l}}{\alpha} \sum_{m=0}^{n-1} \frac{(-1)^m (2n-1-m)!}{m!(n-1-m)!} \sum_{k=0}^{2n-1-m} \frac{(-\alpha l)^{-k}}{(2n-1-m-k)!}$$

$$+ \sum_{m=0}^{n-1} \frac{(n-1)!(2n-1-m)!}{m!(n-1-m)!} \frac{l^m}{\alpha^{2n-m}}$$

$$f_n(l,\omega) = \frac{(2\alpha\log 2)^n y^n l^{2n-1}}{2\alpha (n!)^2} \left\{ \sum_{m=0}^{n-1} \frac{(-1)^m (2n-1-m)!}{m!(n-1-m)!} \times \sum_{k=0}^{2n-1-m} \frac{(-\alpha l)^{-k}}{(2n-1-m-k)!} + \sum_{m=0}^{n-1} \frac{e^{-\alpha l} l^{m-2n+1}}{\alpha^{2n-m-1}} \frac{(2n-1-m)!}{m!(n-1-m)!} \right\}$$

$n=1$.

$$f_1(l,\omega) = \frac{(2\alpha\log 2) y l}{2\alpha} \left\{ 1 - \frac{1}{\alpha l} + \frac{e^{-\alpha l}}{\alpha l} \right\}$$

$$1 - \frac{1}{\alpha l} + \frac{1}{\alpha l} - 1 + \frac{\alpha l}{2} - \frac{(\alpha l)^2}{6}$$

$$= \frac{\alpha \log 2}{2} y l^2 \left(1 - \frac{\alpha l}{3} \right)$$

inflection point $\quad 2 = \frac{3 \cdot 2}{3} \alpha l \quad l = \frac{1}{\alpha}$

$n=2$.

$$f_2(l,\omega) = \frac{(2\alpha\log 2)^2 y^2 l^3}{2\alpha \cdot 4}$$

$$\times \left\{ 6 \sum_{k=0}^{3} \frac{(-\alpha l)^{-k}}{(3-k)!} - 2 \sum_{k=0}^{2} \frac{(-\alpha l)^{-k}}{(2-k)!} + e^{-\alpha l} \left(\frac{6}{(\alpha l)^3} + \frac{2}{(\alpha l)^2} \right) \right\}$$

$$= \alpha (\log 2)^2 y^2 l^3 \left\{ 3 \left(\frac{1}{6} - \frac{1}{2\alpha l} + \frac{1}{(\alpha l)^2} - \frac{1}{(\alpha l)^3} \right) - \frac{1}{2} + \frac{1}{\alpha l} - \frac{1}{(\alpha l)^2} + \left(1 - \alpha l + \frac{(\alpha l)^2}{2} - \frac{(\alpha l)^3}{6} + \frac{(\alpha l)^4}{24} - \frac{(\alpha l)^5}{120} \right) \left(\frac{3}{(\alpha l)^3} + \frac{1}{(\alpha l)^2} \right) \right\}$$

$=$

$(\alpha l)^{-3}$: $-3+3=0$

$(\alpha l)^{-2}$: $3-1-3+1=0$

$(\alpha l)^{-1}$: $-\frac{3}{2}+1-1+\frac{3}{2}=0$

$(\alpha l)^{0}$: $\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=0$

$(\alpha l)^{1}$: $-\frac{1}{6}+\frac{1}{8}=-\frac{1}{24}$

$(\alpha l)^{2}$: $\frac{1}{24}-\frac{1}{40}=\frac{1}{60}$

$$f_2(l,\omega)=\frac{\alpha(\log 2)^2 g^2 l^4}{24}\left(1-\frac{2\alpha l}{5}\right)$$

inflection point $4\cdot 3=\frac{5\cdot 4\cdot 2}{5}\alpha l$

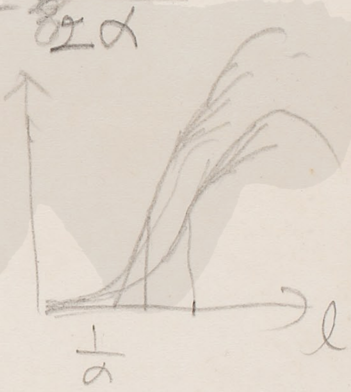
$\alpha l=\frac{3}{2\alpha}$

$\alpha \doteqdot 0.6$

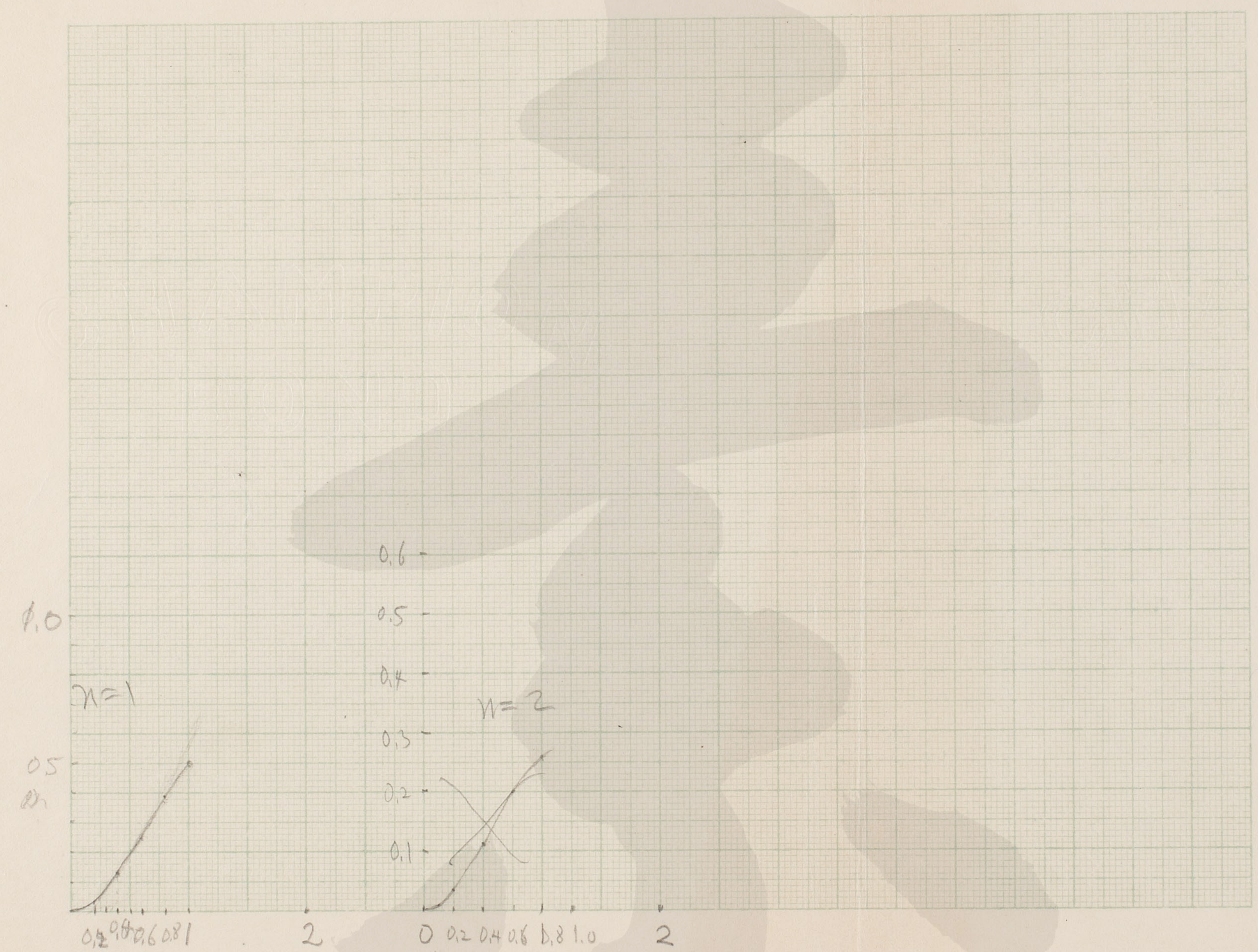
$l=\frac{\lambda}{\lambda_0}$ $\lambda_0=0.4$ cm

$l=\frac{1}{\alpha}\doteqdot\frac{1}{0.6}$ $\lambda=\frac{0.4}{0.6}$ cm

$=0.7$ cm



$l=\frac{3}{2\alpha}\doteqdot\lambda=1$ cm



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