Pair production by scattering of hard γ-ray

$$i\hbar\frac{\partial\psi}{\partial t} = [\alpha p + \rho m c^2]\psi + W\psi$$

$\frac{p}{c}$: Impuls

$$W = V + H = -\frac{Ze^2}{r} + e(\alpha A)$$

$$\psi_\rho^{(k)} = \frac{1}{\sqrt{V}} u_\rho^{(k)}(p_n) e^{-\frac{i}{\hbar c}(E_n^{(k)} ct - p_n x)}$$

$$\sum_l u_{\rho\sigma}^{(l)} u_\rho^{(l)*} u_\rho^{(k)} = \delta_{\rho\sigma} u_\sigma^{(k)}$$

$$E_n^{1,2} = \sqrt{p_n^2 + m^2c^4}$$

$$\sum_k u_\rho^{(l)*} u_\rho^{(k)} u_\sigma^{*(k)} = u_\sigma^{(l)}$$

$\rho \to \delta_{\rho\sigma}$

$$E_n^{3,4} = -\sqrt{p_n^2 + m^2c^4}$$

$$(u^{(l)}(p)\, u^k(p')) = \sum_\rho u_\rho^{(l)}(p)^* u_\rho^{(k)}(p) = \delta_{lk} \qquad u_\rho^{(l)} u_\sigma^{(k)}$$

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \sum_i q^i A_i \qquad A_i = \sqrt{\frac{4\pi c^2}{V}}\, e_i\, e^{\frac{i(k_i r)}{\hbar c}}$$

$$q^i_{n_i, n_i+1} = \sqrt{\frac{\hbar(n_i+1)}{4\pi\nu_i}}, \qquad q^i_{n_i, n_i-1} = \sqrt{\frac{\hbar n_i}{4\pi\nu_i}}$$

$$H_{\mathfrak{p}'1_i,\mathfrak{p}0} = e\sqrt{\frac{\hbar c^2}{\nu_i V}}\,(u'^*(\alpha e_i)u), \qquad \mathfrak{p}'+k_i=\mathfrak{p}$$

$$H_{\mathfrak{p}'0,\mathfrak{p}1_i} = e\sqrt{\frac{\hbar c^2}{\nu_i V}}\,(u'^*(\alpha e_i)u) \qquad \mathfrak{p}'=\mathfrak{p}+k_i$$

$$V_{\mathfrak{p}'\mathfrak{p}} = -4\pi\hbar^2c^2\,\frac{Ze^2}{V}\,\frac{(u'^*u)}{|\mathfrak{p}'-\mathfrak{p}|^2} \qquad \begin{array}{l} u=u(\mathfrak{p}) \\ u'=u(\mathfrak{p}') \end{array}$$

$$\sum_{k=1}^{4} u_\rho^{k*}(\mathfrak{p})\,u_\sigma^{k}(\mathfrak{p}) = \delta_{\rho\sigma}$$

$$\int \psi^{k*}(\mathfrak{p},x)\,\psi^{l}(\mathfrak{p}x)\,dv = \delta(\mathfrak{p}-\mathfrak{p}')\,\delta_{kl}$$

$$\begin{tikzcd}
\begin{array}{c}[A]\\(\mathfrak{p},k)\\E,h\nu\end{array} \arrow[r, "H"] \arrow[dr, "V"] & \begin{array}{c}[1]\\(\mathfrak{p},0)\\E_1\end{array} \arrow[r, "H"] \arrow[dr, "V"] & \begin{array}{c}[3]\\(\mathfrak{p}_3,k')\\E_3\end{array} \arrow[dr, "V"] & \\
& \begin{array}{c}[2]\\(\mathfrak{p}_2,k)\\E_2\end{array} \arrow[r, "H"] & \begin{array}{c}[4]\\(\mathfrak{p}_4,0)\\E_4\end{array} \arrow[r, "H"] & \begin{array}{c}[E]\\(\mathfrak{p}'k')\\E'h\nu'\end{array}
\end{tikzcd}$$

(3)

[A] $E, \mathbf{p}$ $\quad h\nu$ $\quad \mathbf{k}$ $(k = h\nu)$

[E] $E', \mathbf{p}'$ $\quad h\nu'$ $\quad \mathbf{k}'$ $(k' = h\nu')$

$$E + h\nu = E' + h\nu'$$

$$E^2 = p^2 + (mc^2)^2$$

Zwischenzustand [1] $\mathbf{p}_1 = \mathbf{p} + \mathbf{k}$ $\qquad$ [3] $\mathbf{p}_3 = \mathbf{p}_1 - \mathbf{k}' = \mathbf{p} + \mathbf{k} - \mathbf{k}'$

[4] $\mathbf{p}_4 = \mathbf{p}' + \mathbf{k}'$ $\qquad$ [2] $\mathbf{p}_2 = \mathbf{p}_4 - \mathbf{k} = \mathbf{p}' - \mathbf{k} + \mathbf{k}'$

$$\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k} = \mathbf{q}$$

$$i\hbar \frac{\partial}{\partial t}(\mathbf{p}', \mathbf{k}'|) = \sum_{k} (\mathbf{p}' \mathbf{k}' | V | \mathbf{p}_3 \mathbf{k}')(\mathbf{p}_3 \mathbf{k}'|) + \sum_{k} (\mathbf{p}' \mathbf{k}' | H | \mathbf{p}_4 0)(\mathbf{p}_4 0|)$$

$$\begin{cases} i\hbar \frac{\partial}{\partial t}(\mathbf{p}_3 \mathbf{k}'|) = \sum (\mathbf{p}_3 \mathbf{k}' | H | \mathbf{p}_1 0)(\mathbf{p}_1 0|) \quad \sum (\mathbf{p}_4 0 | V | \mathbf{p}_1 0)(\mathbf{p}_1 0|) \\ i\hbar \frac{\partial}{\partial t}(\mathbf{p}_4 0|) = \sum (\mathbf{p}_4 0 | H | \mathbf{p}_2 \mathbf{k})(\mathbf{p}_2 \mathbf{k}|) + \sum (\mathbf{p}_4 0 | V | \mathbf{p}_1 0)(\mathbf{p}_1 0|) \end{cases}$$

$$i\hbar \frac{\partial}{\partial t}(\mathbf{p}_1 0|) = (\mathbf{p}_1 0 | H | \mathbf{p} \mathbf{k})(\mathbf{p} \mathbf{k}|)$$

$$i\hbar \frac{\partial}{\partial t}(\mathbf{p}_2 \mathbf{k}|) = (\mathbf{p}_2 \mathbf{k} | V | \mathbf{p} \mathbf{k})(\mathbf{p} \mathbf{k}|)$$

$$(p_1 0|H|p\,k) = H_{p_1 0,\,p k}\, e^{\frac{i}{\hbar}(E_1 - E - k)t}$$

$$(p_2 k|V|p\,k) = V_{p_2 p}\, e^{\frac{i}{\hbar}(E_2 - E)t}$$

$$(p_3 k'|H|p_1 0) = H_{p_3 k',\,p_1 0}\, e^{\frac{i}{\hbar}(E_3 + k' - E_1)t}$$

$$(p_4 0|H|p_2 k) = H_{p_2 0,\,p_2 k}\, e^{\frac{i}{\hbar}(E_4 - E_2 - k)t}$$

$$(p_4 0|V|p_1 0) = V_{p_4 p_1}\, e^{\frac{i}{\hbar}(E_4 - E_1)t}$$

$$(p' k'|V|p_3 k') = V_{p' p_3}\, e^{\frac{i}{\hbar}(E' - E_3)t}$$

$$(p' k'|H|p_4 0) = H_{p' k',\,p_4 0}\, e^{\frac{i}{\hbar}(E' + k' - E_4)t},$$

$t = 0 \;:\; (p\,k|) = 1$

$$(p_1 0|)_t = -H_{p_1 0,\,p k}\,\frac{e^{\frac{i}{\hbar}(E_1 - E - k)t} - 1}{E_1 - E - k}$$

$$(p_2 k|)_t = -V_{p_2 p}\,\frac{e^{\frac{i}{\hbar}(E_2 - E)t} - 1}{E_2 - E}$$

$$i\hbar \frac{\partial}{\partial t}(p_3 k' \,|\,) = \sum H_{p_3 k', p_1 0}\, e^{\frac{i}{\hbar}(E_3 + k' - E_1)t}$$

$$\times(-1)\, H_{p_1 0, p_0 k} \frac{e^{\frac{i}{\hbar}(E_1 - E - k)t} - 1}{E_1 - E - k}$$

$$= -\sum \frac{H_{p_3 k', p_1 0}\, H_{p_1 0, p_0 k}}{E_1 - E - k}\left\{ e^{\frac{i}{\hbar}(E_3 + k' - E - k)t} - e^{\frac{i}{\hbar}(E_3 + k' - E_1)t} \right\}$$

etc

$$\rho_E = \frac{\Omega_E\, pE}{h^3 c^3} V, \quad \rho_{E'} = \frac{\Omega_{E'}\, p'E'}{h^3 c^3} V \quad \rho_{k'} = \frac{\Omega_{k'}\, k'^2}{h^3 c^3} V$$

$$w = \frac{2\pi}{\hbar} \rho_{E'} \rho_E\, dE\, \rho_{k'}\, dk' \left| \sum_3 \sum_1 \frac{V_{E3} H_{31} H_{1A}}{(E_1 - E - k)(E_3 - E')} + \sum_2 \sum_4 \frac{H_{E4} H_{42} V_{2A}}{(E_2 - E)(E_4 - E - k)} \right.$$

$$\left. + \sum_4 \sum_1 \frac{H_{E4} V_{41} V_{1A}}{(E_1 - E - k)(E_4 - E - k)} \right|^2$$

$$E + k = E' + k'$$

$$P(p'k', p_0 k) = 2 |\quad|^2 \frac{1 - \cos \frac{E' + k' - k - E}{\hbar} t}{(E' + k' - E - k)^2}$$

$$w = \frac{2}{T} \rho_{E'} \rho_E \rho_k\, dE\, dk' \,|\;|^2 \int \left\{ \frac{1 - \cos(\quad)}{(\quad)^2} \right\} dE'$$

$$V_{E3} = -\frac{4\pi\hbar^2c^2Ze^2}{V}(u'^*(\mathfrak{p}')\,u_3(\mathfrak{p}_3))\frac{1}{|\mathfrak{p}'-\mathfrak{p}_3|^2}$$

$$= -\frac{4\pi\hbar^2c^2Ze^2}{V}\frac{(u'^*(\mathfrak{p}')\,u_3(\mathfrak{p}_3))}{q^2}$$

$$V_{2A} = -\frac{4\pi\hbar^2c^2Ze^2}{V}\frac{(u_2^*(\mathfrak{p}_2)\,u(\mathfrak{p}))}{q^2}$$

$$V_{41} = -\frac{4\pi\hbar^2c^2Ze^2}{V}\frac{(u_4^*(\mathfrak{p}_4)\,u_1^*(\mathfrak{p}_1))}{q^2}$$

$$H_{31} = e\sqrt{\frac{\hbar c^2}{\nu' V}}(u_3^*(\mathfrak{p}_3)\,\alpha e'\,u_1(\mathfrak{p}_1))$$

$$H_{E4} = e\sqrt{\frac{\hbar c^2}{\nu V}}(u_3^*(\mathfrak{p}')\,\alpha e'\,u_4(\mathfrak{p}_4))$$

$$H_{42} = e\sqrt{\frac{\hbar c^2}{\nu V}}(u_4^*(\mathfrak{p}_4)\,\alpha e\,u_2(\mathfrak{p}_2))$$

$$H_{1A} = e\sqrt{\frac{\hbar c^2}{\nu V}}(u_1^*(\mathfrak{p}_1)\,\alpha e\,u(\vec{\mathfrak{p}}))$$

$u_1 = u(\mathfrak{p}_1) = u(\mathfrak{p}+\mathfrak{k})$
$u_2 = u(\mathfrak{p}_2) = u(\mathfrak{p}'+\mathfrak{k}'-\mathfrak{k})$
$u_3 = u(\mathfrak{p}_3) = u(\mathfrak{p}+\mathfrak{k}-\mathfrak{k}')$
$u_4 = u(\mathfrak{p}_4) = u(\mathfrak{p}'+\mathfrak{k}')$

$$d\Phi = \frac{V}{c}\frac{2\pi}{\hbar}\frac{pEp'E'k'^2\,\Omega_E\Omega_{E'}\Omega_{k'}\,V^3}{h^9c^9}\,dE\,dk'$$

$$\times\frac{16\pi^2\hbar^4c^4Z^2e^4}{V^2q^4}\,\frac{e^2\hbar^2c^4}{V^2}\,\frac{1}{VV'}\,S_{AE}\left|\quad\right|^2$$

$$d\Phi = \frac{1}{\frac{c}{V}}\,S_{E,A}\,w = \frac{V}{c}\,S_{E,A}\,w$$

$S_{E,A}$: Endzustand と Anfangszustand の spin の方向に関する和

$$d\Phi = \frac{(\alpha Z)^2e^4}{4\pi^4}\,\frac{\Omega_E\Omega_{E'}\Omega_{k'}\,pEp'E'k'\,dE\,dk'}{kq^4}$$

$$\times S_AS_E\left|\sum_3\sum_1\frac{(u'^*u_3)(u_3^*\alpha e'u_1)(u_1^*\alpha e u)}{(E_3-E')(E_1-E-k)}+\sum_4\sum_2\frac{(u'^*\alpha e'u_4)(u_4^*\alpha e u_2)(u_2^*u)}{(E_4-E-k)(E_2-E)}\right.$$

$$\left.+\sum_4\sum_1\frac{(u'^*\alpha e'u_4)(u_4^*u_1)(u_1^*\alpha e u)}{(E_4-E-k)(E_1-E-k)}\right|^2$$

S: [A] に対しては $k=3,4$　[E] に対しては $k=1,2$ に関する和

Σ: $k=1,2,3,4$ に関しての和

$$\left.\begin{aligned} H u_1^{(k)} &= H_1 u_1^{(k)} = E_1^{(k)} u_1^{(k)} \\ H u_2^{(k)} &= H_2 u_2^{(k)} = E_2^{(k)} u_2^{(k)} \\ H u_3^{(k)} &= H_3 u_3^{(k)} = E_3^{(k)} u_3^{(k)} \\ H u_4^{(k)} &= H_4 u_4^{(k)} = E_4^{(k)} u_4^{(k)} \end{aligned}\right\} \qquad \left.\begin{aligned} H_1 &= \alpha p_1 + \beta mc^2 \\ H_2 &= \alpha p_2 + \beta mc^2 \\ H_3 &= \alpha p_4 + \beta mc^2 \\ H_4 &= \alpha p_4 + \beta mc^2 \end{aligned}\right\}$$

$$\sum_3 \sum_1 \frac{(u'^* u_3)(u_3^* \alpha e' u_1)(u_1^* \alpha e u)}{(E_3 - E')(E_1 - E - k)}$$

$$= \frac{\sum_3 \sum_1 (u'^* \cdot H_3 + E' \cdot u_3)(u_3^* \cdot \alpha e' \cdot H_1 + E + k \cdot u_1)(u_1^* \alpha e u)}{(E_3^2 - E'^2)(E_1^2 - (E+k)^2)}$$

$$= \frac{(u'^* \cdot (H_3 + E') \alpha e' \cdot H_1 + E + k \cdot \alpha e \cdot u)}{(E_3^2 - E'^2)(E_1^2 - (E+k)^2)}$$

$$|\quad|^2 = \left| \frac{\{u'^*(H_3 + E')(\alpha e')(H_1 + E + k)(\alpha e) u\}}{(E_3^2 - E'^2)(E_1^2 - [E+k]^2)} \right.$$

$$\left. + \frac{\{u'^*(\alpha e')(H_4 + E + k)(\alpha e')(H_2 + E) u\}}{(E_4^2 - [E+k]^2)(E_2^2 - E^2)} + \frac{\{u'^*(\alpha e')(H_4 + E + k)(H_1 + E + k)(\alpha e) u\}}{(E_4^2 - [E+k]^2)(E_1^2 - [E+k]^2)} \right|^2$$

$$A^{(1)} = (H_3+E)(\alpha e')(H_1+E+k)(\alpha e)$$

$$A^{(2)} = (\alpha e')(H_4+E+k)(\alpha e)(H_2+E)$$

$$A^{(3)} = (\alpha e')(H_4+E+k)(H_1+E+k)(\alpha e)$$

$$S_A S_E (u'^* A^{(i)} u)^* (u'^* A^{(j)} u) = S_A S_E (u'_\rho(k) A^{(i)*}_{\rho\sigma} u^{*(l)}_\sigma)(u'^{*(k)}_\lambda A^{(j)}_{\lambda\mu} u^{(l)}_\mu)$$

$$= S_A S_E (u^{*(l)}_\sigma A^{(i)\dagger}_{\sigma\rho} u'^{(k)}_\rho)(u'^{(k)*}_\lambda A^{(j)}_{\lambda\mu} u^{(l)}_\mu)$$

$$= \sum_A \sum_E \left(u^* A^{(i)\dagger} \frac{H'+E'}{2E'} u'\right)\left(u'^* A^{(j)} \frac{H+E}{2E} u\right) = \frac{1}{4EE'} \mathrm{Spur}\{A^{(i)\dagger}(H'+E')A^{(j)}(H+E)\}$$

$$\frac{H'+E'}{2E'} u' = \begin{cases} u'^{(k)} & \text{für } k=1,2 \\ 0 & \text{für } k=3,4 \end{cases}$$

$\dagger$: adjungiert

$$\frac{H+E}{2E} u^{(k)} = \begin{cases} 0 & \text{für } k=1,2 \\ u^{(k)} & \text{für } k=3,4 \end{cases}$$

$$E' = E'^{(1)} (= E'^{(2)}) > 0 \qquad E = E^{(3)} (= E^{(4)}) < 0$$

~~$(\theta)^\dagger = -\theta$ $\quad (\gamma^5)^\dagger = -\gamma^5$ $\quad$ $\dagger$: adjungiert~~

~~$\theta = (\alpha_1\alpha_2$~~

$$d\Phi = \frac{(\alpha Z)^2}{4\pi}\left(\frac{e^2}{mc^2}\right)^2 \cdot \Omega_E \Omega_{E'} \Omega_{k'} \frac{p p' k' dE dk'}{k q^4} \frac{1}{4}\{ \cdots \}$$

$$d\Phi = \frac{(\alpha Z)^2 c}{4\pi^4}\frac{d\Omega_E\, d\Omega_{E'}\, d\Omega_{k'}\, p\, p'\, k'\, dE\, dk'}{k q^4}\frac{1}{4}\Big\{\frac{\mathrm{Spur}\{A^{(1)\dagger}(H'+E')A^{(1)}(H+E)\}}{(E_3^2-E'^2)^2(E_1^2-[E+k]^2)^2}$$

$$+\frac{\mathrm{Spur}\{A^{(2)\dagger}(H'+E')A^{(2)}(H+E)\}}{(E_4^2-[E+k]^2)^2(E_2^2-E^2)^2}+\frac{\mathrm{Spur}\{A^{(3)\dagger}(H'+E')A^{(1)}(H+E)\}}{(E_1^2-[E+k]^2)^2(E_4^2-[E+k]^2)^2}$$

$$+\frac{\mathrm{Spur}\{A^{(1)\dagger}(H'+E')A^{(2)}(H+E)\}}{(E_1^2-[E+k]^2)(E_2^2-E^2)(E_2^2-E'^2)(E_4^2-[E+k]^2)}+\cdots\Big\}$$

$$E_1^2-[E+k]^2 = \mathfrak{p}_1^2+m^2c^4-\{E+k\}^2 = [\mathfrak{p}+\mathfrak{k}']^2+m^2c^4-[E+k]^2$$

$$= -2kE+2\mathfrak{p}\mathfrak{k} = -2k(E-p\cos\theta)$$

$$E_2^2-E^2 = (\mathfrak{p}'+\mathfrak{k}'-\mathfrak{k})^2+m^2c^4-E^2 = -2\{(E'k'-\mathfrak{k}\mathfrak{p})-(E'k-\vec{\mathfrak{p}}\mathfrak{k})$$

$$-(\mathfrak{k}\mathfrak{k}'-\mathfrak{k}\mathfrak{k}')\} = (\mathfrak{q}+\mathfrak{p})^2+m^2c^4-E^2 = q^2+2\mathfrak{p}\mathfrak{q}$$

$$E_4^2-[E-k]^2 = -2k'(E'-p'\cos\theta')$$

$$E_3^2-E'^2 = -2\{(Ek-\mathfrak{p}\mathfrak{k})-(E'k'-\mathfrak{p}\mathfrak{k}')-(\mathfrak{k}\mathfrak{k}'-\mathfrak{k}\mathfrak{k}')\}$$

$$= q^2-2\mathfrak{p}'\mathfrak{q}$$

$$E_1^2-[E+k]^2 = \mathfrak{p}_1^2+m^2c^4-[E+k]^2 =$$

$$= -2k(E-\mathfrak{p}\mathfrak{n})$$

$$E_2^2 - E^2 = \cdots = q^2 + 2pq$$

$$E_3^2 - E'^2 = \quad = q^2 - 2p'q$$

$$E_4^2 - [E' + k']^2 = \cdots = -2k'(E' - p'n')$$