Tensor renormalization group approach for lattice gauge theories

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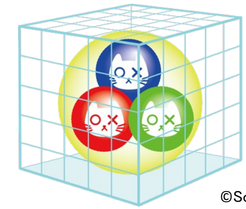
2025.7.31 @ YITP

Recent Developments and Challenges in Tensor Networks: Algorithms, Applications to science, and Rigorous theories

- I’m going to talk about the tensor renormalization group (TRG) approach and its application to the **high-energy physics**, particularly to the **lattice gauge theories (LGT)**
  - Traditionally, the LGT community has been working on the path integral formalism using the MC methods. The sign problem is inevitable, unfortunately
  - The community has been recently paying more attention to some alternative approaches, involving tensor networks and quantum computing
- A goal for the community is to investigate the **quantum chromodynamics (QCD)** using tensor networks, which will be allow us to understand the QCD phase diagram at finite temperature and density
  - Two ways are available in tensor networks:
    - Working on the canonical quantization ⇨ **3D PEPS**, **TTN**, …
    - Working on the path integral quantization ⇨ **4D TNR, TRG, …**
  - Interdisciplinary efforts should be needed

A quick look at the QCD



- The QCD describes **the strong interaction** btw quarks and gluons

Quarks = Grassmann fields

$$S = -\frac{1}{2}\sum_{n\in\Lambda_d}\sum_{\nu=1}^{d}\sum_{f=1}^{N_f}\left[\mathrm{e}^{\mu\delta_{\nu,d}}\bar{\psi}^{(f)}(n)(\mathbb{1}-\gamma_\nu)U_\nu(n)\psi^{(f)}(n+\hat{\nu}) + \mathrm{e}^{-\mu\delta_{\nu,d}}\bar{\psi}^{(f)}(n+\hat{\nu})(\mathbb{1}+\gamma_\nu)U_\nu(n)^\dagger\psi^{(f)}(n)\right]$$

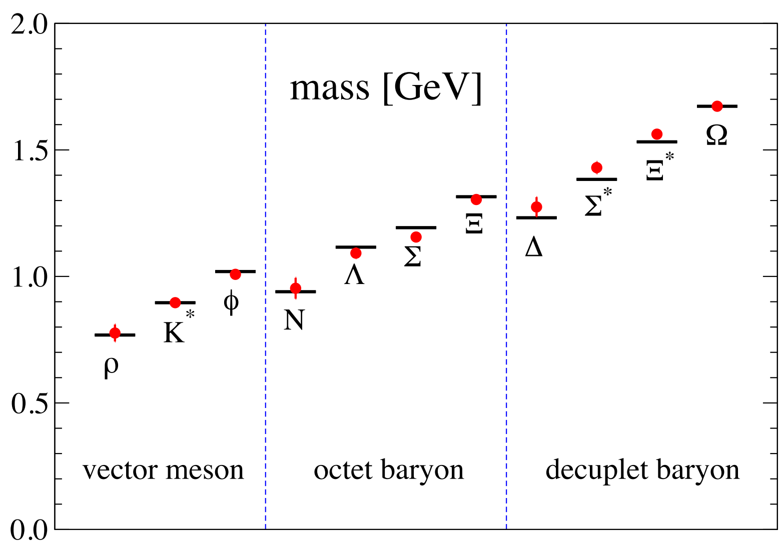
$$+\sum_n\sum_f M^{(f)}\bar{\psi}^{(f)}(n)\psi^{(f)}(n)$$

$$+\frac{\beta}{N_c}\sum_n\sum_{\nu<\rho}\Re\mathrm{Tr}\left[\mathbb{1}-U_\nu(n)U_\rho(n+\hat{\nu})U_\nu(n+\hat{\rho})^\dagger U_\rho(n)^\dagger\right]$$

Gluons = SU(N_c) fields

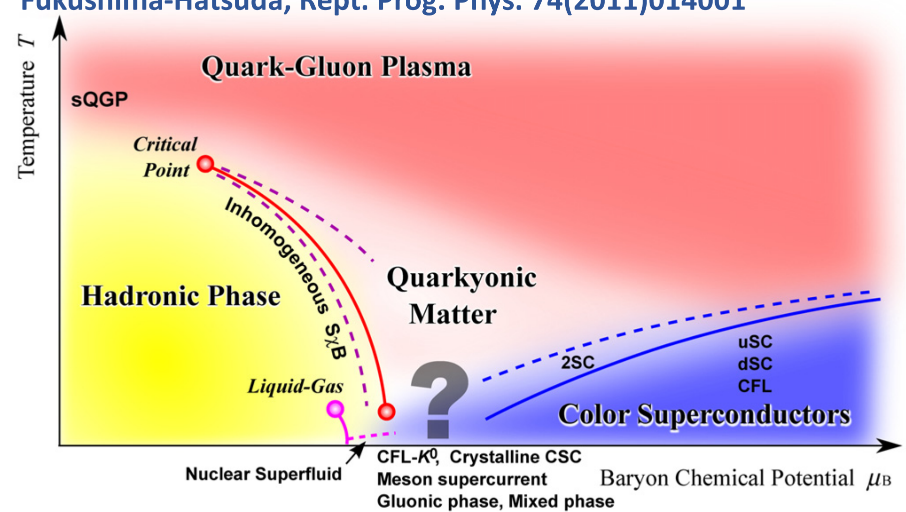
Dimensionality: d = 3+1
\# of flavors: N_f = 2+1(+1)
\# of colors: N_c = 3
Quarks: 4 × N_f × N_c-component spinor
Gluons: N_c × N_c matrix

PACS-CS Collaboration, Lattice2008



μ≠0 ⇨ the sign problem

Fukushima-Hatsuda, Rept. Prog. Phys. 74(2011)014001



"Kogut's sequence" toward QCD

- Can be a good roadmap
  - Gradually try to enlarge symmetry and to increase dimensionality
- Advantage of TN over MC
  - No sign problem
  - Can directly deal with **fermions**
  - Thermodynamic limit can be handled directly when the system has the translational symmetry
- What should be addressed
  - How can we regularize **bosons?**
  - Accuracy vs computational cost

II. LATTICE FIELD THEORY

A. The Kogut sequence: From Ising to QCD

In the early 1970s, QCD appeared to be a strong candidate for a theory of strong interactions involving quarks and gluons. However, the perturbative methods that provided satisfactory ways to handle the electroweak interactions of leptons failed to explain confinement, mass gaps, and chiral symmetry breaking. A nonperturbative definition of QCD was needed. In 1974, Wilson proposed (Wilson, 1974) a lattice formulation of QCD where the SU(3) local symmetry is exact. As this four-dimensional model is fairly difficult to handle numerically, a certain number of research groups started considering simpler lattice models in lower dimensions and then increased symmetry and dimensionality. This led to a sequence of models, sometimes called the "Kogut ladder," that appears in the reviews of Kogut (1979, 1983) and was later addressed with small modifications by Polyakov (1987) and Itzykson and Drouffe (1991).

The sequence is approximately the following:
(1) $D = 2$ Ising model
(2) $D = 3$ Ising model and its gauge dual
(3) $D = 2$ O(2) spin and Abelian Higgs models
(4) $D = 2$ fermions and the Schwinger model
(5) $D = 3$ and 4U(1) gauge theory
(6) $D = 3$ and 4 non-Abelian gauge theories
(7) $D = 4$ lattice fermions
(8) $D = 4$ QCD

Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94 (2022)025005

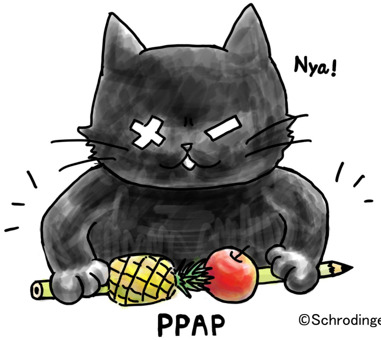
What is TRG?

Levin-Nave, PRL99 (2007) 120601
Cf. Talk by Naoki Kawashima on Jul. 30

- TRG is a variant of RSRG to approximately carry out tensor contractions
  - What do we have to do within the TRG approach?

① **Derive a TN representation for the path integral we want to solve**

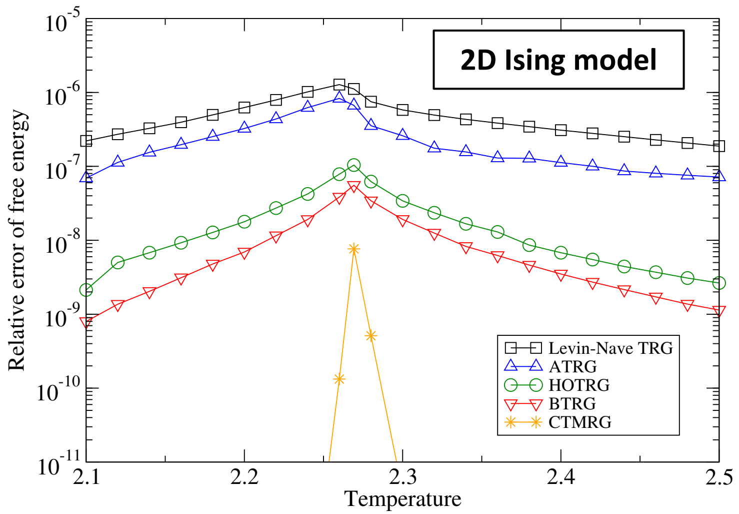
② **Carry out the tensor contractions employing a certain TRG algorithm**



Pros and Cons

Cf. Talk by Naoki Kawashima on Jul. 30

- Several advantages:
  - **No sign problem**
  - **Thermodynamic limit**
  - **Grassmann variables**
  - **Path integral, Transfer matrix**
  - **Higher-dimensional extension**



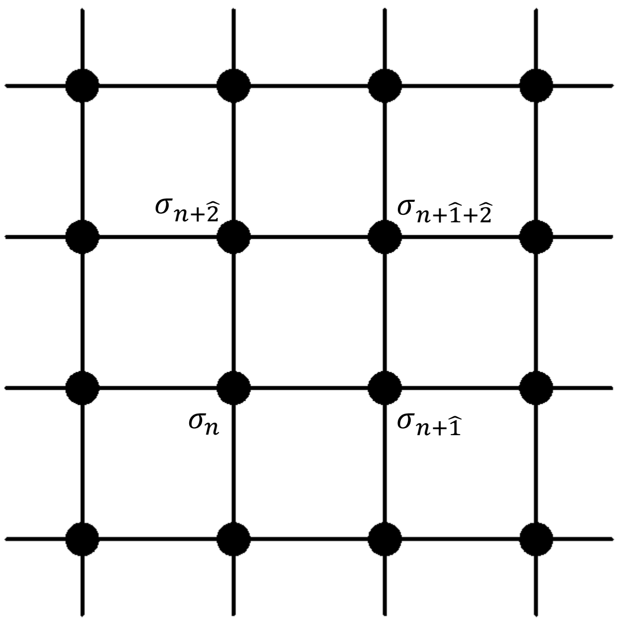
- For the 2D TNs, there are much more efficient contraction algorithms are available
  - Corner Transfer Matrix RG (CTMRG)
    Baxter, J. Math. Phys. 9 (1968) 650
    Nishino, J. Phys. Soc. Jpn, 64 (1995) 3598-3601
    Nishino-Okunishi, J. Phys. Soc. Jpn, 65 (1996) 891-894
  - infinite Time-Evolving Block Decimation
    G. Vidal, PRL91 (2003) 147902, PRL98 (2007) 070201
- Naïve TRG algorithms suffer from the **short-range correlation**

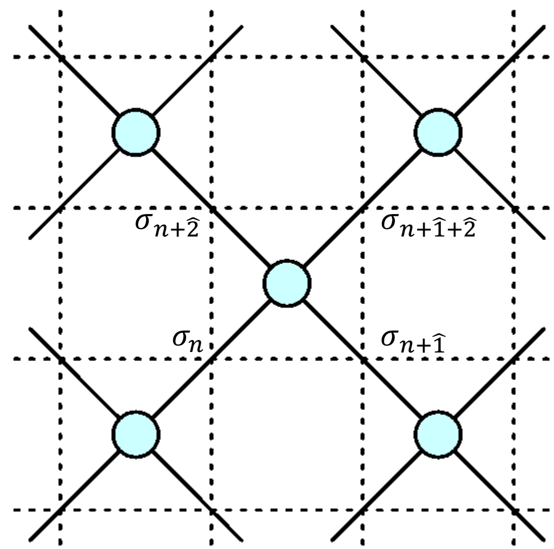
Cf. Evenbly-Vidal, PRL115 (2015) 180405, Yang+, PRL118 (2017) 110504, Hauru+, PRB97 (2018) 045111, Harada, PRB97 (2018) 045125, Homma-Okubo-Kawashima, PRR6 (2024) 043102, …

TN formulation of spin systems 1/2

- In the case of the nearest-neighbor interacting model on a square lattice
  - $Z = \Sigma_{\{\sigma=\pm 1\}} \Pi_{n,\mu} \exp\left[K \sigma_n \sigma_{n+\hat{\mu}}\right]$
  - $Z = \Sigma_{\{\sigma=\pm 1\}} \prod T_{\sigma_n \sigma_{n+\hat{1}} \sigma_{n+\hat{2}} \sigma_{n+\hat{1}+\hat{2}}}$
    - $T_{\sigma_n \sigma_{n+\hat{1}} \sigma_{n+\hat{2}} \sigma_{n+\hat{1}+\hat{2}}} \coloneqq \exp[K(\sigma_n \sigma_{n+\hat{1}} + \sigma_{n+\hat{1}} \sigma_{n+\hat{1}+\hat{2}} + \sigma_{n+\hat{1}+\hat{2}} \sigma_{n+\hat{2}} + \sigma_{n+\hat{2}} \sigma_n)]$

Real Space





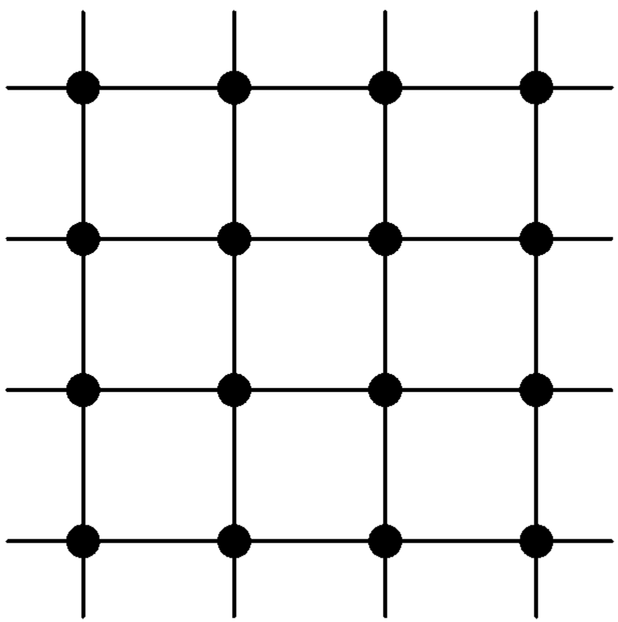
TN rep. of Z

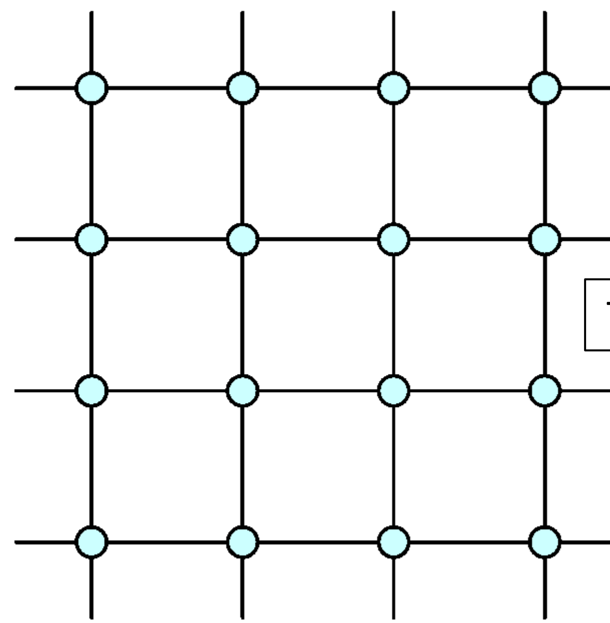
Cf. Nakayama-Schneider, PRD110(2024)094501

TN formulation of spin systems 2/2

- Another way to derive a TN rep. by integrating out original spin variables
  - $M_{\sigma_n \sigma_{n+\hat{\mu}}} := \exp\left[K \sigma_n \sigma_{n+\hat{\mu}}\right] = \sum_\alpha U_{\sigma_n \alpha}\, s_\alpha V^*_{\sigma_{n+\hat{\mu}} \alpha}$
  - $(T_n)_{xyx'y'} := \sqrt{s_x s_y s_{x'} s_{y'}} \sum_{\sigma_n} U_{\sigma_n x} U_{\sigma_n y} V^*_{\sigma_n x'} V^*_{\sigma_n y'}$
    - $Z = \mathrm{tTr}[\prod_n T_n]$

Real Space





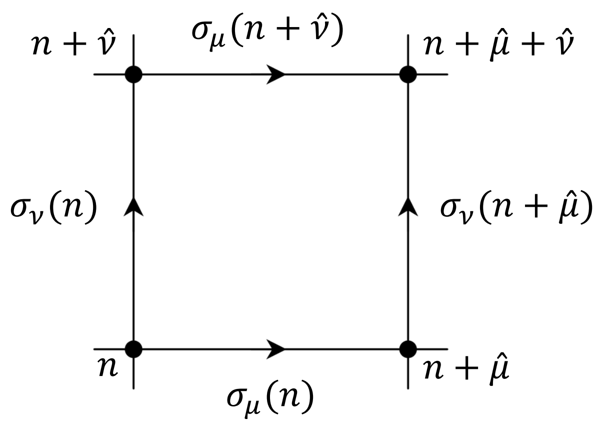
TN rep. of Z

Cf. Song-Zhang, PRB105(2022)134516

TN formulation of gauge fields 1/2

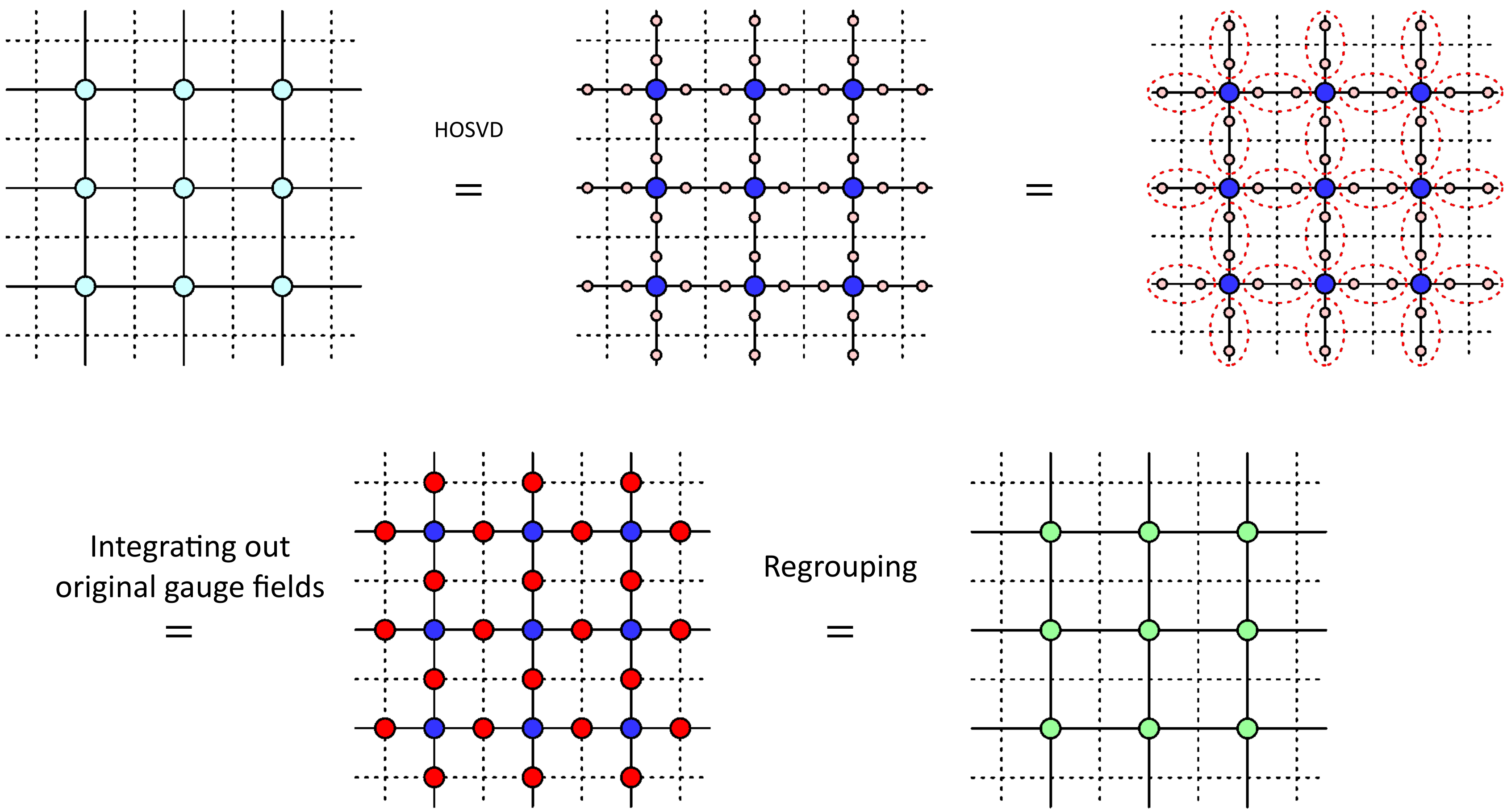
- Let us consider the Ising gauge theory on a square lattice
  - $Z = \sum_{\{\sigma=\pm1\}} \Pi_{n,\mu>\nu} \exp\left[K\sigma_\mu(n)\sigma_\nu(n+\hat{\mu})\sigma_\mu(n+\hat{\nu})\sigma_\nu(n)\right]$
  - $Z = \sum_{\{\sigma=\pm1\}} \prod T_{\sigma_\nu(n+\hat{\mu})\sigma_\mu(n+\hat{\nu})\sigma_\nu(n)\sigma_\mu(n)}$
    - $T_{\sigma_\nu(n+\hat{\mu})\sigma_\mu(n+\hat{\nu})\sigma_\nu(n)\sigma_\mu(n)} \coloneqq \exp\left[K\sigma_\mu(n)\sigma_\nu(n+\hat{\mu})\sigma_\mu(n+\hat{\nu})\sigma_\nu(n)\right]$

Cf. Kuramashi-Yoshimura, JHEP04(2020)089



TN formulation of gauge fields 2/2

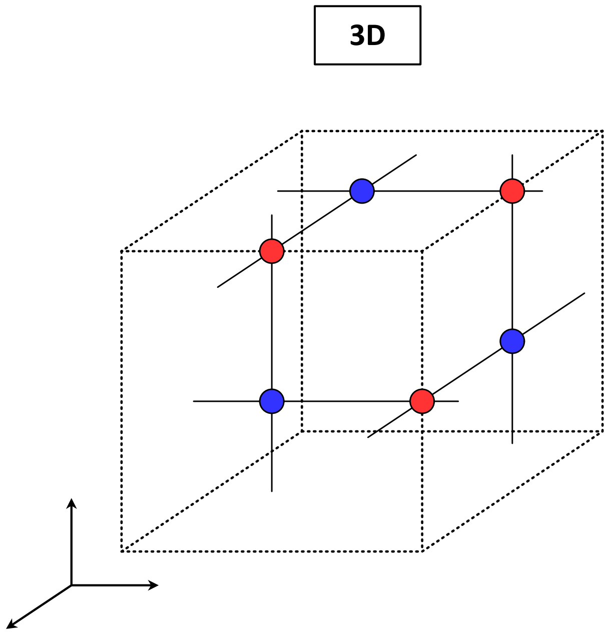
- One can move on to another TN rep. by integrating out original Ising gauge fields

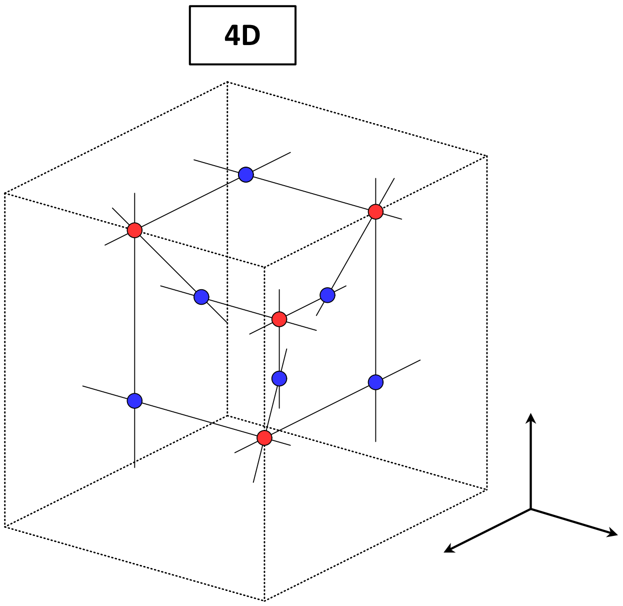


Cf. Liu+, PRD88(2013)056005, SA-Kuramashi, JHEP10(2023)077

TN formulation of higher-dimensional LGTs

- Fundamental tensors are available by integrating out the original gauge fields
  - **Asymmetric formulation** Liu+, PRD88(2013)056005
- Another TN formulation without integrating out the original fields
  - Plaquette gauge action ⇨ Can be regarded as local four-leg tensors





Cf. Meurice, PRD102(2020)014506

Cf. Tagliacozzo-Vidal, PRB83(2011)115127, Canals-Chepiga-Tagliacozzo, arXiv:2412.16961

How about non-Abelian gauge theories?

- **Character expansion** Liu+, PRD88(2013)056005, Hirasawa+, JHEP12(2021)011
  - Fourier transformation on a group manifold
  - **Reduced TN formulation** Yosprakob, PTEP2024(2024)073B05, Yosprakob-Okunishi, PTEP2025(2025)033B06
    - Removes redundancy from the initial TN rep.
- **Sampling method**
  - Tensor indices label the gauge field configurations
    - Random sampling Fukuma-Kadoh-Matsumoto, PTEP2021(2021)123B03
    - Trial action method Kuwahara-Tsuchiya, PTEP2022(2022)093B02
- **Numerical quadrature**
  - Replaces integrals by discretized summations Luo-Kuramashi, PRD107(2023)094509

Grassmann TN formulation for fermions

Gu-Verstraete-Wen, arXiv:1004.2563

- Path integrals involving the fermionic DoFs are defined as the Grassmann integrals

Gu, PRB88(2013)115139
Shimizu-Kuramashi, PRD90(2014)014508
Takeda-Yoshimura, PTEP2015(2015)043B01
Meurice, PoS LATTICE2018(2018)231
Bao's thesis, PhD, Uwaterloo

  - $\mathcal{T}_{\eta_1\eta_2\eta_3\cdots} = \sum_{i_1,i_2,i_3,\cdots} T^{i_1 i_2 i_3\cdots}\eta_1^{i_1}\eta_2^{i_2}\eta_3^{i_3}\cdots$

  - **A straightforward derivation of the GrassmannTN rep. for the ferminonic path integral is available by introducing auxiliary Grassmann fields to decompose the hopping structure**

SA-Kadoh, JHEP10(2021)188

    - $\mathrm{e}^{A\bar{\psi}_n\psi_{n+\mu}} = \left(\int\int \mathrm{d}\bar{\eta}_n \mathrm{d}\eta_n \mathrm{e}^{-\bar{\eta}_n\eta_n}\right)\exp\left[-\sqrt{A}\bar{\boldsymbol{\psi}}_{\boldsymbol{n}}\eta_n + \sqrt{A}\bar{\eta}_n\boldsymbol{\psi}_{\boldsymbol{n+\mu}}\right]$

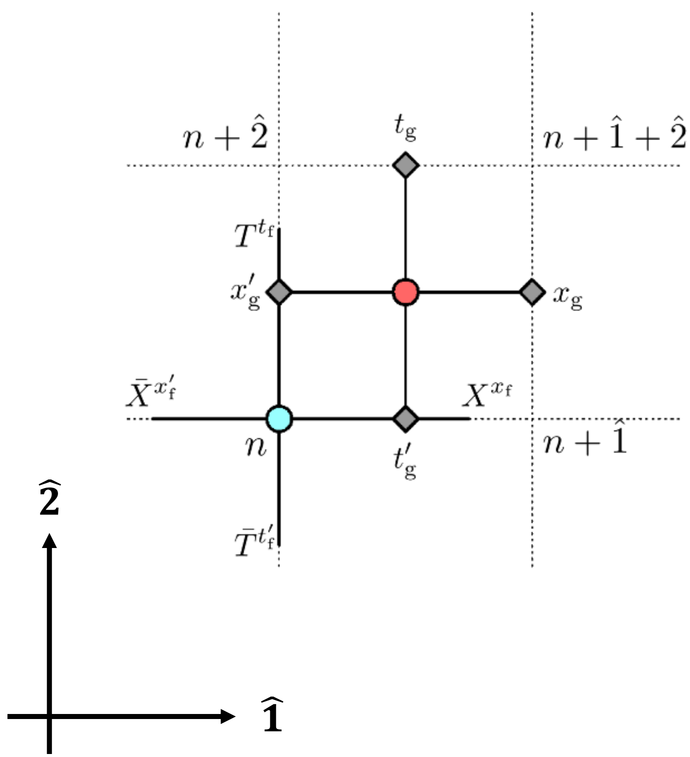
    - Explicit correspondence btw TN and GrassmannTN

	Normal TN	**Grassmann TN**
Fundamental tensor	$(\boldsymbol{T}_{\boldsymbol{n}})_{\boldsymbol{xyx'y'}}$	$(\boldsymbol{\mathcal{T}}_{\boldsymbol{n}})_{\boldsymbol{XY\bar{X}\bar{Y}}}$
Contraction	$\sum_{\alpha}\cdots$	$\int_{\bar{\Theta},\Theta}\cdots$
TN rep.	$\mathbf{tTr}[\prod_n \boldsymbol{T}_{\boldsymbol{n}}]$	$\mathbf{gTr}[\prod_n \boldsymbol{\mathcal{T}}_{\boldsymbol{n}}]$

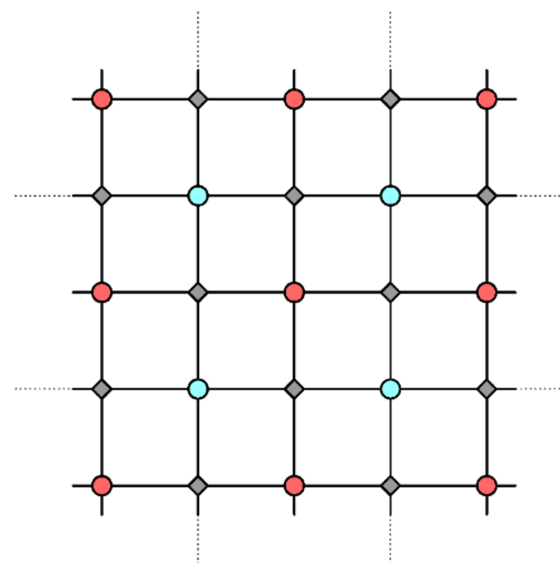
TN formulation of LGTs w/ fermions

- Normal TN formulation for the gauge part and GrassmannTN formulation for the fermion part

Fundamental unit-cell tensor



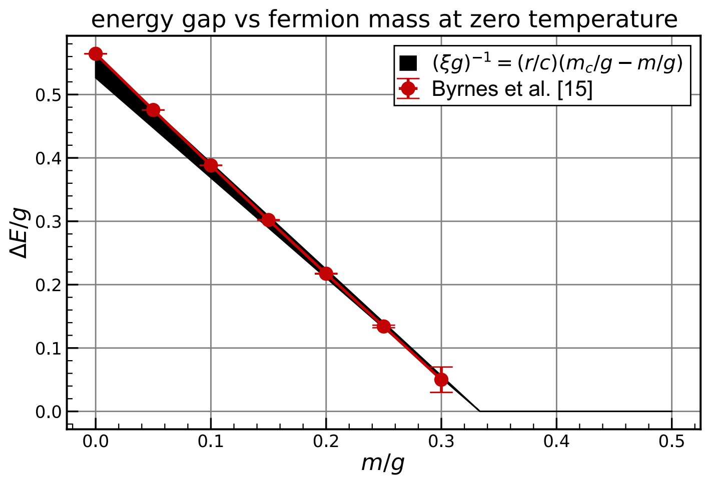
TN rep. of partition function

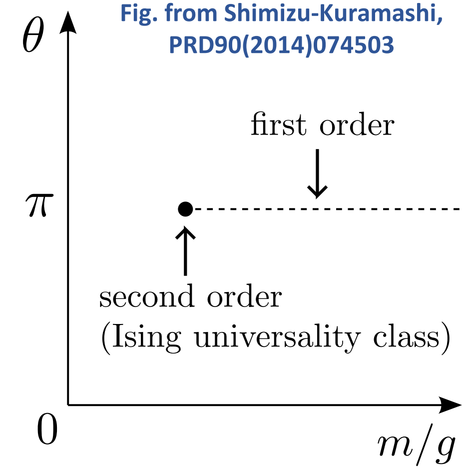


Schwinger model

- The most important QCD toy model in 2D
  - Exactly solvable in the massless limit
  - Mass perturbation, Bosonization Coleman, Ann.Phys.101(1976)239
- Various TN studies particularly for the model with a θ term
  - "θ≠0 ⇨ The sign problem"
  - However, **this sign problem can be avoided by the bosonization!**

Ohata, JHEP12(2023)007, PTEP2023(2023)013B02





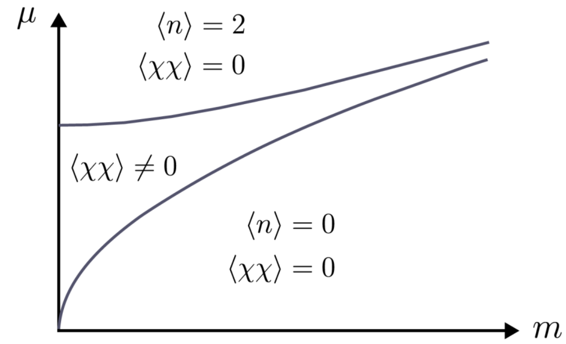
Byrnes-Sriganesh-Bursill-Hamer, PRD66(2002)013002
Bañuls-Cichy-Jansen-Cirac, JHEP13(2013)158
Byrnes-Haegeman-Acoleyen-Verschelde-Verstraete, PRL113(2014)091601
Shimizu-Kuramashi, PRD90(2014)014508, 074503, PRD97(2018)034502
Bañuls-Cichy-Cirac-Jansen-Kühn, PRL118(2017)071601
Pichler-Dalmonte-Rico-Zoller-Montangero, PRX6(2016)011023
Saito-Bañuls-Cichy-Cirac-Jansen, Lattice2014, Lattice2015
Zapp-Orús, PRD95(2017)114508
Magnifico-Vodola-Ercolessi-Kumar-Müller-Bermudez, PRD99(2019)014503, PRB100(2019)115152
Funcke-Janse-Kühn, PRD101(2020)054507
Butt-Catterall-Meurice-Sakai-Unmuth–Yockey, PRD101(2020)094509
Honda-Itou-Tanizaki, JHEP11(2022)141
Angelides-Funcke-Janse-Küh, PRD108(2023)0145156
Dempsey-Klebanov-Benjamin-Søggard-Zan, PRL132(2024)031603
Yosprakob-Nishimura-Okunishi, JHEP11(2023)187
Itou-Matsumoto-Tanizaki, JHEP11(2023)231, JHEP09(2024)155
Kanno-SA-Murakami-Takeda, arXiv:2412.08959
Fujii-Fujikura-Kikukawa-Okuda-Pedersen, PRD111(2025)094505
Cruz-Tarnopolsky-Xin, arXiv:2412.01902

Two-color QCD in two dimensions

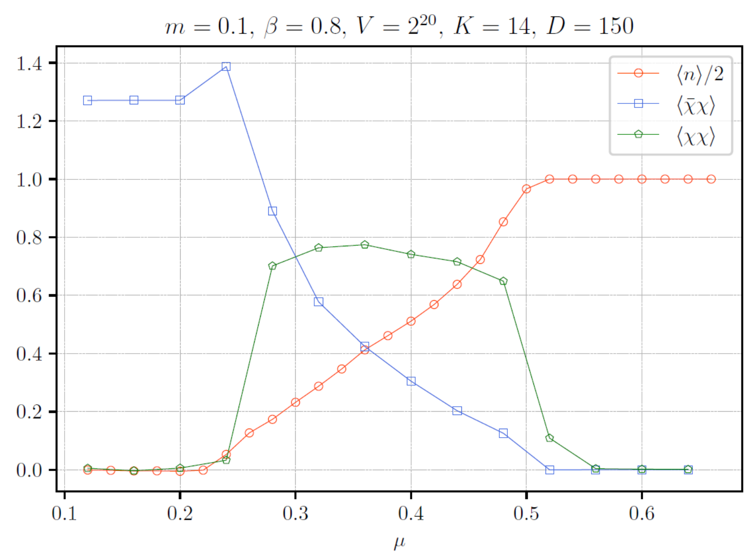
Kwok Ho Pai-SA-Todo, JHEP03(2025)027, PoS(LATTICE2024)364

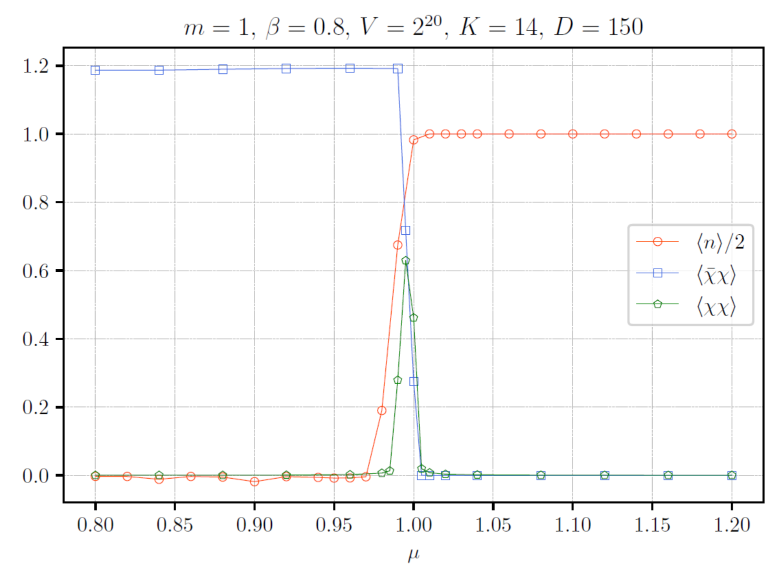
- 2D non-Abelian gauge theory is a non-trivial testbed for tensor networks
  - N_c=2, **finite gauge coupling, dynamical fermions,** finite density
    - The typical initial bond dimension is $\chi=4^{N_c}\times K$
  - The DMRG study has been also reported recently

Hayata-Hidaka-Nishimura, JHEP07(2024)106



Nishida+, Phys. Rept. 398(2004)281-300



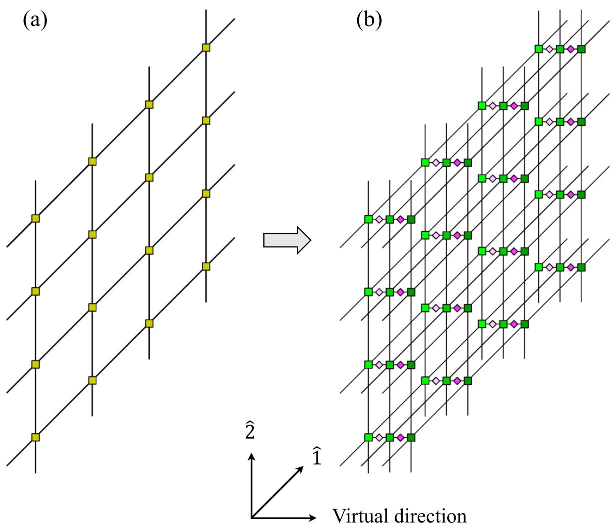


Multi-flavor theory

- Different flavor DoFs can be arranged in the virtual dimension direction
  - Pseudo-site approach
  - The bond dimension can be reduced **from $\chi_f^{N_f}$ to $N_f \times \chi_f$**
    - $\chi_f = 4^{N_c}$ in 2D
    - $\chi_f = 16^{N_c}$ w/ the 4D Wilson fermion

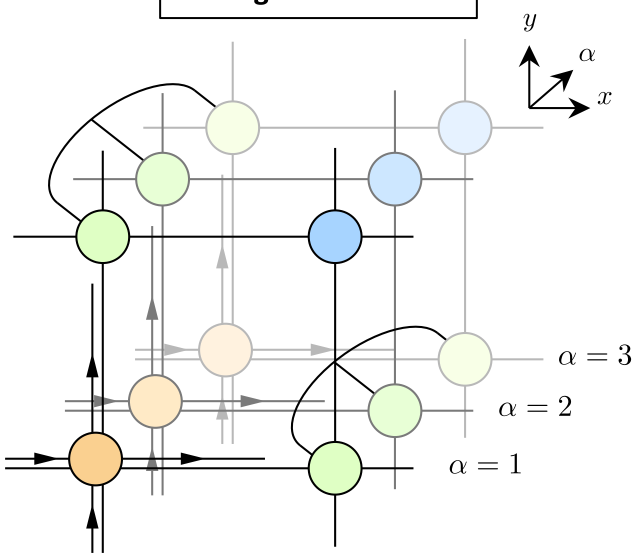
Ho Pai-SA-Todo, JHEP03(2025)027, PoS(LATTICE2024)364, Cf. Asaduzzama+, JHEP05(2024)195

Four-fermi interaction



SA, PRD108(2023)034514

Gauge interaction



Yosprakob-Nishimura-Okunishi, JHEP11(2023)187

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Topical Review

Tensor renormalization group for fermions

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Abstract

We review the basic ideas of the tensor renormalization group method and show how they can be applied for lattice field theory models involving relativistic fermions and Grassmann variables in arbitrary dimensions. We discuss recent progress for entanglement filtering, loop optimization, bond-weighting techniques and matrix product decompositions for Grassmann tensor networks. The new methods are tested with two-dimensional Wilson–Majorana fermions and multi-flavor Gross–Neveu models. We show that the methods can also be applied to the fermionic Hubbard model in 1+1 and 2+1 dimensions.

Keywords: tensor networks, lattice gauge theory, relativistic lattice fermions, Fermi Hubbard model, Grassmann path integrals, sign problems

Contents

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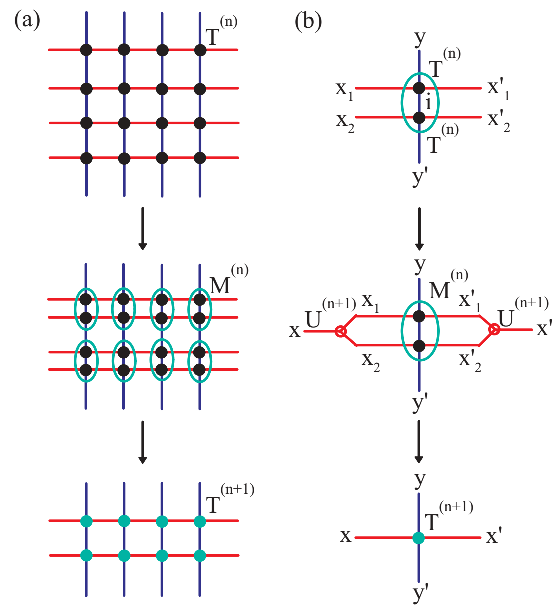
Higher-dimensional TRG algorithms

- Real-space RG on tensor networks, whose accuracy can be systematically improved by increasing the bond dimension χ
  - **Higher-Order TRG (HOTRG)** Xie-Chen-Qin-Zhu-Yang-Xiang, PRB86(2012)045139
    - Computationally demanding: $O\left(\chi^{4d-1}\right)$
  - **Anisotropic TRG (ATRG)** Adachi-Okubo-Todo, PRB102(2020)054432
    - Less demanding than HOTRG: $O\left(\chi^{2d+1}\right)$
  - **Triad TRG** Kadoh-Nakayama, arXiv:1912.02414
    - Based on three-leg tensors: $O\left(\chi^{d+3}\right)$
  - **Minimally-Decomposed TRG** Nakayama, arXiv:2307.14191
    - Comparable accuracy to HOTRG: $O\left(\chi^{2d+1}\right)$
  - **Triad ATRG** Sugimoto-Sasaki, PoS(LATTICE2024)038, arXiv:2507.21909
    Cf. P28 by Yuto Sugimoto on Aug. 6



Xie+, PRB86(2012)045139

Cf. TNRKit.jl, P1 by Adwait Naravane on Aug. 5

Anisotropic TRG

Adachi-Okubo-Todo, PRB102 (2020) 054432

- Decomposition of fundamental tensors

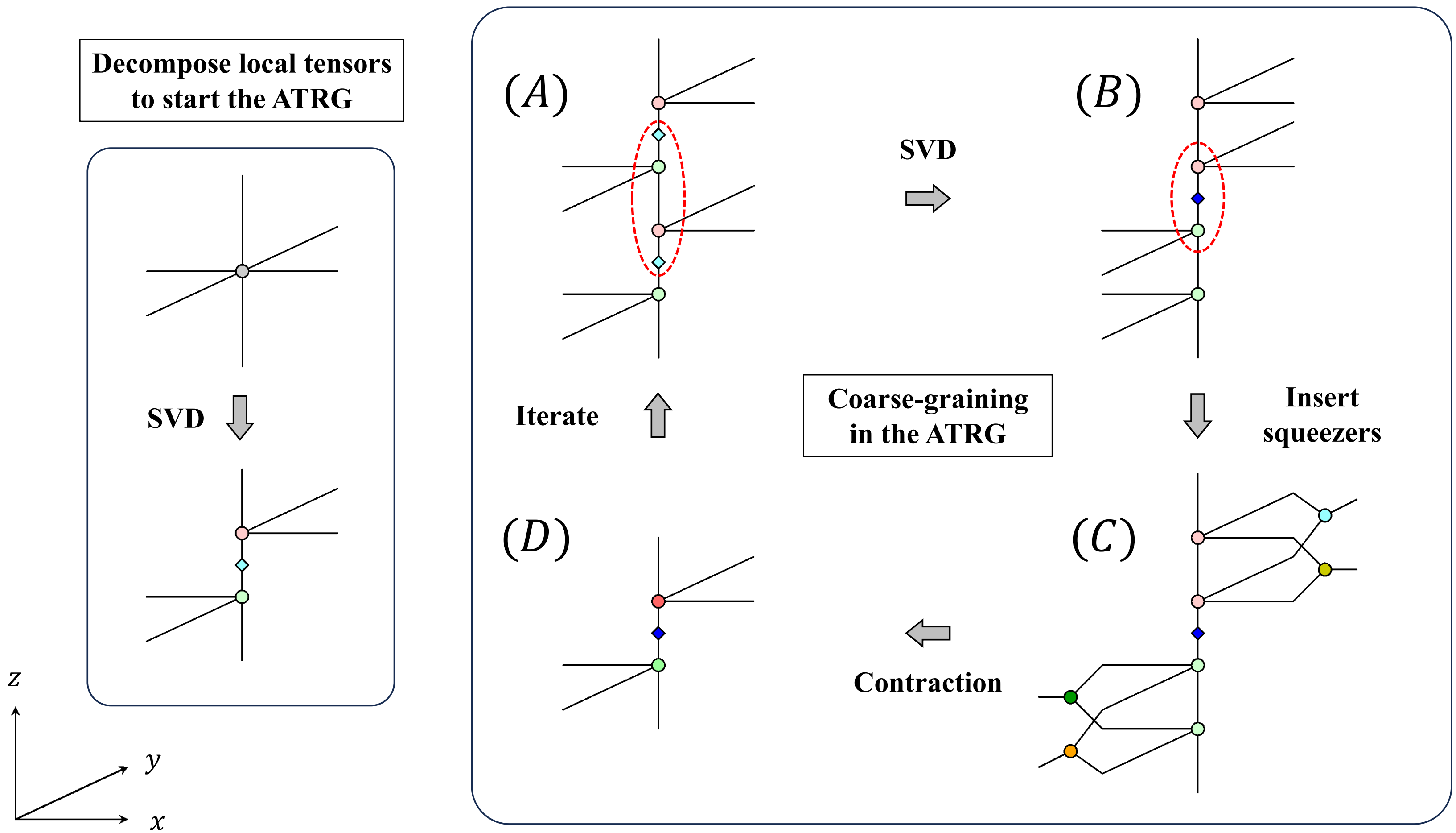
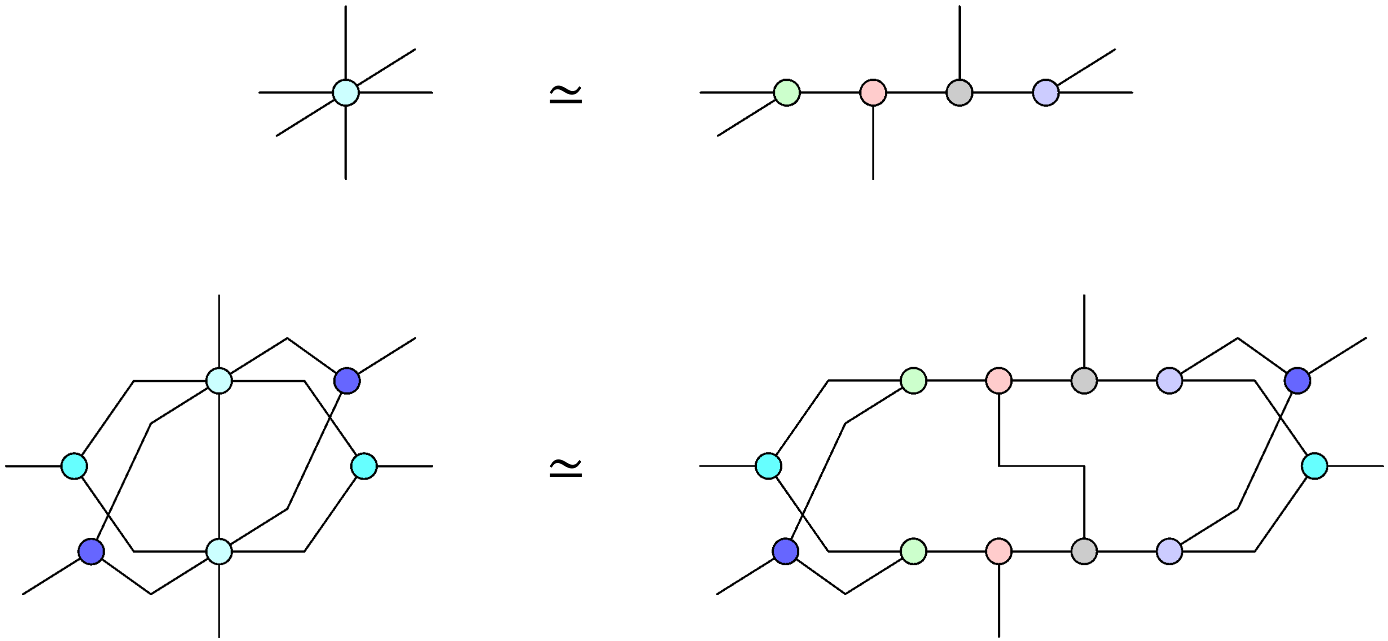


Figure is adapted from SA-Jha-Umuth-Yockey, PoS (LATTICE2023) 2024

Triad TRG

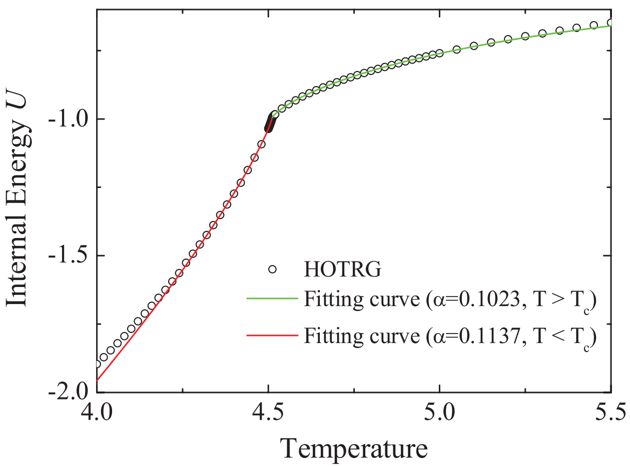
Kadoh-Nakayama, arXiv:1912.02414

- Further decomposition of fundamental tensors
  - Triad TRG is based on the HOTRG recursion formula



Status in the (2+1)D systems

3D Ising model



Method	T_c
HOTRG ($D = 16$, from U)	4.511544
HOTRG ($D = 16$, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

Xie+, PRB86(2012)045139

Algorithm	Cost	Model
HOTRG Xie+, PRB86(2012)045139	$\chi^{4d-1}\ln L$	Ising Xie+ ★Ising Lyu-Kawashima Potts model Wang+ free Wilson fermion Sakai+ Z_2 gauge theory Dittirich+, Kuramashi-Yoshimura U(1) gauge theory Unmuth-Yockey ★: **w/ entanglement filtering**
ATRG Adachi-Okubo-Todo, PRB102(2020)054432	$\chi^{2d+1}\ln L$	SU(2) LGT Kuwahara-Tsuchiya Real ϕ^4 theory SA+ Hubbard model SA-Kuramashi Z_2 gauge-Higgs model SA-Kuramashi SU(2) principal chiral model SA-Jha-Unmuth-Yockey
Triad RG Kadoh-Nakayama, arXiv:1912.02414	$\chi^{d+3}\ln L$	O(2) model Bloch+ Z_3 extended clock model Bloch+ Potts models Jha SU(2) principal chiral model SA-Jha-Unmuth-Yockey
MDTRG Nakayama, arXiv:2307.14191	$\chi^{2d+1}\ln L$	Complex ϕ^4 theory Aizawa+

The SU(2) principal chiral model on a cubic lattice

- Critical phenomena is in the O(4) universality class
  - Critical point and exponents are well known
  - $S = -\frac{J}{2}\sum_{n,\nu}\mathrm{Tr}[U(n)U(n+\hat{\nu})^{\dagger}]$

Pisarski-Wilczek, PRD29(1984)338
Kanaya-Kaya, PRD51(1995)2402
Toussaint, PRD55(1997)362
Hasenbuch, JPA34(2001)8221
Engels+, PoS LAT2005(2006)148
Ejiri+, PRD80(2009)094505

- Investigation of the truncation effect in the **character expansion**

$$\exp\left[\frac{J}{2}\mathrm{Tr}[U(n)U(n+\hat{\nu})^{\dagger}]\right] \simeq \sum_{r=0}^{r_{\max}} F_r(J)\chi_r(U(n)U(n+\hat{\nu})^{\dagger})$$

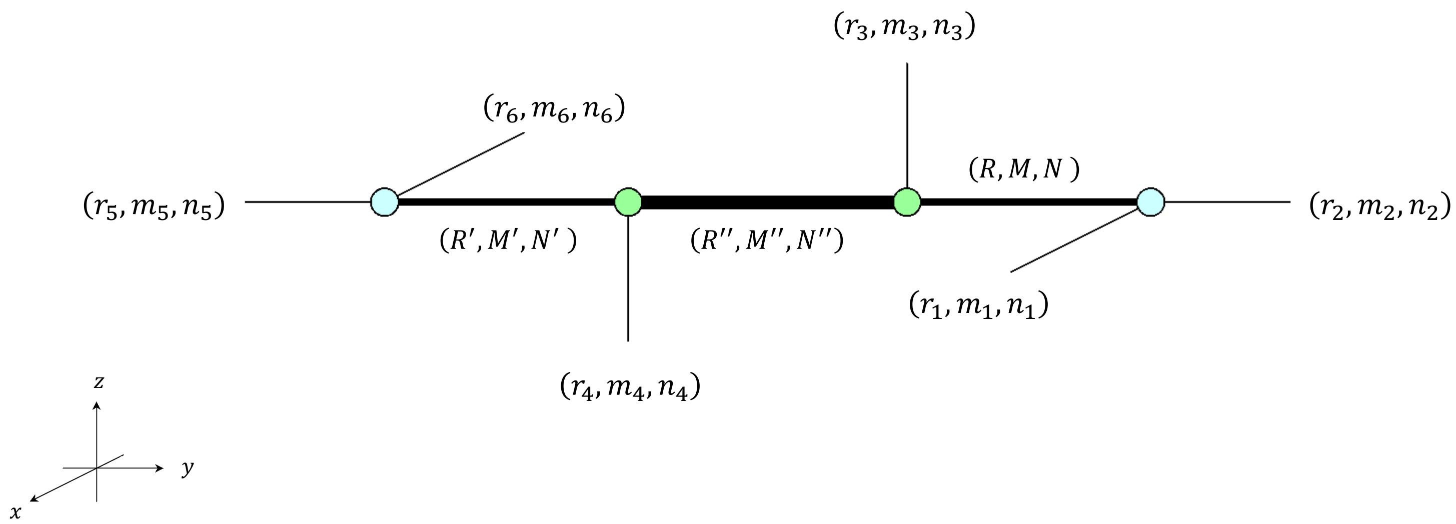
  - SU(2) symmetry is preserved even after the truncation
  - **O(4) universality should be observed even with the finite cutoff**
- Comparison of TRG algorithms beyond the Ising model
  - Triad TRG and Anisotropic TRG

Cf. TRG studies of the 2D model
Luo-Kuramashi, PRD107(2023)094509

The TN formulation

Liu+, PRD88(2013)056005

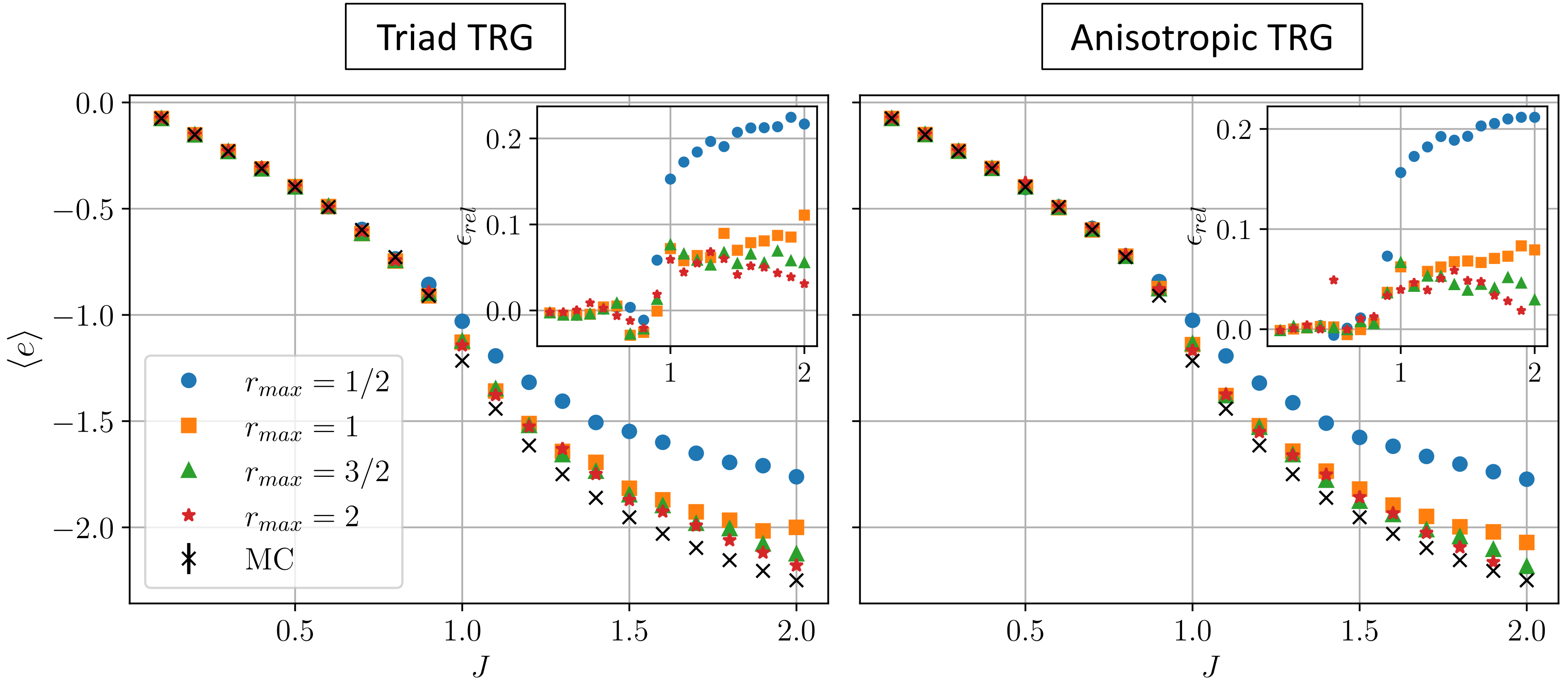
- Character expansion immediately gives us the triad representation
  - **Network of the CG coefficients**
  - Fundamental tensor at each site is as follows
  - Path integral is restored by contracting these six-leg tensors



The internal energy

SA-Jha-Unmuth–Yockey, PRD110(2024)034519

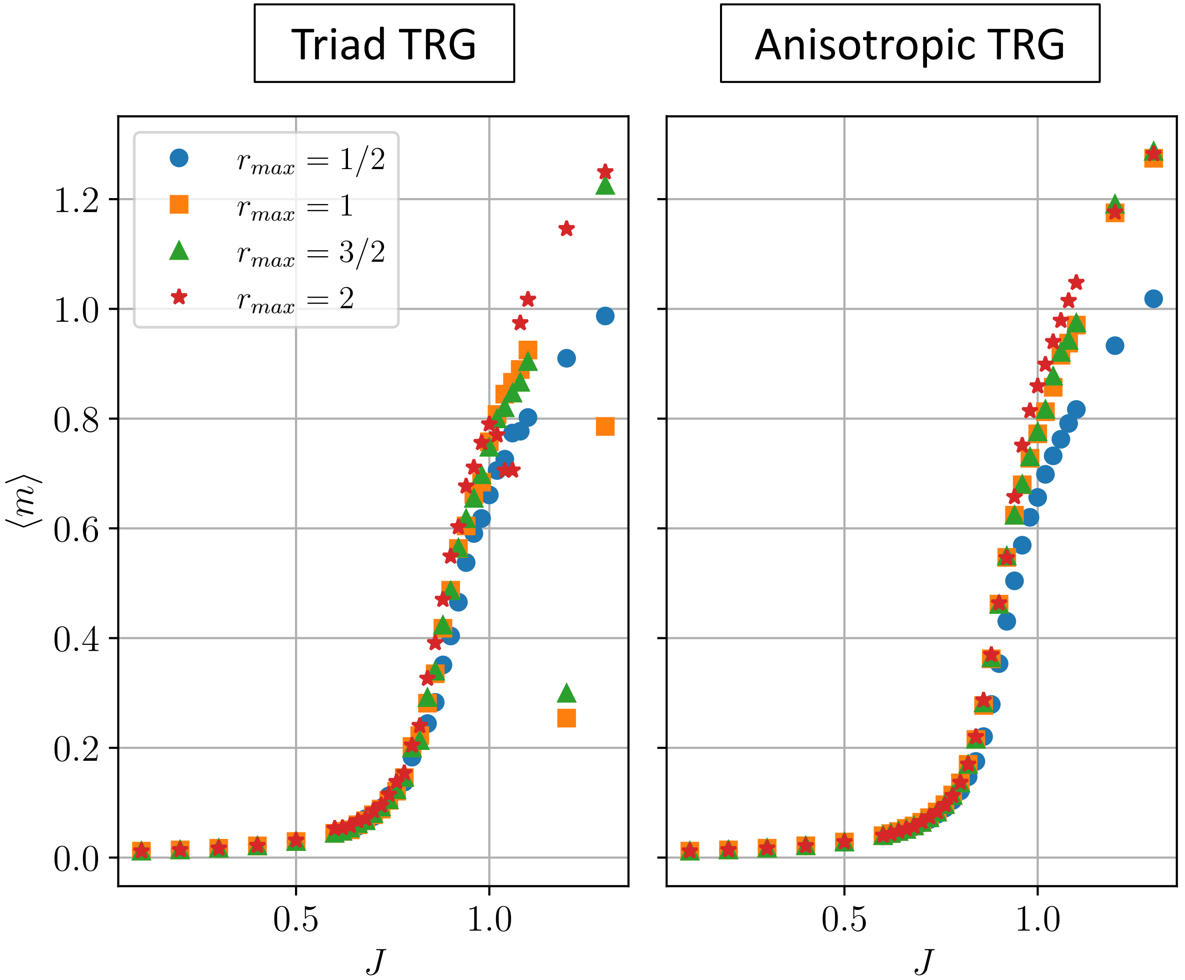
- Faster convergence in the strong-coupling regime



The magnetization

SA-Jha-Unmuth–Yockey, PRD110(2024)034519

- The magnetization is available, introducing a finite external field



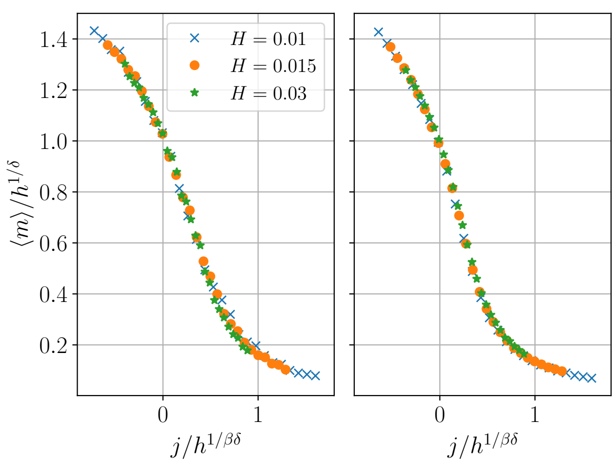
O(4) scaling collapse

SA-Jha-Unmuth–Yockey, PRD110(2024)034519

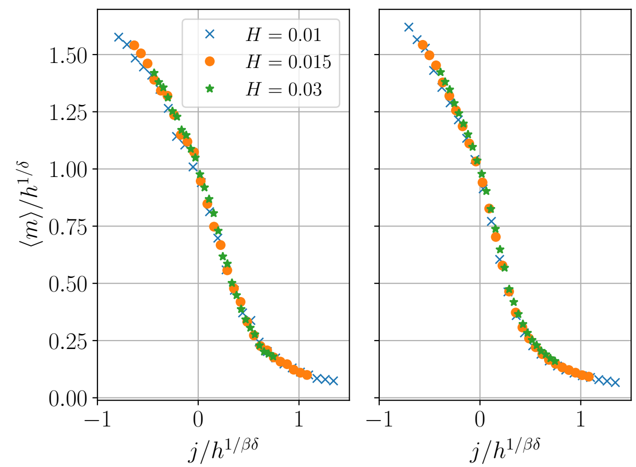
Cf. Kanaya-Kaya, PRD51(1995)2402
Toussaint, PRD55(1997)362

- $\langle m \rangle h^{-\frac{1}{\delta}} = f\left(j/h^{1/\beta\delta}\right)$ with $\beta = 0.3836$ and $\delta = 4.851$
  - **O(4) universality holds even with the finite cutoff**

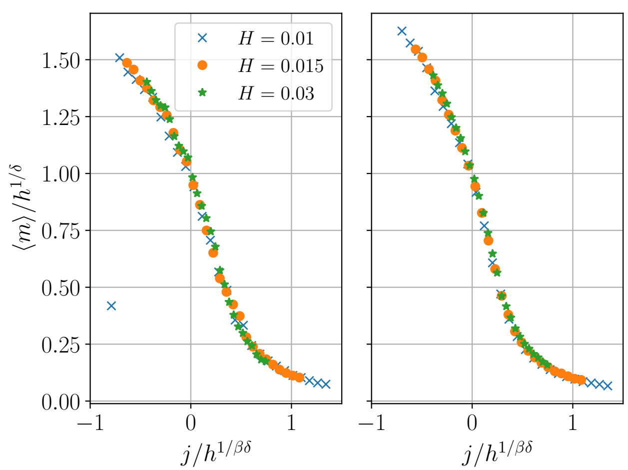
$r_{max} = 1/2$



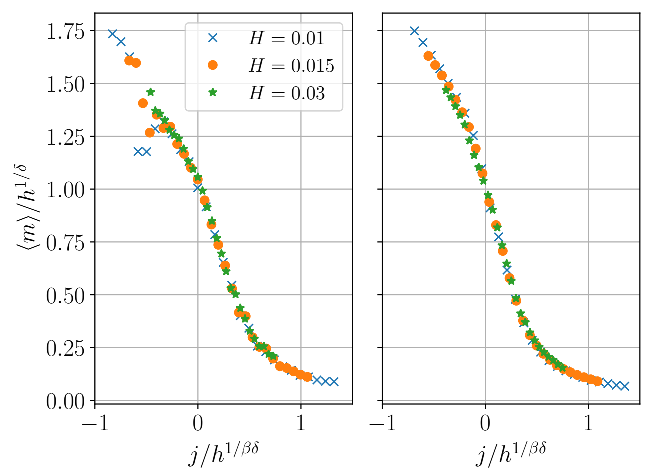
$r_{max} = 1$



$r_{max} = 3/2$



$r_{max} = 2$



Status of TNs in the (3+1)D systems

- More applications have been made by the TRG approach
  - Designing parallel computing methods for individual algorithms SA+, PoS(LATTICE2019)138 Yamashita-Sakurai, CPC278(2022)108423
  - Application of the machine learning techniques Liao+, PRX9(2019)031041
  - GPU acceleration Jha-Samlodia, CPC(2023)108941, Sugimoto-Sasaki, PoS(LATTICE2024)038

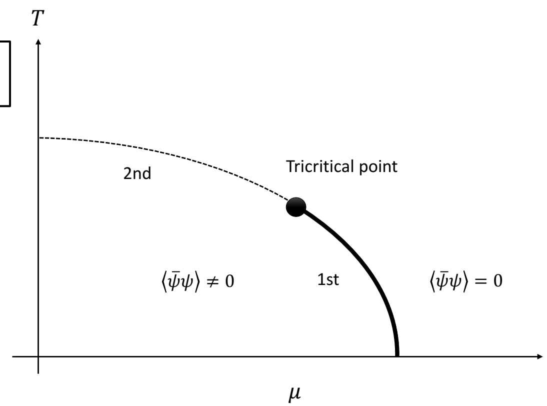
Variational approach	TRG approach
・QED at finite density★ Magnifico+ (2021)	・Ising model SA+ (2019), Sugimoto-Sasaki (2024) ・Staggered fermion w/ strongly coupled U(N) Milde+ (2022) ・Complex ϕ^4 theory★ SA+ (2020) ・Nambu—Jona-Lasinio model★ SA+ (2021) ・Real ϕ^4 theory SA+ (2021) ・Z_2 & Z_3 gauge-Higgs★ SA-Kuramashi (2022, 2023)

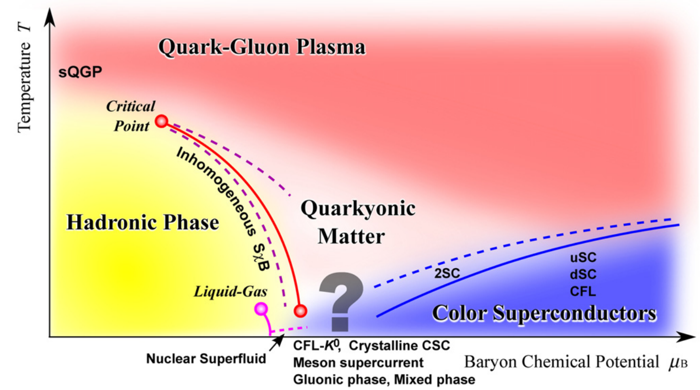
★ : system w/ the sign problem

The cold-dense NJL model

- NJL model at finite density is an effective theory of the QCD at finite density

NJL

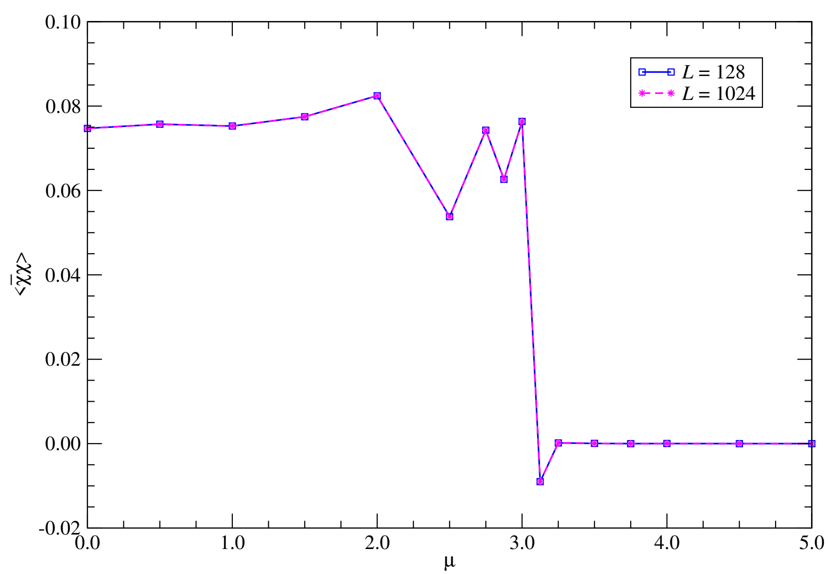




QCD

Fukushima-Hatsuda, Rept. Prog. Phys. 74(2011)014001

- **TRG has confirmed that the chiral symmetry is restored in the cold dense regime, where the MC is hindered by the severe sign problem**
  - Pressure and number density have also been obtained SA+, JHEP01(2021)121



Z_n gauge-Higgs model

- The simplest lattice gauge theory coupling to a matter field

$$S = -\beta \sum_n \sum_{\nu>\rho} \mathrm{Re}[U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu^*(n+\hat{\rho}) U_\rho^*(n)]$$
$$-\eta \sum_n \sum_\nu \left[\mathrm{e}^{\mu\delta_{\nu,d}} \sigma^*(n) U_\nu(n) \sigma(n+\hat{\nu}) + \mathrm{e}^{-\mu\delta_{d,4}} \sigma^*(n) U_\nu^*(n-\hat{\nu}) \sigma(n-\hat{\nu}) \right]$$

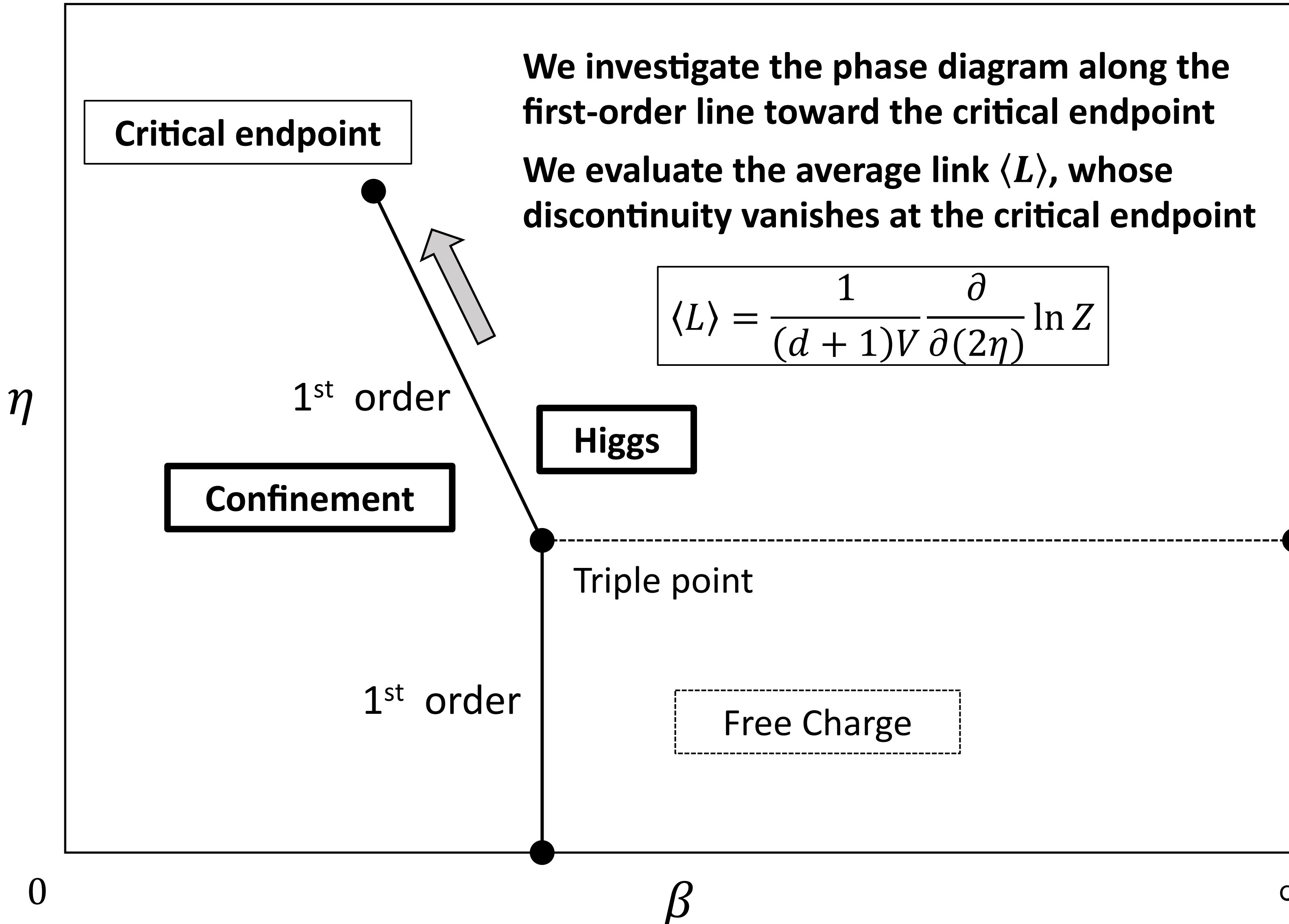
- LGT's path integral measure is always defined as the Haar measure
  - Unitary gauge fixing: $\sigma^*(n) U_\nu(n) \sigma(n+\hat{\nu}) \mapsto U_\nu(n)$

$$S = -\beta \sum_n \sum_{\nu>\rho} \mathrm{Re}[U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu^*(n+\hat{\rho}) U_\rho^*(n)]$$
$$-2\eta \sum_n \sum_\nu \left[\cosh(\mu\delta_{\nu,4}) \mathrm{Re} U_\nu(n) + \mathrm{i}\sinh(\mu\delta_{\nu,4}) \mathrm{Im} U_\nu(n) \right]$$

- **Although the model looks quite simply, the reliable MC simulation is not always available**
  - The sign problem takes place when n=3 and $\mu \neq 0$

Cf. TRG studies of gauge-Higgs models in 2D
Unmuth–Yockey+, PRD98(2018)094511
Bazavov+, PRD99(2019)114507
Butt+, PRD101(2020)094509

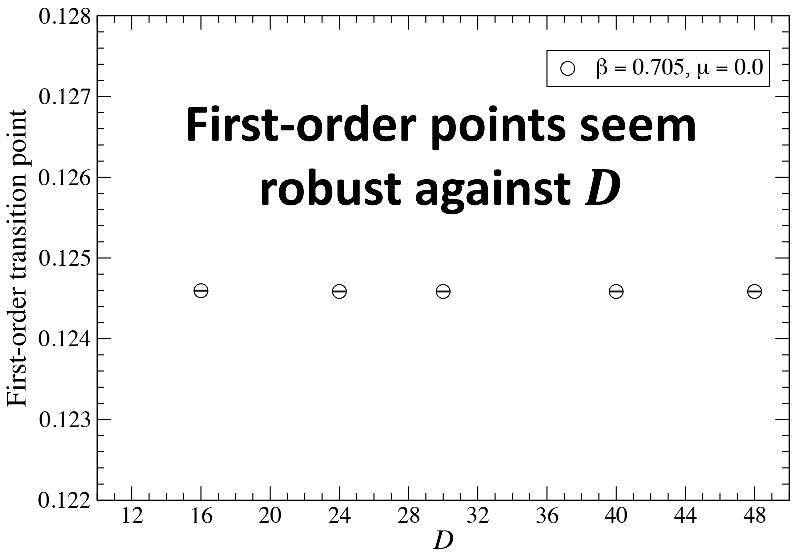
Phase diagram of the (3+1)D model at $\mu = 0$

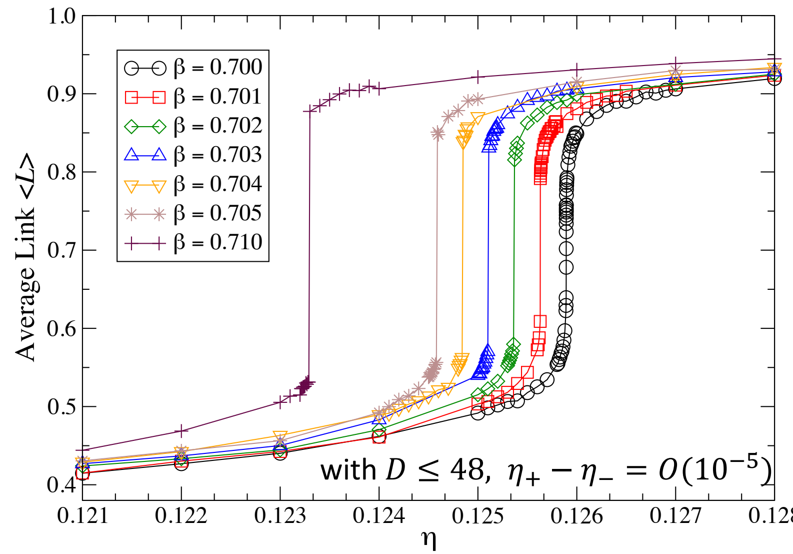


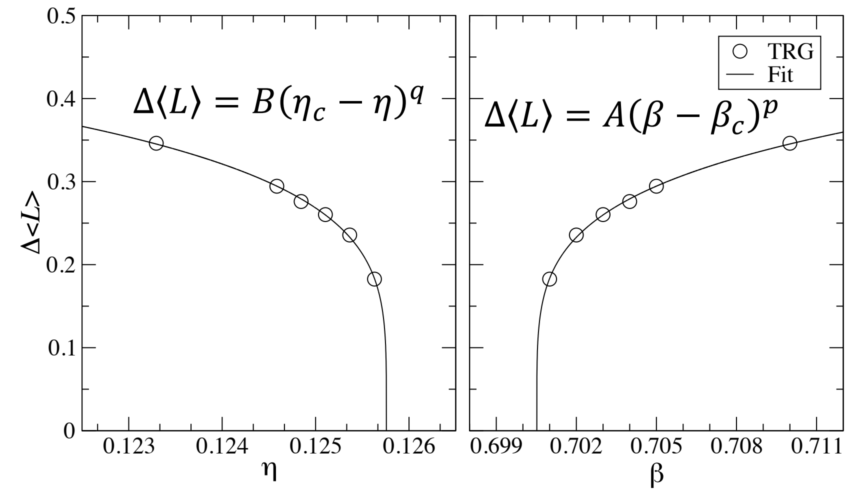
Cf. Fradkin-Shenker, PRD19(1979)3682

The (2+1)D model at $\mu = 0$

SA-Kuramashi, JHEP05(2022)102





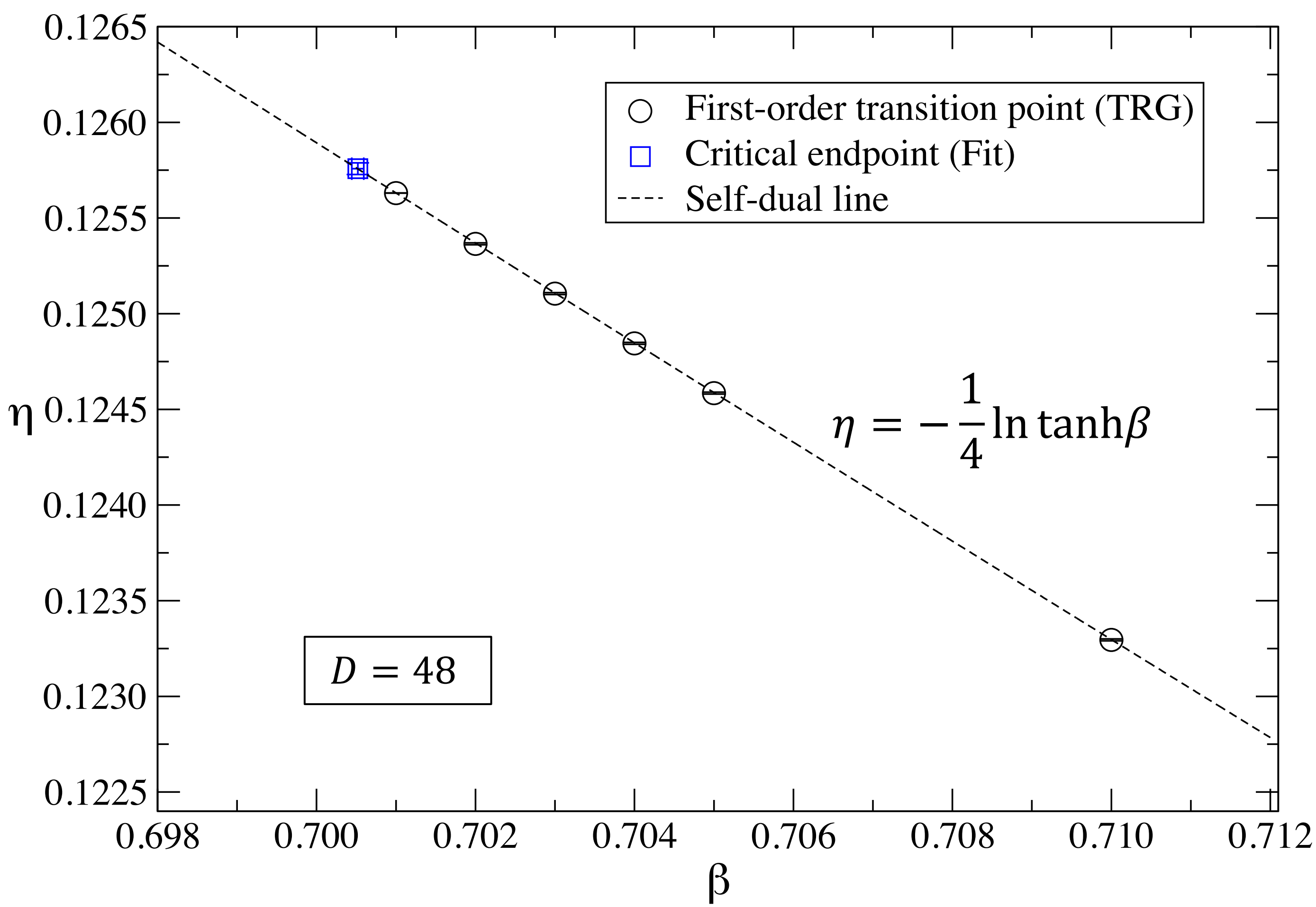


MC Somoza+, PRX11(2021)041008	$\beta_c \approx 0.701$
TRG this work	$\beta_c = 0.70051(7)$

Comparison with the self-dual line

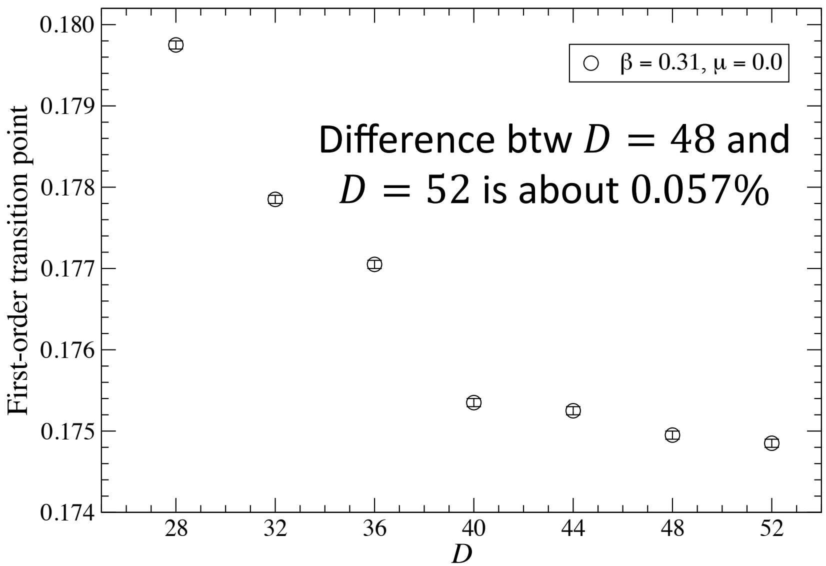
SA-Kuramashi, JHEP05(2022)102

- **All transition points are precisely located on the self-dual line**

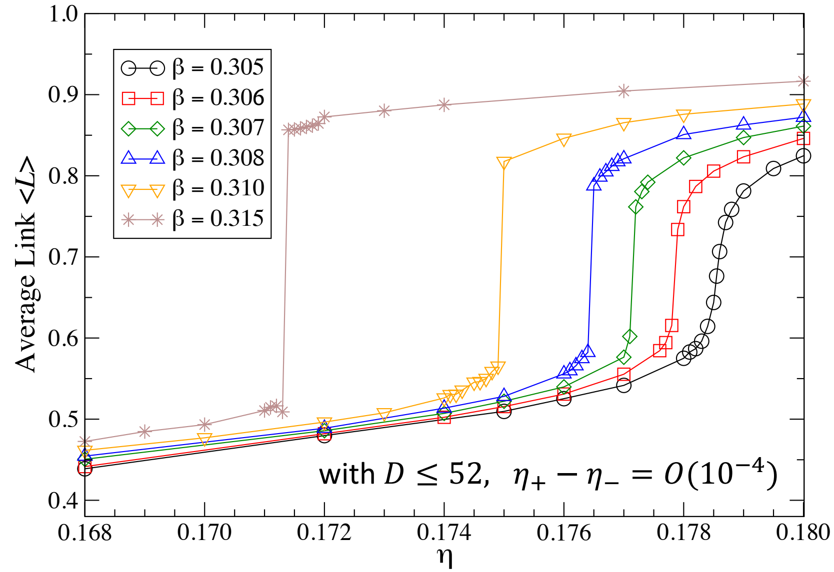


The (3+1)D model at vanishing density

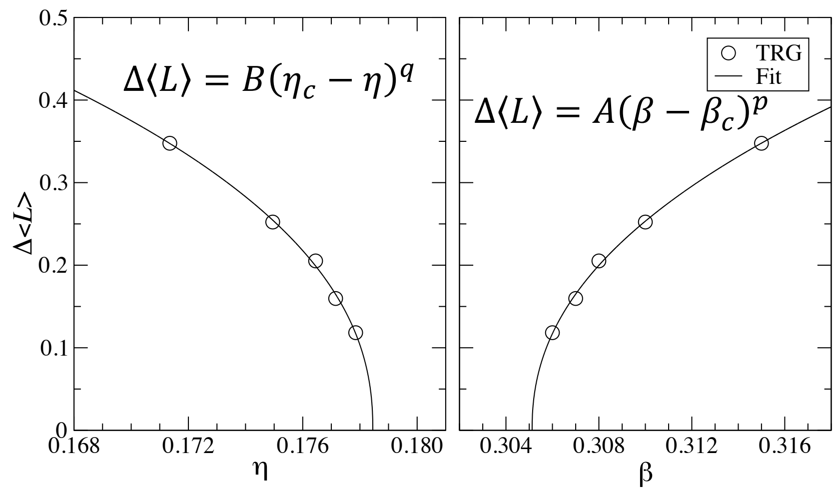
SA-Kuramashi, JHEP05(2022)102



Bond dimension



Mean-field Brezin-Drouffe, NPB200(1982)93	$(\beta_c, \eta_c) = (0.22, 0.205)$
MC on $V = 8^4$ Creutz, PRD21(1980)1006	$(\beta_c, \eta_c) = (0.22(3), 0.24(2))$
TRG w/ $D = 52$ this work	$(\beta_c, \eta_c) = (0.3051(2), 0.1784(2))$



The (3+1)D Z_3 model at vanishing density 1/2

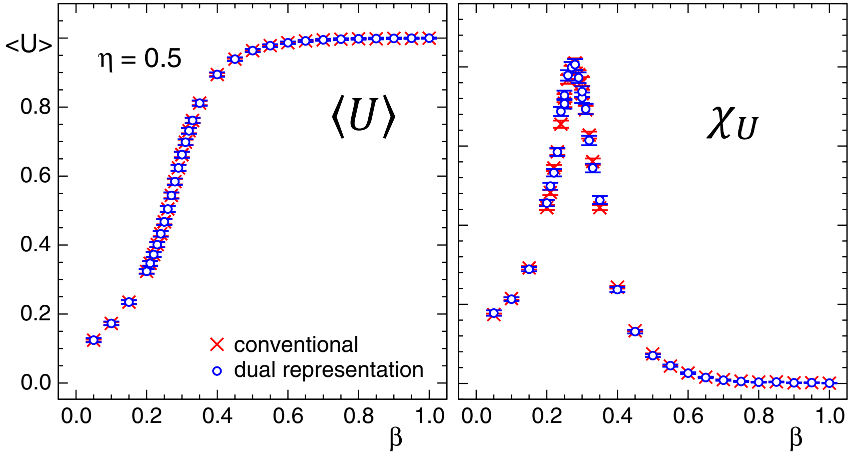
SA-Kuramashi, JHEP10(2023)077

- Comparison btw MC and TRG via the average plaquette $\langle U \rangle$ and its susceptibility
  - Good agreement just w/ $D = 45$ at finite-η regime
  - The susceptibility of $\langle U \rangle$ is obtained by numerical difference in case of TRG

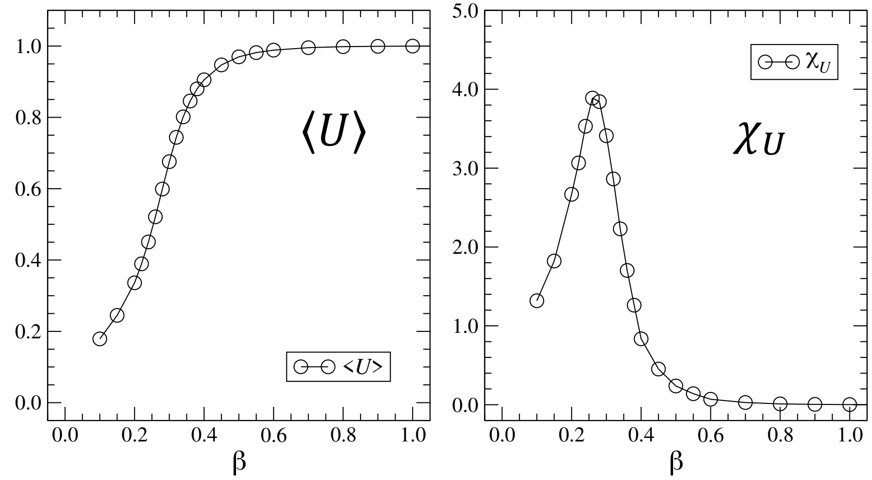
$$\langle U \rangle = -\frac{1}{6V}\frac{\partial}{\partial \beta}\ln Z$$

Monte Carlo

Gattringer-Schmidt, PRD86(2012)094506



TRG w/ $D = 45$



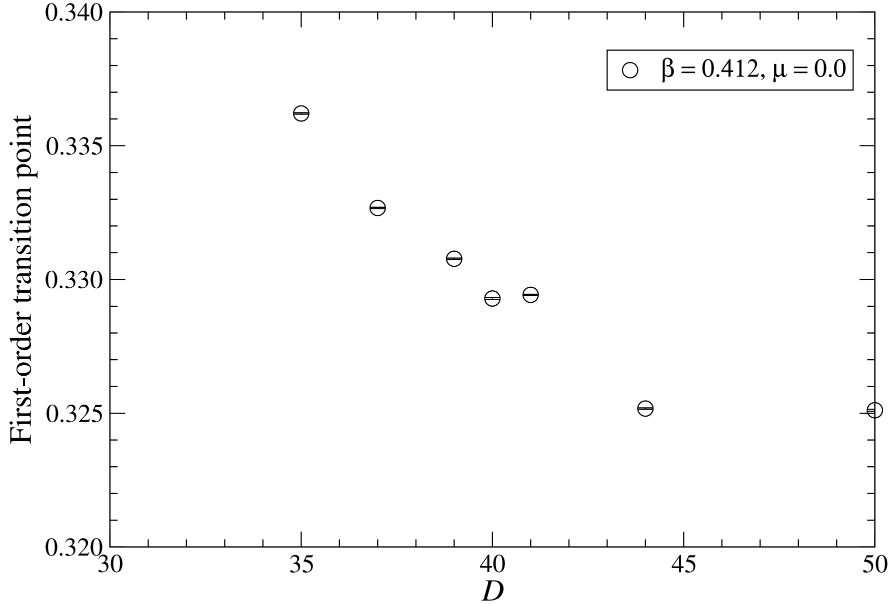
$\eta = 0.5, \mu = 0$

The (3+1)D Z_3 model at vanishing density 2/2

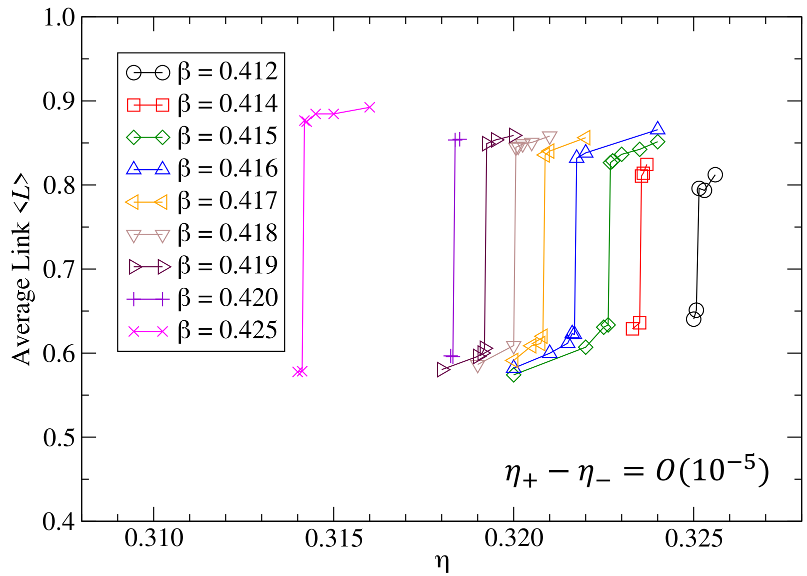
SA-Kuramashi, JHEP10(2023)077

- Location of the transition point seems converging w.r.t the bond dim.
  - Relative error btw $D = 44$ and $D = 50$ is **0.019%**
  - $\Delta\langle L\rangle$ becomes smaller when β becomes smaller, as expected

D-dependence in the first-order transition point



Average link@$D = 50$

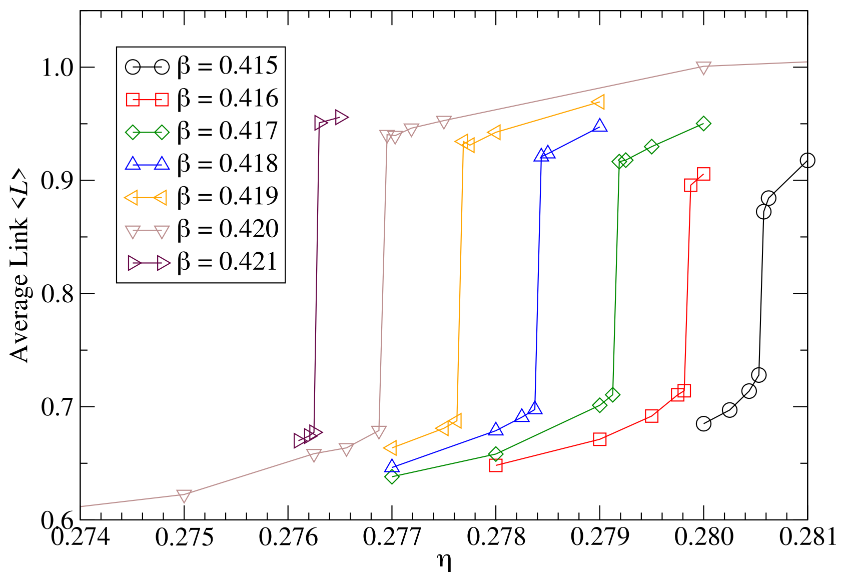


(3+1)D Z_3 model at finite density

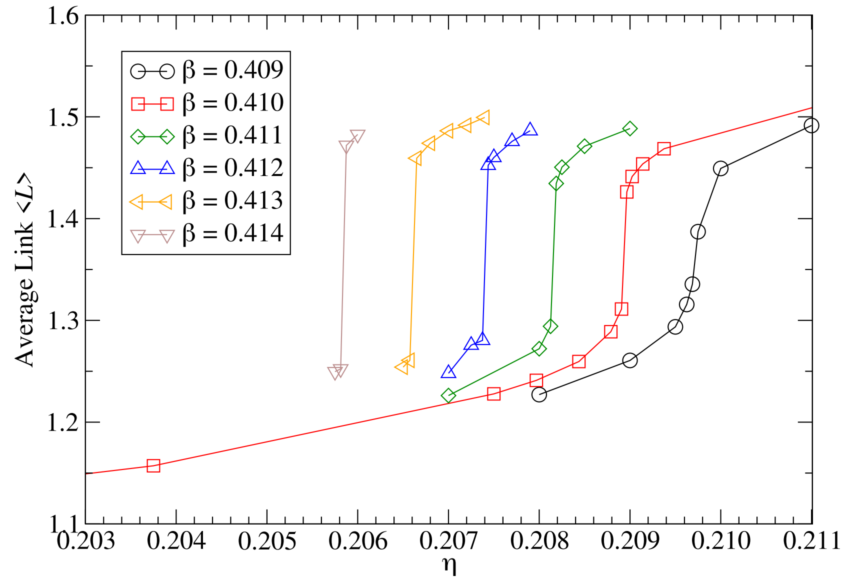
SA-Kuramashi, JHEP10(2023)077

- Again, $\Delta\langle L\rangle$ becomes smaller when β becomes smaller, as expected

Average link@$\mu = 1$



Average link@$\mu = 2$

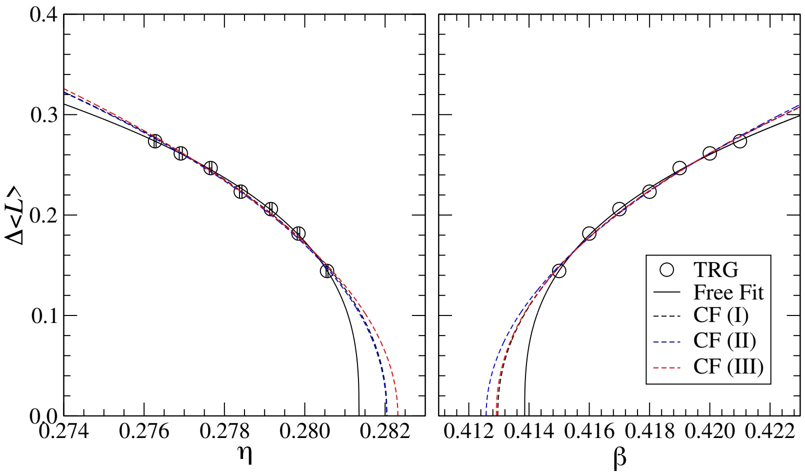


CEP in (3+1)D Z_3 model at finite μ

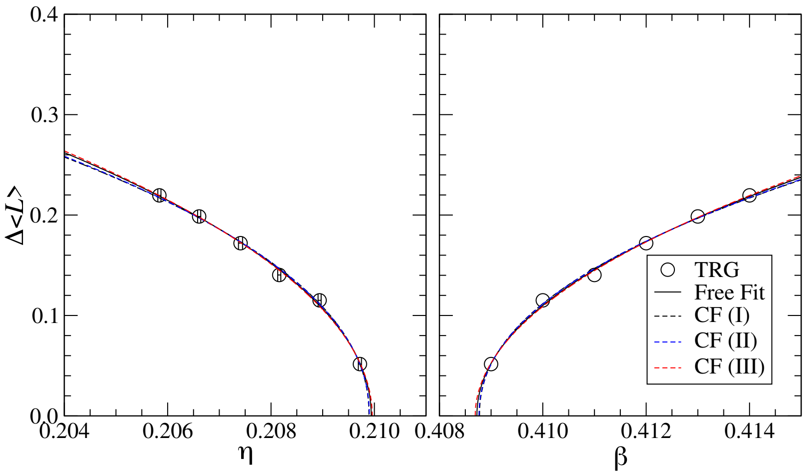
SA-Kuramashi, JHEP10(2023)077

- CEP is determined via the similar fit to the Z_2 model
  - $\Delta\langle L\rangle = A(\beta-\beta_c)^p$ and $\Delta\langle L\rangle = B(\eta_c-\eta)^q$
  - According to the mean field theory, $p = q = 0.5$
  - The simultaneous fit among different μ suggests $\boldsymbol{p = 0.46(2)}$, $\boldsymbol{q = 0.46(3)}$

$\mu = 1$



$\mu = 2$

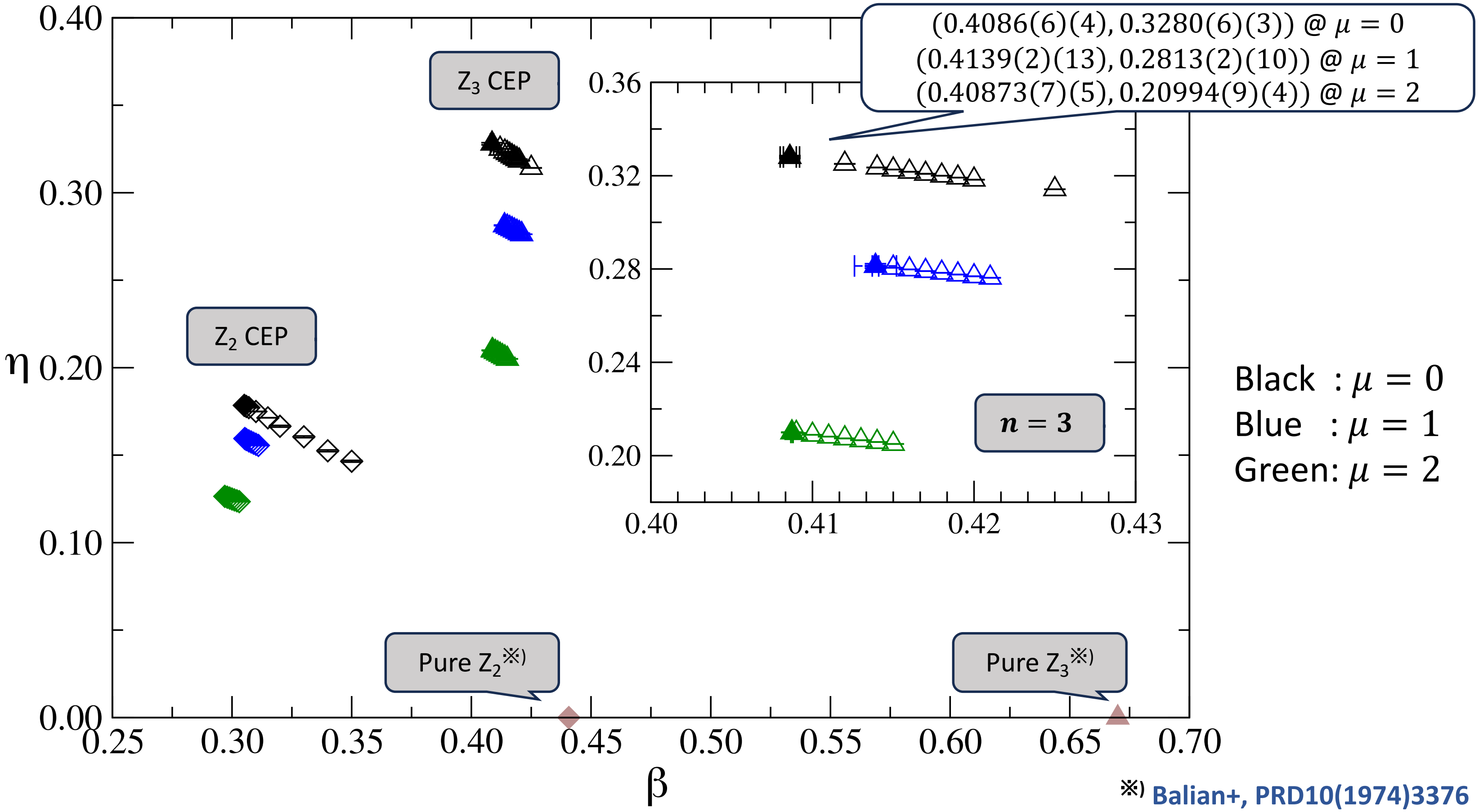


Phase diagram of Z_n gauge-Higgs model (n=2,3)

SA-Kuramashi, JHEP10(2023)077

- n-dependence in the resulting CEP is consistent w/ previous studies

Cf. U(1) gauge-Higgs studies, Baig-Clua, PRD57(1998)3902, Franzki+, PRD57(1998)6625



※) Balian+, PRD10(1974)3376

Lüscher's admissibility condition

- **Beyond the standard Wilson gauge action?**
  - Lüscher's gauge action for the U(1) gauge fields:

$$\beta S_g = \beta \sum_{n,\mu>\nu} \frac{1-\mathrm{Re}P_{\mu\nu}(n)}{1-\|1-P_{\mu\nu}(n)\|/\epsilon} \qquad \text{if} \qquad \|1-P_{\mu\nu}(n)\| < \epsilon$$

    and $\beta S_g = \infty$, otherwise

  - The gauge fields are separated into disconnected subspaces, corresponding to topological charge
    Lüscher, NPB549(1999)295-334
  - In the MC simulation, however, the topological change is substantially suppressed ⇨ Topological freezing
    Cf. Fukaya-Onogi, PRD68(2003)074503

Why don't we take the advantage of TRG?

- TRG allows us to compute the path integral w/o suffering from the sign problem and **the full contributions from every topological sector should be automatically included**
- As an example, we consider the U(1) gauge-Higgs model w/ a θ term in (1+1)D

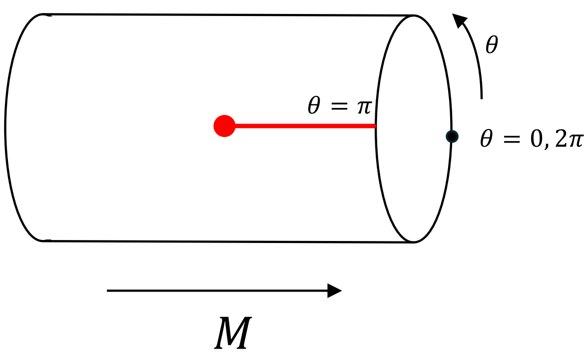
$$S = \beta S_g + S_h + S_\Theta$$

$$S_h = -\sum_n [\sum_\nu \{\phi^*(n) U_\nu \phi(n+\hat{\nu}) + \phi^*(n+\hat{\nu}) U_\nu^* \phi(n)\} + M|\phi(n)|^2 + \lambda |\phi(n)|^4]$$

$$S_\Theta = -\frac{\mathrm{i}\theta}{2\pi} \sum_n \log P_{12}(n)$$

- At θ=π, the first-order transition takes place w/ $M > M_\mathrm{c}$ and the critical behavior at $M = M_\mathrm{c}$ is in the **2D Ising universality class**

Gattringer+, NPB935(2018)344-364
Komargodski+, SciPost Phys. 6(2019)003



- The MC simulation for this model is extremely difficult due to **the complex action problem and the topological freezing**

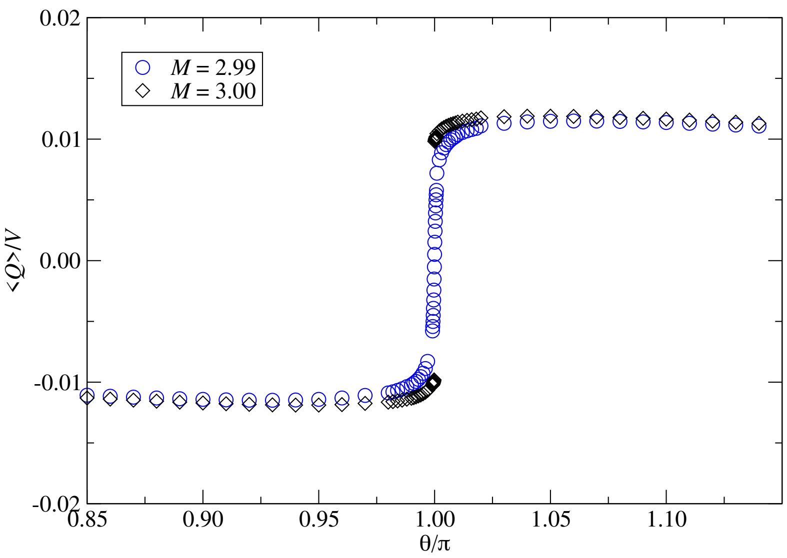
Hunting the critical endpoint

SA-Kuramashi, JHEP09(2024)086

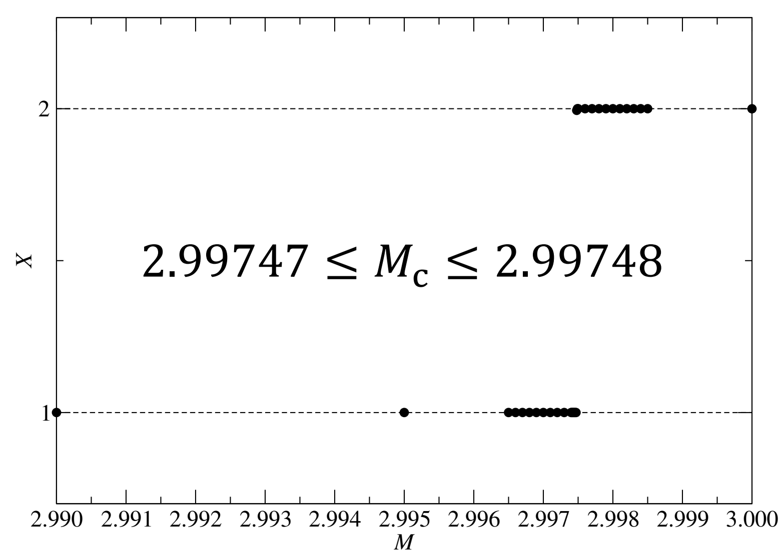
- Discontinuity in the topological charge is vanishing by decreasing the mass M
- Computing the ground-state degeneracy, we can bound the critical mass M_{c}

Gu-Wen, PRB80(2009)155131

Topological charge density



Ground-state degeneracy

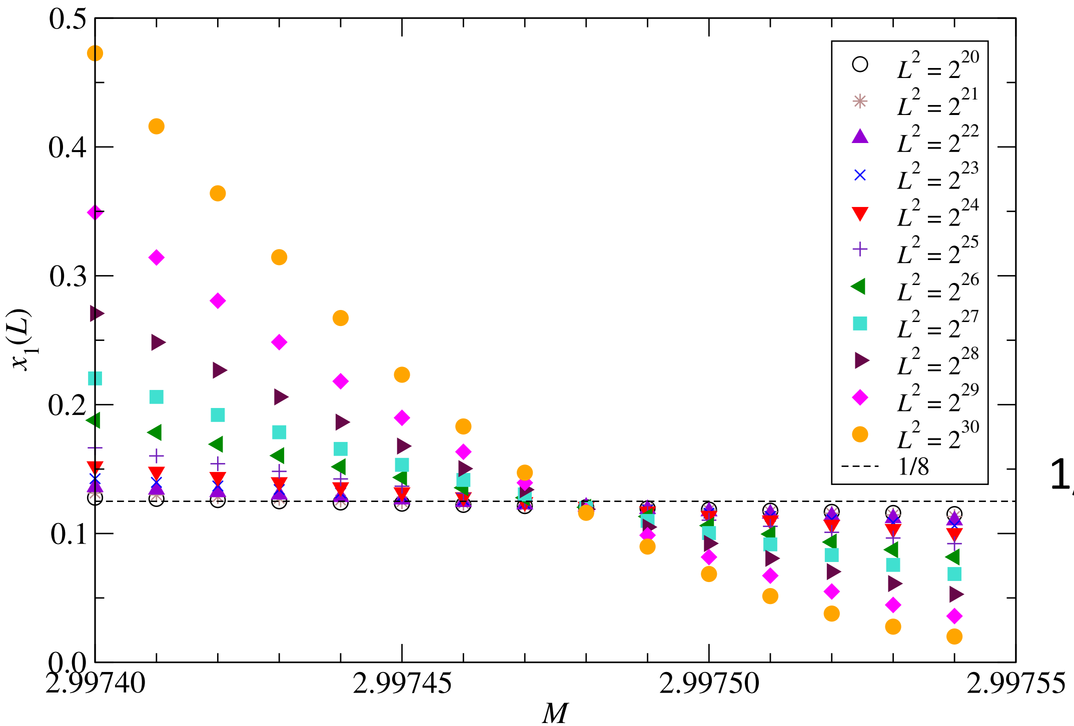


w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

Identification of the universality class

- Transfer matrix from the TN representation Gu-Wen, PRB80(2009)155131
  - **Scaling dimensions are available via** $x_n(L) = \frac{1}{2\pi} \ln \frac{\lambda_0(L)}{\lambda_n(L)}$
  - The volume independence in $x_1(L)$ is observed w/ $x_1(L) = 1/8$, which agrees with the 2D Ising universality class

SA-Kuramashi, JHEP09(2024)086



1/8

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

Tensor-network-based level spectroscopy

Ueda-Oshikawa, PRB108(2023)024413

- Assuming the 2D Ising universality class, we employ the level spectroscopy to determine the critical mass $M_{\rm c}$ from scaling dimensions intersections
  - The algorithmic-parameter dependence of $M_{\rm c}$ seems well suppressed
  - Consistent not only with the previous bound from the ground-state degeneracy, but also comparable with the previous MC result based on dual representation employing the Villain-type gauge action: $\boldsymbol{M_{\rm c} = 2.989(2)}$ Gattringer+, NPB935(2018)344-364

SA-Kuramashi, JHEP09(2024)086

K_g	K_h	χ	D	$M_{\rm c}$
24	20	8	192	2.9982886(1)
22	20	8	176	2.9998263(13)
20	20	8	160	2.9974765(14)
24	10	6	144	2.9929635(1)
22	10	6	132	2.9945222(9)
20	10	7	140	2.9921698(6)

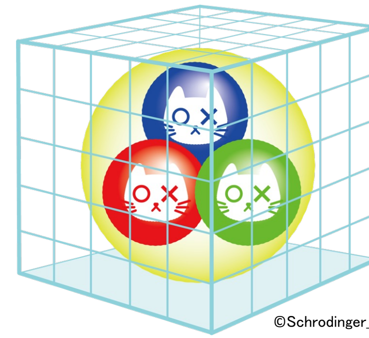
χ is another algorithmic parameter to compress the initial bond dimension from $K_g K_h$ to $K_g \chi$

Summary & Future Perspectives

- TRG approach for the higher-dimensional LGTs
  - Gauge sector ⇨ normal TN formulation, but a regularization scheme is necessary
  - Fermion sector ⇨ GrassmannTN formulation
  - **The technology needed to investigate the actual 4D QCD is gradually coming together**
- Enhancing the accuracy of the method is crucial for advancing qualitative insights into quantitative understanding
  - **The finite-bond effect should be under control**
    - Any lesson from 2D systems? Tagliacozzo+, PRB78(2008)024410 , Pollmann+, PRL102(2009)255701 Ueda-Oshikawa, PRB108(2023)024413, Huang+, PRB107(2023)205123 …
  - Entanglement filtering Hauru+, PRB97 (2018) 045111, Lyu-Kawashima, PRE111(2025)054140, …
  - Lesson from the PEPS/TNS community Vanderstraeten+, PRE98(2018)042145, Emonts, PRD107(2023)014505, Vlaar-Corboz, PRL130(2023)130601, …
  - TN+MC Ferris, arXiv:1507.00767, Zohar-Cirac, PRD97(2018)034516, Arai+, PRD107(2023)114515, Todo, arXiv:2412.02974, …

Appendices

A quick look at the QCD



- The Kogut-Susskind formulation
  - Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{\boldsymbol{n}\in\Lambda_3}\sum_{\sigma=1,2,3}\left[\hat{\psi}(\boldsymbol{n})^\dagger \hat{U}_\nu(\boldsymbol{n})\hat{\psi}(\boldsymbol{n}+\sigma) + \text{h.c.}\right] + m\sum_{\boldsymbol{n}}(-1)^{\boldsymbol{n}}\psi(\boldsymbol{n})^\dagger\psi(\boldsymbol{n}) - \mu\sum_{\boldsymbol{n}}\psi(\boldsymbol{n})^\dagger\psi(\boldsymbol{n})$$
$$+ \frac{g^2}{2}\sum_{\boldsymbol{n}}\hat{E}(\boldsymbol{n})^2 + \frac{1}{g^2}\sum_{\boldsymbol{n},\sigma>\tau}\text{Tr}\left[2 - \hat{P}_{\sigma\tau}(\boldsymbol{n}) - \hat{P}^\dagger_{\sigma\tau}(\boldsymbol{n})\right]$$

  - Lagrangian (Action)

$$S = \frac{1}{2}\sum_{n\in\Lambda_{3+1}}\sum_{\nu=1,2,3,4}\left[\mathrm{e}^{\mu\delta_{\nu,4}}\eta_\nu(n)\bar{\psi}(n)U_\nu(n)\psi(n+\hat{\nu}) - \mathrm{e}^{-\mu\delta_{\nu,4}}\eta_\nu(n+\hat{\nu})\bar{\psi}(n+\hat{\nu})U^\dagger_\nu(n)\psi(n)\right]$$
$$+ m\sum_{n}\bar{\psi}(n)\psi(n) + \frac{2}{g^2}\sum_{n,\nu>\rho}\Re\text{Tr}\left[\mathbb{1} - U_\nu(n)U_\rho(n+\nu)U^\dagger_\nu(n+\rho)U^\dagger_\rho(n)\right]$$