Dirac's Electron in Repulsive Field

Hamiltonian を polar coordinate で書けば

$$-\frac{H}{c} = \rho_1' p_r - i\rho_2' \frac{j\hbar}{r} + \frac{e}{c}A_0 + \rho_3' mc$$

$$j'\hbar = \sigma_r'\left(\sigma_\theta' p_\theta + \frac{\sigma_\varphi' p_\varphi}{\sin\theta}\right)$$

$$j'^2\hbar^2 = \sigma_r'\left(\sigma_\theta' p_\theta + \frac{\sigma_\varphi' p_\varphi}{\sin\theta}\right)\sigma_r'(\quad \cdot \quad)$$

$$= \sigma_r' - (\qquad)^2 - i\hbar\, \sigma_r'\sigma_\theta'(\quad \cdot \cdot \quad)$$

$$= -\sigma_\theta'\left(p_\theta^2 + \frac{(p_\varphi^2 + i\hbar(\sigma_r'\sin\theta + \sigma_\theta'\cos\theta))}{\sin^2\theta}\right)$$

σ_φ', $+\sigma_\varphi'\cancel{\sin\theta}$

~~$-i\hbar\,\sigma_z(xp_y - yp_x) + \frac{\hbar}{2}\sigma_z$ etc~~

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left(-i\hbar\frac{d}{dr}\right) + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{j\hbar}{r} + \frac{e}{c}A_0 + \frac{W}{c} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}mc$$

$$-i\frac{d\psi_2}{dr} - \frac{j}{r}\psi_2 + \left(\frac{eA_0}{\hbar c} + \frac{W}{\hbar c} + \frac{mc}{\hbar}\right)\psi_1 = 0$$

$$-i\frac{d\psi_1}{dr} + \frac{j}{r}\psi_1 + \left(\frac{eA_0}{\hbar c} + \frac{W}{\hbar c} - \frac{mc}{\hbar}\right)\psi_2 = 0$$

$$\psi_1 = e^{-\lambda r}\chi_1, \qquad A_0(r) \underset{r\to\infty}{\to} 0.$$

$$\psi_2 = e^{-\lambda r}\chi_2.$$

$$-i\frac{d\chi_2}{dr} - \frac{j\chi_2}{r} + i\lambda\chi_2 + \left(\frac{eA_0}{\hbar c} + \frac{W}{\hbar c} + \frac{mc}{\hbar}\right)\chi_1 = 0$$

$$-i\frac{d\chi_1}{dr} + \frac{j\chi_1}{r} + i\lambda\chi_1 + (\quad \cdot\cdot \quad - \cdot\cdot\quad)\chi_2 = 0$$

$$\lambda = \sqrt{\left(\frac{mc}{\hbar}\right)^2 - \left(\frac{W}{\hbar c}\right)^2} = \frac{mc}{\hbar}\sqrt{1 - \left(\frac{W}{mc}\right)^2} = \frac{mc}{\hbar}\sqrt{1-\varepsilon^2}$$

$$\chi_1 = \sum_{\nu=0}^{\infty} a_\nu r^{\nu_1 - \nu}$$

$$\chi_2 = \sum_{\nu=0}^{\infty} b_\nu r^{\nu_2 - \nu}$$

a_ν

$$-\frac{H}{c} = \rho_2' p_r \pm \rho_1' \frac{j\hbar}{r} + \frac{e}{c}A_0 + \rho_3' mc$$

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\hbar\frac{d}{dr} \pm \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{j\hbar}{r} + \frac{e}{c}A_0 + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}mc$$

$$\frac{d\chi_2}{dr} + \frac{j\chi_2}{r} = \left\{\frac{mc}{\hbar}(1-\varepsilon) + \alpha V(r)\right\}\chi_1$$

$$\frac{d\chi_1}{dr} \mp \frac{j\chi_1}{r} = \left\{\frac{mc}{\hbar}(1+\varepsilon) + \alpha V(r)\right\}\chi_2$$

$$\chi_1 = \sqrt{1-\varepsilon}\, e^{-\lambda r}(\varphi_1 - \varphi_2) \qquad \varepsilon = \frac{H}{mc^2} \qquad \lambda = \frac{mc}{\hbar}\sqrt{1-\varepsilon^2}$$

$$\chi_2 = \sqrt{1+\varepsilon}\, e^{-\lambda r}(\varphi_1 + \varphi_2)$$

$$\rho = 2\lambda r.$$

$$\frac{d\varphi_1}{d\rho} = \left(1 - \frac{2\alpha\varepsilon\lambda}{\sqrt{1-\varepsilon^2}} \cancel{V(r)} f(\rho)\right)\varphi_1 + \left(-\frac{j}{\rho} - \frac{2\alpha\lambda}{\sqrt{1-\varepsilon^2}} f(\rho)\right)\varphi_2$$

$$\frac{d\varphi_2}{d\rho} = \left(-\frac{j}{\rho} + \frac{2\alpha\lambda}{\sqrt{1-\varepsilon^2}} f(\rho)\right)\varphi_1 + \frac{2\alpha\varepsilon\lambda}{\sqrt{1-\varepsilon^2}} f(\rho)\varphi_2$$

$$f(\rho) = \frac{1}{2\lambda\rho} + f'(\rho) \qquad \varphi_1 = \rho^\gamma \sum_{\nu=0}^{\infty} a_\nu \rho^{-\nu} \qquad \varphi_2 = \rho^\gamma \sum_{\nu=0}^{\infty} b_\nu \rho^{-\nu}$$

$$a_\nu(\gamma-\nu-1)\rho^{\gamma-\nu-1} = b_\nu \rho^{\gamma-\nu} - \frac{2\alpha\varepsilon\lambda}{\sqrt{1-\varepsilon^2}} f(\rho)$$

$$\frac{d\varphi_1}{d\rho} = \{1 - \varepsilon g(\rho)\}\varphi_1 - \left(\frac{j}{\rho} + g(\rho)\right)\varphi_2 \qquad g = \sum_\mu g_\mu \rho^\mu$$

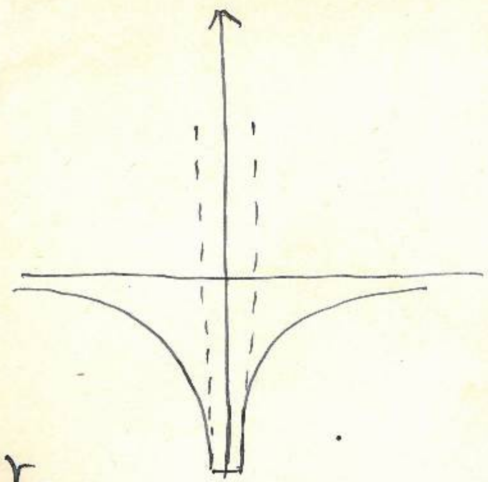
$$\frac{d\varphi_2}{d\rho} = \left(-\frac{j}{\rho} + g(\rho)\right)\varphi_1 + 2\varepsilon g(\rho)\varphi_2 \qquad \cancel{\gamma-\mu-\nu}=\gamma-\nu'$$

$$a_{\nu+1}(\gamma-\nu)\rho^{\gamma-\nu} = \sum_\mu \{\delta_{\mu 0} - \varepsilon g_\mu\} a_{\nu-\mu}\rho^{\gamma-\nu} - (j\,\delta_{\mu,-1} + g_\mu) b_{\nu-\mu}\rho^{\gamma-\nu}$$

$$b_{\nu-1}(\gamma-\nu)\rho^{\gamma-\nu} = \sum(-j\,\delta_{\mu,-1} + g_\mu) a_{\nu-\mu}\rho^{\gamma-\nu}$$

$$+ \sum 2\varepsilon g_\mu b_{\nu-\mu}\rho^{\gamma-\nu}$$

$$\cancel{a_{\nu+1}(\gamma-\nu)} \qquad g = g_{-1}\rho^{-1} + g_{-2}\rho^{-2} + \cdots$$



$$-\frac{H}{c} = \rho_2 p_r - \rho_1 \frac{j\hbar}{r} + \frac{e}{c}A_0 + \rho_3' mc$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hbar \frac{d}{dr} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{j\hbar}{r} + \frac{e}{c}A_0 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc$$

$$\frac{d\chi_2}{dr} + \frac{j\chi_2}{r} = \left\{ \frac{mc}{\hbar}(1-\varepsilon) + \alpha V(r) \right\} \chi_1$$

$$\frac{d\chi_1}{dr} - \frac{j\chi_1}{r} = \left\{ \frac{mc}{\hbar}(1+\varepsilon) - \alpha V(r) \right\} \chi_2$$

$r > r_0$

$$V(r) = \frac{Z}{r} \quad \text{for } r > r_0$$

$$V(r) = V_0 \quad \text{for } r < r_0$$

$$\chi_1 = \sqrt{1-\varepsilon}\; e^{-\lambda r}(\varphi_1 - \varphi_2)$$

$$\chi_2 = \sqrt{1+\varepsilon}\; e^{-\lambda r}(\varphi_1 + \varphi_2) \qquad \varepsilon = \frac{H}{mc^2} \qquad \lambda = \frac{mc}{\hbar}\sqrt{1-\varepsilon^2}$$

$$\rho = 2\lambda r$$

$$\left.\begin{aligned} \frac{d\varphi_1}{d\rho} &= \left(1 - \frac{\alpha\varepsilon}{\sqrt{1-\varepsilon^2}}\frac{Z}{\rho}\right)\varphi_1 + \left(-\frac{j}{\rho} - \frac{\alpha}{\sqrt{1-\varepsilon^2}}\frac{Z}{\rho}\right)\varphi_2 \\ \frac{d\varphi_2}{d\rho} &= \left(-\frac{j}{\rho} + \frac{\alpha}{\sqrt{1-\varepsilon^2}}\frac{Z}{\rho}\right)\varphi_1 + \frac{\alpha\varepsilon}{\sqrt{1-\varepsilon^2}}\frac{Z}{\rho}\varphi_2 \end{aligned}\right\}$$

$$\varphi_1 = \rho^\gamma \sum_{\nu=0}^{\infty} a_\nu \rho^{-\nu} \qquad \varphi_2 = \rho^\gamma \sum_{\nu=0}^{\infty} b_\nu \rho^{-\nu}$$

$\rho^{\gamma-\nu-1}$:

$$a_\nu(\gamma-\nu) = a_{\nu+1} - \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}} a_\nu - \left(j + \frac{\alpha Z}{\sqrt{1-\varepsilon^2}}\right) b_\nu$$

$$b_\nu(\gamma-\nu) = \left(-j + \frac{\alpha Z}{\sqrt{1-\varepsilon^2}}\right) a_\nu + \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}} b_\nu$$

$\nu = \cancel{0}1$: $a_{0} = 0$.

$$\cancel{a_\nu(\gamma-\nu) =}$$

$$\cancel{b_\nu}\; \frac{a_\nu}{b_\nu} = \frac{\gamma - \nu - \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}}}{-j + \frac{\alpha Z}{\sqrt{1-\varepsilon^2}}} = \frac{-\nu}{-j + \frac{\alpha}{\varepsilon}}$$

$$\left\{ (\gamma-\nu) + \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}} + \frac{\left(j + \frac{\alpha Z}{\sqrt{1-\varepsilon^2}}\right)\left(-j + \frac{\alpha Z}{\sqrt{1-\varepsilon^2}}\right)}{\gamma - \nu - \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}}} \right\} a_\nu = a_{\nu+1}.$$

$$\gamma = \cancel{\nu + \frac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}}}$$

$\nu = 0$; $a_0 = 0$ 故, $b_0 \neq 0$; $\gamma = \dfrac{\alpha\varepsilon Z}{\sqrt{1-\varepsilon^2}}$

$$\left.\begin{aligned} \cancel{\nu}\left\{ 2\gamma - \nu + \frac{j^2 - \frac{\gamma^2}{\varepsilon^2}}{\nu} \right\} a_\nu &= a_{\nu+1}. \\ b_\nu &= \frac{j - \frac{\alpha}{\varepsilon}}{\nu} a_\nu \end{aligned}\right\}$$

$r < r_0$: $\quad \chi_1 = e^{-\lambda' r} f_1, \ \chi_2 = e^{-\lambda' r} f_2$

$$\frac{df_2}{dr} + \frac{jf_2}{r} = \left\{ \frac{mc}{\hbar}(1-\varepsilon) + \alpha V_0 \right\} f_1 \qquad \alpha = \frac{e^2}{\hbar c}.$$

$(-\lambda' f_2)$

$$\frac{df_1}{dr} - \lambda' f_1 - \frac{jf_1}{r} = \left\{ \frac{mc}{\hbar}(1+\varepsilon) - \alpha V_0 \right\} f_2$$

$$\lambda'^2 = \left\{ \frac{mc}{\hbar}(1-\varepsilon) + \alpha V_0 \right\} \left\{ \frac{mc}{\hbar}(1+\varepsilon) - \alpha V_0 \right\}$$

$$\lambda' = \frac{mc}{\hbar}\sqrt{1 - \left(\varepsilon - \frac{e^2}{mc^2}V_0\right)^2}$$

$$\chi_1 = \sqrt{1 - \varepsilon' \cancel{+ \frac{e^2}{mc^2}V_0}}, \quad \chi_2 = \sqrt{1 + \varepsilon' \cancel{- \frac{e^2}{mc^2}V_0}}$$

$$\varepsilon' = \varepsilon - \frac{e^2}{mc^2}V_0 \qquad e^{-\lambda' r}(\varphi_1' - \varphi_2') \qquad e^{-\lambda' r}(\varphi_1' + \varphi_2')$$

$$a_\nu'(\gamma' + \nu) = a_{\nu-1}' - j b_\nu' \qquad \varphi_1' = \rho^{\gamma'} \sum a_\nu' \rho^{-\nu}$$

$$b_\nu'(\gamma' + \nu) = -j a_\nu' \qquad \varphi_2' = \rho^{\gamma'} \sum b_\nu' \rho^{-\nu}$$

$$\gamma' a_0' = -j b_0'$$

$$\gamma' b_0' = -j a_\nu'$$

$$\gamma' = \pm j \qquad (+j)$$

$$b_\nu' = \frac{-j}{\gamma' + \nu} a_\nu' \qquad \left(\gamma' + \nu - \frac{j^2}{\gamma' + \nu}\right) a_\nu' = a_{\nu-1}'$$

$$a_\nu' = \left(\frac{(j+\nu)^2 - j^2}{j+\nu}\right)^{-1} a_{\nu-1}' = \left(\frac{(2j+\nu)\nu}{j+\nu}\right)^{-1} a_{\nu-1}'$$

$$= \left(\frac{(2j+\nu)!\,\nu!\,j!}{2j!\,(j+\nu)!}\right)^{-1} a_0'$$

$$= \left(\frac{j \cdot j+1 \cdots (j+\nu)}{\nu!\ 2j\,(2j+1)\cdots(2j+\nu)}\right) a_0'$$

H

$$\frac{W}{c}\psi = \{-\frac{e}{c}A_0 - \rho_1\alpha(p+\frac{e}{c}A) - \alpha_m mc\}\psi$$

$$\frac{W}{c}\bar{\Psi} = \{-\frac{e}{c}A_0 - \alpha^*(-p+\frac{e}{c}A) - \alpha_m^* mc\}\bar{\Psi}$$

$$\frac{W}{c}\rho_2^*\bar{\Psi} = \{-\frac{e}{c}A_0 - \alpha^*(p-\frac{e}{c}A) + \alpha_m^* mc\}\rho_2^*\bar{\Psi}$$

$$\frac{W}{c}\rho_1^*{}_{(\sigma_2)}\bar{\Psi} = \{-\frac{e}{c}A_0 + \alpha^*(p-\frac{e}{c}A) + \alpha_m^* mc\}\rho_1^*{}_{(\sigma_2)}\bar{\Psi}$$

or $$-\frac{W}{c}\rho_1{}_{(\sigma_2)}\bar{\Psi} = \{\frac{e}{c}A_0 \pm \alpha^*(p-\frac{e}{c}A) \pm \alpha_m^* mc\}\rho_1{}_{(\sigma_2)}\bar{\Psi}$$

$$\frac{W}{c}\boxed{\rho_1\sigma_2\bar{\Psi}} = \{-\frac{e}{c}A_0 - \alpha(p-\frac{e}{c}A) - \alpha_m mc\}\rho_1\sigma_2\bar{\Psi}$$

~~current~~ field

$$\begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix}$$

$\rho_1\sigma_2$

$$\frac{W}{c}\psi = \{-\frac{e}{c}A_0 - \rho_1(\sigma, p+\frac{e}{c}A) - \rho_3 mc\}\psi$$

$$\frac{W}{c}\bar{\Psi} = \{-\frac{e}{c}A_0 - \rho_1\{(\sigma_1, -p_x+\frac{e}{c}A_x) - (\sigma_2, -p_y+\frac{e}{c}A_y) + (\sigma_3, -p_z+\frac{e}{c}A_z)\} - \rho_3 mc\}\bar{\Psi}$$

$W = -W'$

$$-\frac{W'}{c}\rho_2\sigma_2\bar{\Psi} = \{-\frac{e}{c}A_0 + \rho_1(\sigma, p-\frac{e}{c}A) + \rho_3 mc\}\bar{\Psi}\,\boxed{\rho_2\sigma_2}$$

$$\rho_2\sigma_2 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$\Psi_{\rho_2\sigma_2} H_{\rho_2\sigma_2}\bar{\Psi} =$$

$$\Psi \alpha \rho_2\sigma_2 \bar{\Psi} = -\bar{\Psi}\alpha\Psi.$$