An area law for the maximally-mixed ground state in arbitrarily degenerate systems with good AGSP

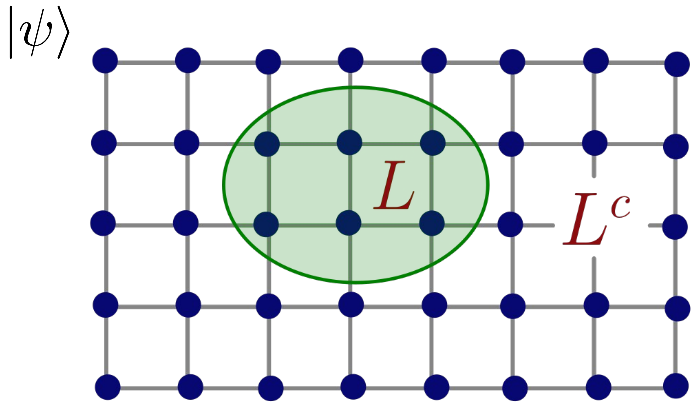
IA[1], **Raz Firanko**[2], **Rahul Jain**[1]

1. The Centre for Quantum Technologies (**CQT**) Singapore
2. The Technion, Haifa, Israel

arXiv:2310.19028

Area laws

Many-body system on D-dim lattice



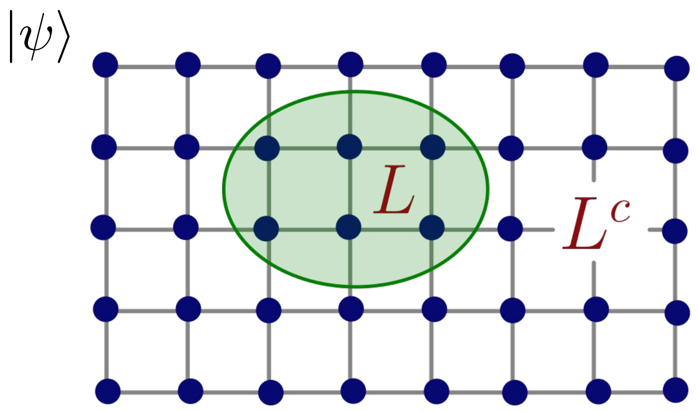
$$\rho_L = \operatorname*{Tr}_{L^c} |\psi\rangle\langle\psi|$$

Volume law: $S(\rho_L) \propto |L|$ (general case)

Area law: $S(\rho_L) \propto |\partial L|$ (much less entanglement)

Area laws

Many-body system on D-dim lattice



$$\rho_L = \operatorname*{Tr}_{L^c} |\psi\rangle\langle\psi|$$

Volume law: $S(\rho_L) \propto |L|$ (general case)

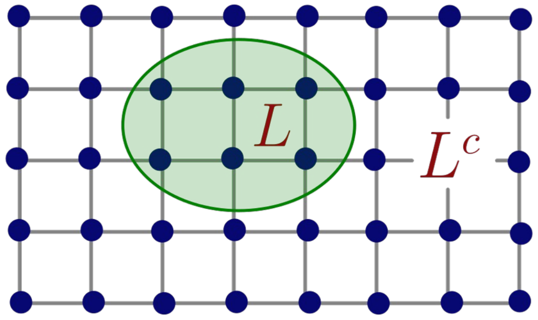
Area law: $S(\rho_L) \propto |\partial L|$ (much less entanglement)

Area-law conjecture:

Ground states of **gapped** local Hamiltonians on a lattice satisfy an area-law

Previous results

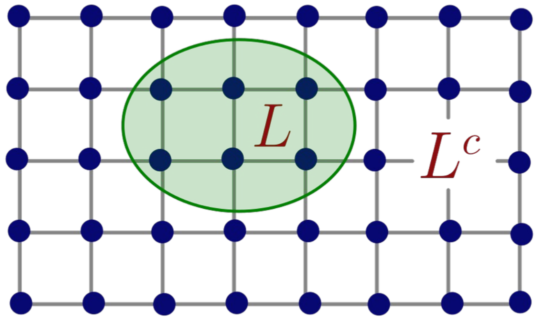
For unique ground states ($\dim \mathcal{H}_0 = 1$):



- Proven for 1D systems (Hastings ’07, Arad et al ’13)
- Proven for 2D under additional assumptions:

  Particle-like excitations (Hastings '07),
  Low density of low energy states (Masanes '09),
  FF spin 1/2 with 2-body interactions (N. de Beaudrap et al 2010),
  Adiabatic assumption (Cho '14), Specific heat assumption (Brandao & Cramer '15),
  FF + local gap (Anshu et al '21)

Previous results



★ **For unique ground states ($\dim \mathcal{H}_0 = 1$):**

- Proven for 1D systems (Hastings ’07, Arad et al ’13)
- Proven for 2D under additional assumptions:

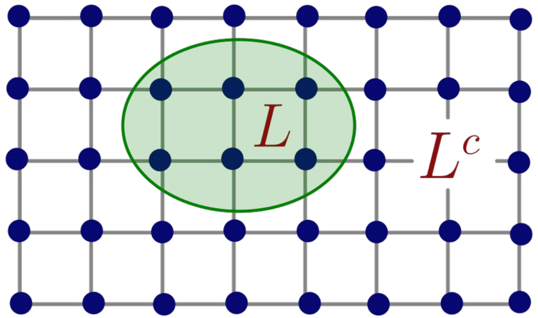
  Particle-like excitations (Hastings '07),
  Low density of low energy states (Masanes '09),
  FF spin 1/2 with 2-body interactions (N. de Beaudrap et al 2010),
  Adiabatic assumption (Cho '14), Specific heat assumption (Brandao & Cramer '15),
  FF + local gap (Anshu et al '21)

★ **For degenerate ground spaces ($\dim \mathcal{H}_0 = r$):**

- For 1D and 2D (F.F. + local gap):

$$S(\rho_L) = O(|\partial L|) + \log r$$

Previous results



★ **For unique ground states ($\dim \mathcal{H}_0 = 1$):**

- Proven for 1D systems (Hastings '07, Arad et al '13)
- Proven for 2D under additional assumptions:

  Particle-like excitations (Hastings '07),
  Low density of low energy states (Masanes '09),
  FF spin 1/2 with 2-body interactions (N. de Beaudrap et al 2010),
  Adiabatic assumption (Cho '14), Specific heat assumption (Brandao & Cramer '15),
  FF + local gap (Anshu et al '21)

★ **For degenerate ground spaces ($\dim \mathcal{H}_0 = r$):**

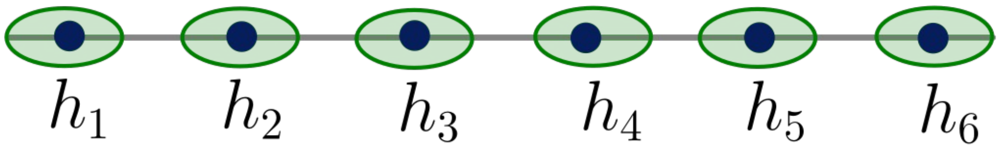
- For 1D and 2D (F.F. + local gap):

$$S(\rho_L) = O(|\partial L|) + \log r$$

becomes trivial when $r = 2^{\Omega(n)}$

Example

Non-interacting 1D system of qudits $\{|0\rangle, |1\rangle, |2\rangle\}$



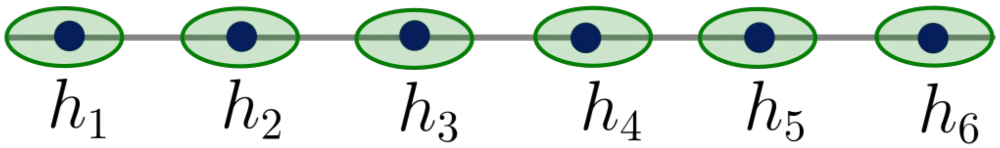
$$H = \sum_i |2\rangle\langle 2|_i$$

$\Longrightarrow$ $\mathcal{H}_0 = \mathrm{span}\{|0\rangle, |1\rangle\}^{\otimes n}$

$\Longrightarrow$ Not all ground states satisfy an area-law

Example

Non-interacting 1D system of qudits $\{|0\rangle, |1\rangle, |2\rangle\}$



$$H = \sum_i |2\rangle\langle 2|_i$$

$\Longrightarrow$ $\mathcal{H}_0 = \text{span}\{|0\rangle, |1\rangle\}^{\otimes n}$

$\Longrightarrow$ Not all ground states satisfy an area-law

What about the maximally-mixed state in $\mathcal{H}_0$:

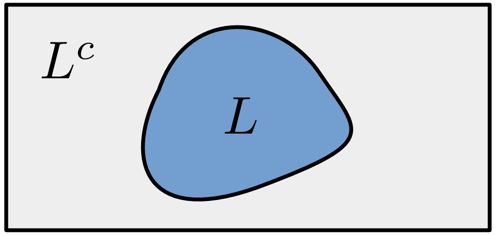
$$\rho_{\max} = \frac{1}{D} \sum_{\psi_i \in \mathcal{H}_0} |\psi_i\rangle\langle\psi_i|$$

?

The maximally-mixed ground state

For mixed states, we use **mutual information** instead of entanglement entropy:

$$I(L : L_c)_\rho \stackrel{\text{def}}{=} S(\rho_L) + S(\rho_{L^c}) - S(\rho)$$

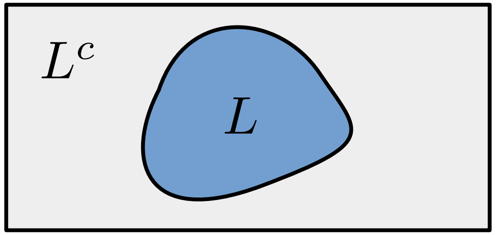


$$H = \sum_i h_i \qquad \mathcal{H}_0 = \text{span}\{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_{i=1}^{r} |\psi_i\rangle\langle\psi_i|$$

The maximally-mixed ground state

For mixed states, we use **mutual information** instead of entanglement entropy:

$$I(L : L_c)_\rho \stackrel{\text{def}}{=} S(\rho_L) + S(\rho_{L^c}) - S(\rho)$$



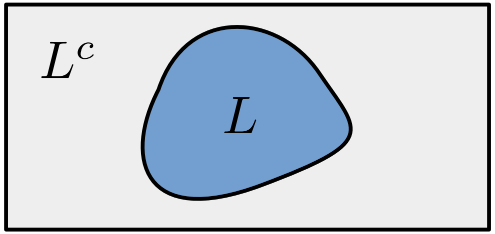
$$H = \sum_i h_i \qquad \mathcal{H}_0 = \text{span}\{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_{i=1}^{r} |\psi_i\rangle\langle\psi_i|$$

Is there an area-law in the mutual information for ρ_{max}?

$$I(L : L^c)_{\rho_{max}} = O(|\partial L|)$$

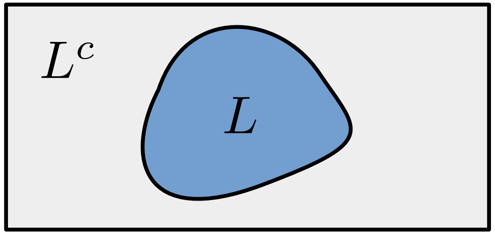
Motivation (and why it might be true)

$$H = \sum_i h_i \qquad \mathcal{H}_0 = \mathrm{span}\{|\psi_1\rangle, \dots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$



Motivation (and why it might be true)

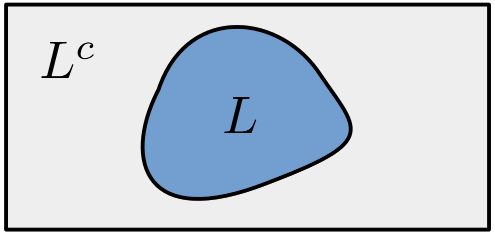
$$H = \sum_i h_i \qquad \mathcal{H}_0 = \mathrm{span}\{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$



- ρ_{max} is the most "natural" state in $\mathcal{H}_0$ (zero temp' Gibbs state $\lim_{\beta\to\infty} \frac{1}{Z} e^{-\beta H}$)

Motivation (and why it might be true)

$$H = \sum_i h_i \qquad \mathcal{H}_0 = \text{span}\{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$



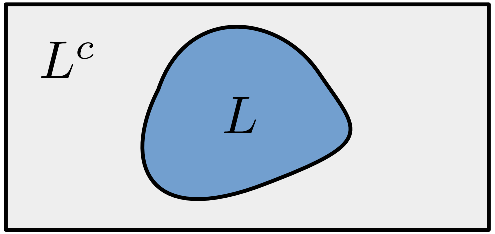
- ρ_{max} is the most "natural" state in $\mathcal{H}_0$ (zero temp' Gibbs state $\lim_{\beta\to\infty} \frac{1}{Z} e^{-\beta H}$)
- Holds for finite temperature Gibbs state $\rho_\beta = \frac{1}{Z} e^{-\beta H}$ we have:

$$I(L : L_c)_{\rho_\beta} = O(\beta \cdot |\partial L|)$$

(Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac '08)

Motivation (and why it might be true)

$H = \sum_i h_i \qquad \mathcal{H}_0 = \text{span}\{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$



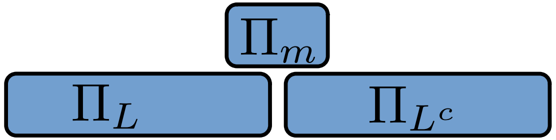
- ρ_{max} is the most "natural" state in $\mathcal{H}_0$ (zero temp' Gibbs state $\lim_{\beta\to\infty} \frac{1}{Z} e^{-\beta H}$)
- Holds for finite temperature Gibbs state $\rho_\beta = \frac{1}{Z} e^{-\beta H}$ we have:

$$I(L : L_c)_{\rho_\beta} = O(\beta \cdot |\partial L|)$$ (Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac '08)

- Holds in the commuting case:

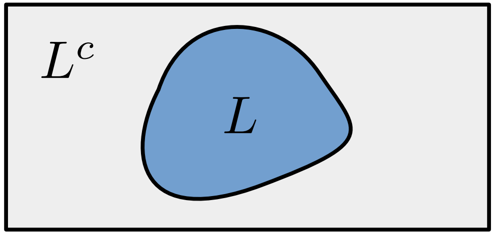
$$\rho_{max} = \lim_{\beta\to 0} e^{-\beta H} / \, \text{Tr}(\ldots) = \frac{1}{\text{Tr}(\Pi_0)}\Pi_0$$

$$\Pi_0 = \Pi_L \cdot \Pi_{L^c} \cdot \Pi_{mid}$$



Main result

$$H = \sum_i h_i \qquad \mathcal{H}_0 = \{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$

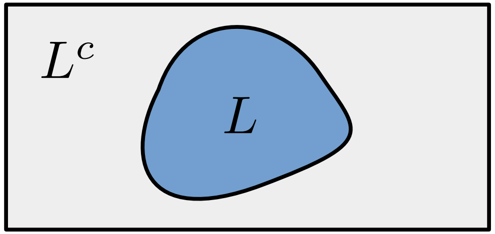


If there exists a “good” (D, Δ)-AGSP with $D^2 \cdot \Delta < 1$, then

$$I(L : L_c)_{\rho_{max}} \leq 2\log D + 24\log(|L|) + O(1)$$

Main result

$$H = \sum_i h_i \qquad \mathcal{H}_0 = \{|\psi_1\rangle, \dots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$



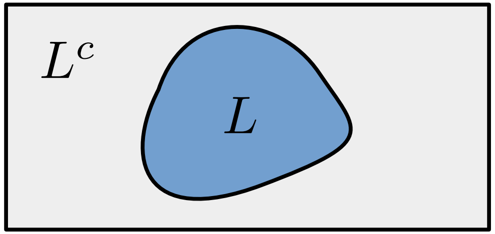
If there exists a “good” (D, Δ)-AGSP with $D^2 \cdot \Delta < 1$, then

$$I(L : L_c)_{\rho_{max}} \leq 2\log D + 24\log(|L|) + O(1)$$

★ Gives an area law with logarithmic corrections for systems with a good AGSP with $\log D = O(|\partial L|)$

Main result

$$H = \sum_i h_i \qquad \mathcal{H}_0 = \{|\psi_1\rangle, \ldots, |\psi_r\rangle\} \qquad \rho_{max} = \frac{1}{r}\sum_i |\psi_i\rangle\langle\psi_i|$$



If there exists a “good” (D, Δ)-AGSP with $D^2 \cdot \Delta < 1$, then

$$I(L : L_c)_{\rho_{max}} \leq 2\log D + 24\log(|L|) + O(1)$$

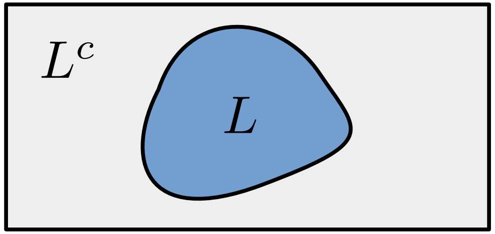
- Gives an area law with logarithmic corrections for systems with a good AGSP with $\log D = O(|\partial L|)$
- Holds for gapped 1D systems and for F.F. 2D systems with a local gap

Outline

- Background: the AGSP framework + min/max smooth entropy
- An (almost) formal statement of the main result
- Proof idea

Background I: AGSP framework

AGSP = **A**pproximate **G**round **S**pace **P**rojector



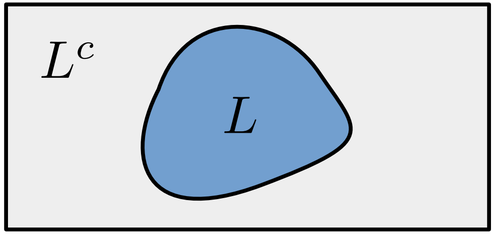
$$H = \sum_i h_i$$

Given a bi-partite $L \cup L^c$ system, an operator K is a (D, Δ)-AGSP if:

1. $K|\Omega\rangle = K^\dagger|\Omega\rangle = |\Omega\rangle$ for every $|\Omega\rangle \in \mathcal{H}_0$
2. $\| K|\Omega^\perp\rangle^2 \| \leq \Delta$ for every $|\Omega^\perp\rangle \in \mathcal{H}_0^\perp$
3. $K = \sum_{i=1}^{D} A_i \otimes B_i \qquad (\mathrm{SR}(K) \leq D)$

Background I: AGSP framework

AGSP = **A**pproximate **G**round **S**pace **P**rojector



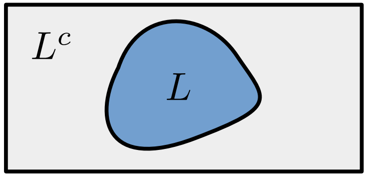
$$H = \sum_i h_i$$

Given a bi-partite $L \cup L^c$ system, an operator K is a (D, Δ)-AGSP if:

1. $K|\Omega\rangle = K^\dagger|\Omega\rangle = |\Omega\rangle$ for every $|\Omega\rangle \in \mathcal{H}_0$
2. $\| K|\Omega^\perp\rangle^2 \| \leq \Delta$ for every $|\Omega^\perp\rangle \in \mathcal{H}_0^\perp$
3. $K = \sum_{i=1}^{D} A_i \otimes B_i$ $\qquad$ $(\mathrm{SR}(K) \leq D)$

K is a **good AGSP** if $D^2 \cdot \Delta < 1/2$

The overlap bootstrapping lemma: (Arad et al '11)

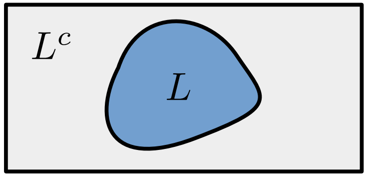


Suppose K is a good (D, Δ)-AGSP and the ground state $|\Omega\rangle$ is **unique**.

Then $\exists$ a product state $|L\rangle \otimes |L^c\rangle$ such that

$$|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$$

The overlap bootstrapping lemma: (Arad et al '11)



Suppose K is a good (D, Δ)-AGSP and the ground state $|\Omega\rangle$ is **unique**.
Then $\exists$ a product state $|L\rangle \otimes |L^c\rangle$ such that

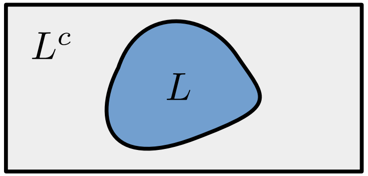
$$|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$$

Intuition: K approaches the g.s. faster than it creates entanglement

Good AGSP $\Longrightarrow$ Area law:

$$|\psi_0\rangle = |L\rangle \otimes |L^c\rangle \rightarrow K|\psi_0\rangle \rightarrow K^2|\psi_0\rangle \rightarrow \ldots \rightarrow |\Omega\rangle$$

The overlap bootstrapping lemma: (Arad et al ‘11)



Suppose K is a good (D, Δ)-AGSP and the ground state $|\Omega\rangle$ is **unique**.
Then $\exists$ a product state $|L\rangle \otimes |L^c\rangle$ such that

$$|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$$

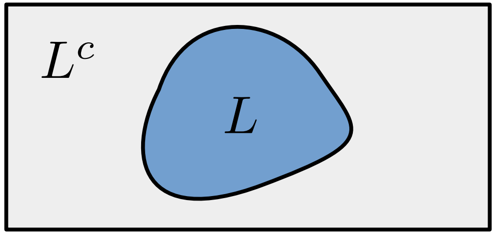
Intuition: K approaches the g.s. faster than it creates entanglement

Good AGSP $\Longrightarrow$ Area law:

$$|\psi_0\rangle = |L\rangle \otimes |L^c\rangle \to K|\psi_0\rangle \to K^2|\psi_0\rangle \to \ldots \to |\Omega\rangle$$

- Good AGSPs for 1D gapped systems (Arad et al ’13)
- Good AGSPs for F.F. 2D locally gapped systems (Anshu et al, ’21)

Proof of the overlap bootstrapping lemma

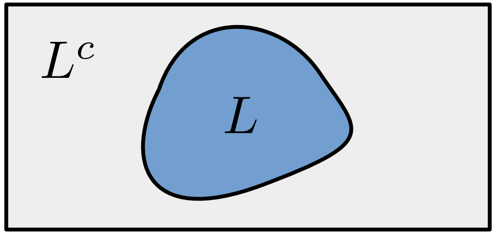


$|\Omega\rangle$ — ground state of H

K — A (D, Δ) AGSP with $D \cdot \Delta < 1/2$

We want to show: $\exists\ |L\rangle \otimes |L^c\rangle$ such that $|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$

Proof of the overlap bootstrapping lemma



$|\Omega\rangle$ — ground state of H

K — A (D, Δ) AGSP with $D \cdot \Delta < 1/2$

We want to show: $\exists\ |L\rangle \otimes |L^c\rangle$ such that $|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$

Proof:

Let $|L\rangle \otimes |L^c\rangle$ be the optimal product state state. Write:

$$|L\rangle \otimes |L^c\rangle = \mu|\Omega\rangle + (1-\mu^2)^{1/2}|\Omega^\perp\rangle$$

Apply K: $\quad |\phi\rangle = K|L\rangle \otimes |L_c\rangle = \sum_{i=1}^{D} \lambda_i |A_i\rangle \otimes |B_i\rangle = \mu|\Omega\rangle + (1-\mu^2)^{1/2} K|\Omega^\perp\rangle$

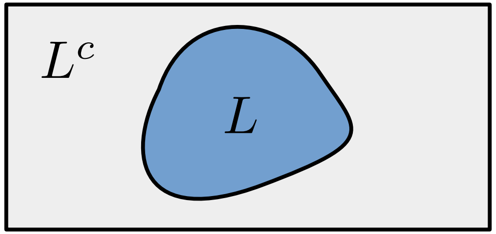
$\Longrightarrow \quad \mu = |\langle\Omega|\phi\rangle| = \sum_{i=1}^{D} \lambda_i |\langle\Omega|A_i, B_i\rangle| \leq \mu \sum_{i=1}^{D} \lambda_i$

$\Longrightarrow \quad \sum_{i=1}^{D} \lambda_i \geq 1$

$\Longrightarrow \quad$ By Cauchy-Schwartz: $D \cdot \sum_{i=1}^{D} \lambda_i^2 \geq 1$

Proof of the overlap bootstrapping lemma

$|\Omega\rangle$ — ground state of H



K — A (D, Δ) AGSP with $D \cdot \Delta < 1/2$

We want to show: $\exists\ |L\rangle \otimes |L^c\rangle$ such that $|\langle L, L^c|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$

Proof:

Let $|L\rangle \otimes |L^c\rangle$ be the optimal product state state. Write:

$$|L\rangle \otimes |L^c\rangle = \mu|\Omega\rangle + (1-\mu^2)^{1/2}|\Omega^\perp\rangle$$

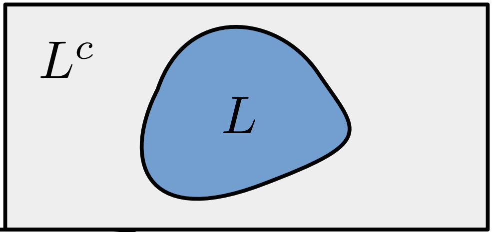
Apply K: $\quad |\phi\rangle = K|L\rangle \otimes |L_c\rangle = \sum_{i=1}^{D} \lambda_i |A_i\rangle \otimes |B_i\rangle = \mu|\Omega\rangle + (1-\mu^2)^{1/2} K|\Omega^\perp\rangle$

$\Longrightarrow$ $\quad \mu = |\langle\Omega|\phi\rangle| = \sum_{i=1}^{D} \lambda_i |\langle\Omega|A_i, B_i\rangle| \leq \mu \sum_{i=1}^{D} \lambda_i$

$\Longrightarrow$ $\quad \sum_{i=1}^{D} \lambda_i \geq 1$

$\Longrightarrow$ $\quad$ By Cauchy-Schwartz: $D \cdot \sum_{i=1}^{D} \lambda_i^2 \geq 1$ $\quad$ ($\|\phi\|^2 \leq \mu^2 + \Delta$)

Proof of the overlap bootstrapping lemma



$|\Omega\rangle$ — ground state of H

K — A (D, Δ) AGSP with $D \cdot \Delta < 1/2$

We want t

$$\Longrightarrow \quad D \cdot (\mu^2 + \Delta) \geq 1$$

$D \cdot \Delta < 1/2$

$$\Longrightarrow \quad \mu^2 \geq \frac{1}{D} - \Delta = \frac{1}{D}(1 - D \cdot \Delta) \geq \frac{1}{2D}$$

Proof:

Let $|L\rangle \otimes |L$

Apply K: $\quad)^{1/2} K|\Omega^{\perp}\rangle$

$$\Longrightarrow \quad \mu = |\langle \Omega|\phi\rangle| = \sum_{i=1}^{D} \lambda_i |\langle \Omega|A_i, B_i\rangle| \leq \mu \sum_{i=1}^{D} \lambda_i$$

$$\Longrightarrow \quad \sum_{i=1}^{D} \lambda_i \geq 1$$

$\|\phi\|^2 \leq \mu^2 + \Delta$

$\Longrightarrow$ By Cauchy-Schwartz: $D \cdot \sum_{i=1}^{D} \lambda_i^2 \geq 1$

Background II: I_{max} and I^{ϵ}_{max} mutual information

Assume $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$

Relative entropy: $D(\rho||\sigma) \stackrel{\text{def}}{=} \text{Tr}\, \rho(\log \rho - \log \sigma)$

Background II: I_{max} and I^{ϵ}_{max} mutual information

Assume $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

Relative entropy: $D(\rho||\sigma) \stackrel{\text{def}}{=} \operatorname{Tr} \rho(\log \rho - \log \sigma)$

Mutual information: $I(A:B)_\rho \stackrel{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A \otimes \sigma_B} D(\rho_{AB}||\sigma)$

Background II: I_{max} and I^{ϵ}_{max} mutual information

Assume $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

Relative entropy: $D(\rho||\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$

Mutual information: $I(A:B)_\rho \stackrel{\mathrm{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A\otimes\sigma_B} D(\rho_{AB}||\sigma)$

Min/Max relative entropy:

$$D_{min}(\rho||\sigma) \stackrel{\mathrm{def}}{=} -\log \mathrm{Tr}(\Pi_\rho \sigma) \qquad \Pi_\rho = \text{projection to } \mathrm{supp}(\rho)$$

Background II: I_{max} and I^{ϵ}_{max} mutual information

Assume $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

Relative entropy: $D(\rho||\sigma) \stackrel{\text{def}}{=} \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$

Mutual information: $I(A:B)_\rho \stackrel{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A\otimes\sigma_B} D(\rho_{AB}||\sigma)$

Min/Max relative entropy:

$D_{min}(\rho||\sigma) \stackrel{\text{def}}{=} -\log \mathrm{Tr}(\Pi_\rho \sigma)$ $\qquad$ $\Pi_\rho =$ projection to $\mathrm{supp}(\rho)$

$D_{max}(\rho||\sigma) \stackrel{\text{def}}{=} \log \min_t (t \in \mathbb{R}; \rho \preceq t\sigma)$

Intuition: $\rho \preceq (1+\epsilon)\sigma \implies \|\rho - \sigma\|_1 \leq 2\epsilon$

Background II: I_{max} and I^{ϵ}_{max} mutual information

Assume $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$

Relative entropy: $D(\rho||\sigma) \stackrel{\text{def}}{=} \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$

Mutual information: $I(A:B)_\rho \stackrel{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A\otimes\sigma_B} D(\rho_{AB}||\sigma)$

Min/Max relative entropy:

$D_{min}(\rho||\sigma) \stackrel{\text{def}}{=} -\log\mathrm{Tr}(\Pi_\rho\sigma)$ $\quad\quad$ $\Pi_\rho =$ projection to $\mathrm{supp}(\rho)$

$D_{max}(\rho||\sigma) \stackrel{\text{def}}{=} \log\min_t(t \in \mathbb{R}; \rho \preceq t\sigma)$

Intuition: $\rho \preceq (1+\epsilon)\sigma \implies \|\rho-\sigma\|_1 \leq 2\epsilon$

Fact:

$$D_{min}(\rho||\sigma) \leq D(\rho||\sigma) \leq D_{max}(\rho||\sigma)$$

We can use D_{min}, D_{max} to define I_{min}, I_{max}:

Recall: $$I(A:B)_\rho \stackrel{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A\otimes\sigma_B} D(\rho_{AB}||\sigma)$$

We can use D_{min}, D_{max} to define I_{min}, I_{max}:

Recall: $I(A:B)_\rho \stackrel{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = \min_{\sigma=\sigma_A\otimes\sigma_B} D(\rho_{AB}||\sigma)$

$$I_{min}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\sigma=\sigma_A\otimes\sigma_B} D_{min}(\rho||\sigma)$$

$$I_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\sigma=\sigma_A\otimes\sigma_B} D_{max}(\rho||\sigma)$$

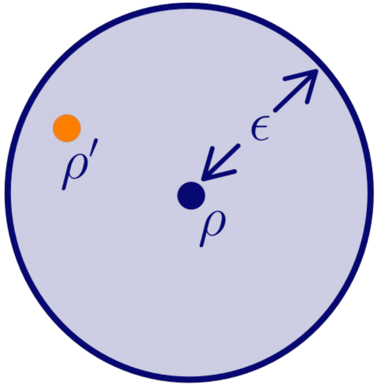
$\Longrightarrow$ For every state ρ and two regions A, B:

$$I_{min}(A:B)_\rho \leq I(A:B)_\rho \leq I_{max}(A:B)_\rho$$

$I_{max}(A:B)_\rho$ is easy to analyze combinatorially, but it depends **non-smoothly** on ρ.

Def: Max smooth mutual info:

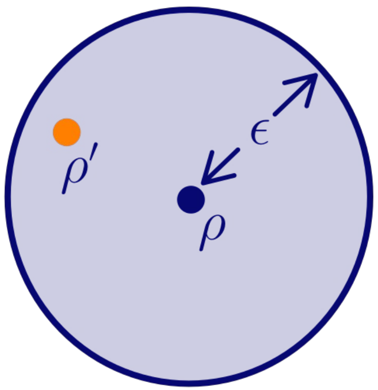
$$I^\epsilon_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\|\rho-\rho'\|_1 \le \epsilon} I_{max}(A:B)_{\rho'}$$



$I_{max}(A:B)_\rho$ is easy to analyze combinatorially, but it depends **non-smoothly** on ρ.

Def: Max smooth mutual info:

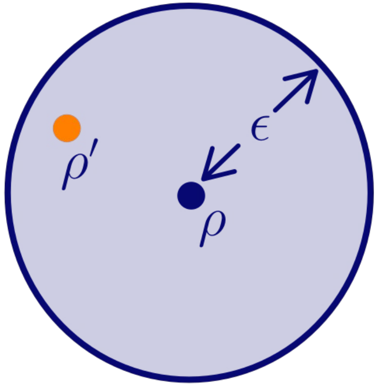
$$I^\epsilon_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\|\rho-\rho'\|_1 \leq \epsilon} I_{max}(A:B)_{\rho'}$$



Fact: $I(L:L^c)_\rho \leq I^\epsilon_{max}(L:L^c)_\rho + O(\epsilon \cdot |L|)$

$I_{max}(A:B)_\rho$ is easy to analyze combinatorially, but it depends **non-smoothly** on ρ.

Def: Max smooth mutual info:



$$I^\epsilon_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\|\rho-\rho'\|_1\leq\epsilon} I_{max}(A:B)_{\rho'}$$

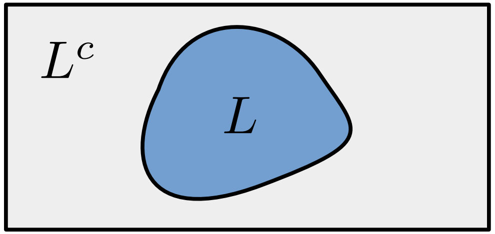
Fact: $I(L:L^c)_\rho \leq I^\epsilon_{max}(L:L^c)_\rho + O(\epsilon\cdot|L|)$

⟹ Bounding $I^\epsilon_{\text{max}}(L:L^c)$ can give us an immediate area-law (take $\epsilon \simeq 1/|L|$)

Main result (more formal)

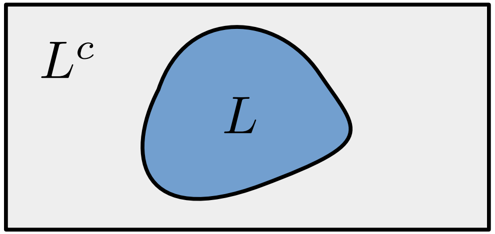
Thm 1:

Let $H = \sum_i h_i$ be a local Hamiltonian on a lattice with a bi-parition $L \cup L^c$, and assume that there exists a good AGSP ($D^2 \cdot \Delta < 1/2$). Then for every $\epsilon > 0$,



$$I^{\epsilon}_{max}(L : L^c)_{\rho_{max}} = O\Big(\log D + \log(|L|/\epsilon)\Big)$$

Main result (more formal)



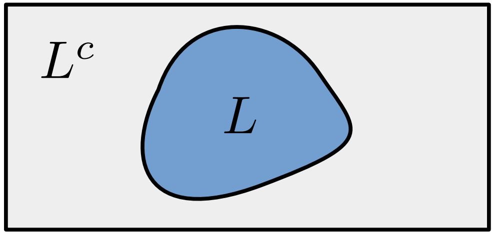
Thm 1:

Let $H = \sum_i h_i$ be a local Hamiltonian on a lattice with a bi-parition $L \cup L^c$, and assume that there exists a good AGSP ($D^2 \cdot \Delta < 1/2$). Then for every $\epsilon > 0$,

$$I^{\epsilon}_{max}(L : L^c)_{\rho_{max}} = O\Big(\log D + \log(|L|/\epsilon)\Big)$$

$\Longrightarrow$ Taking $\epsilon = 1/|L|$ gives us $I(L : L^c) = O\Big(\log D + \log(|L|)\Big)$

Main result (more formal)



Thm 1:

Let $H = \sum_i h_i$ be a local Hamiltonian on a lattice with a bi-parition $L \cup L^c$, and assume that there exists a good AGSP ($D^2 \cdot \Delta < 1/2$). Then for every $\epsilon > 0$,

$$I^{\epsilon}_{max}(L : L^c)_{\rho_{max}} = O\Big(\log D + \log(|L|/\epsilon)\Big)$$

$\Longrightarrow$ Taking $\epsilon = 1/|L|$ gives us $I(L : L^c) = O\Big(\log D + \log(|L|)\Big)$

Using known AGSP constructions:

★ $I(L : L^c)_{\rho_{max}} = O(|\log L|)$ in 1D

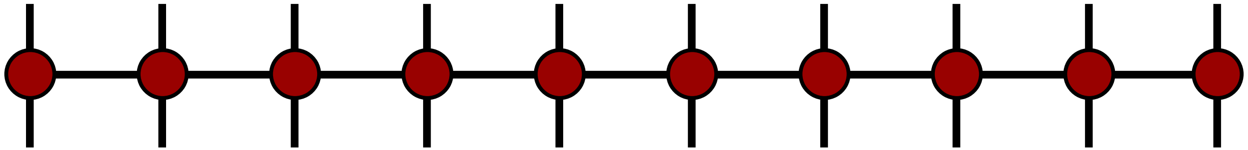
★ $I(L : L^c)_{\rho_{max}} = |\partial L|^{1+O(\log^{-1/5}|\partial L|)}$ in 2D

MPO approximation in 1D

Thm 2:

Let $H = \sum_i h_i$ be a 1D gapped system. Let ρ_{max} be its maximally-mixed ground state and $\epsilon > 0$. Then there exists an MPO Ψ with bond dimension $D = \mathrm{poly}(n/\epsilon)$ s.t.

$$\|\rho_{max} - \Psi\|_1 \leq \epsilon \qquad \rho_{max} = \frac{1}{\mathrm{Tr}\,\Pi_0}\Pi_0$$

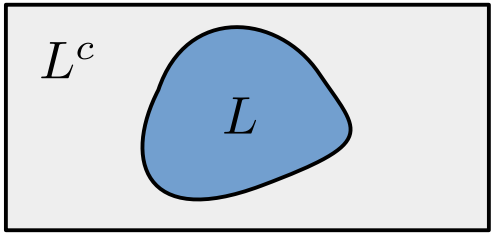


Ψ

Proof idea I:

$I_{max}(L : L_c)$ bootstrapping

Recall:

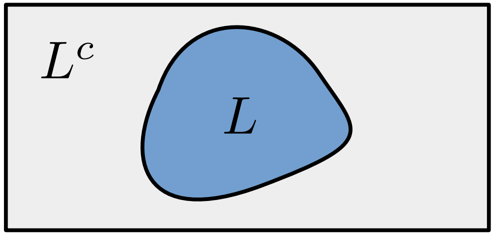


$$D_{max}(\rho||\sigma) \stackrel{\text{def}}{=} \log \min_t (t \in \mathbb{R}; \rho \preceq t\sigma)$$

$$I_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\sigma=\sigma_A \otimes \sigma_B} D_{max}(\rho||\sigma)$$

$$\Longrightarrow \qquad I_{max}(L:L^c)_\rho = \min_{t,\sigma_L,\sigma_{L^c}} \left\{ \log t | \rho \preceq t \cdot \sigma_L \otimes \sigma_{L^c} \right\}$$

Recall:



$$D_{max}(\rho||\sigma) \stackrel{\text{def}}{=} \log \min_t (t \in \mathbb{R}; \rho \preceq t\sigma)$$

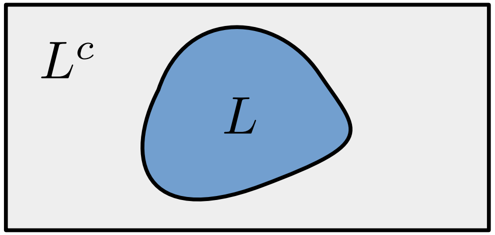
$$I_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\sigma=\sigma_A \otimes \sigma_B} D_{max}(\rho||\sigma)$$

$$\Longrightarrow \quad I_{max}(L:L^c)_\rho = \min_{t,\sigma_L,\sigma_{L^c}} \left\{ \log t | \rho \preceq t \cdot \sigma_L \otimes \sigma_{L^c} \right\}$$

- Start with the optimal $\sigma_L, \sigma_{L^c}, t$:

$$\rho_{max} \preceq t\sigma_L \otimes \sigma_{L^c}, \qquad t = 2^{I_{max}(L:L^c)}$$

Recall:



$$D_{max}(\rho||\sigma) \stackrel{\text{def}}{=} \log \min_t (t \in \mathbb{R}; \rho \preceq t\sigma)$$

$$I_{max}(A:B)_\rho \stackrel{\text{def}}{=} \min_{\sigma=\sigma_A\otimes\sigma_B} D_{max}(\rho||\sigma)$$

$$\Longrightarrow \qquad I_{max}(L:L^c)_\rho = \min_{t,\sigma_L,\sigma_{L^c}} \left\{ \log t | \rho \preceq t\cdot\sigma_L\otimes\sigma_{L^c} \right\}$$

- Start with the optimal $\sigma_L, \sigma_{L^c}, t$:

$$\rho_{max} \preceq t\sigma_L \otimes \sigma_{L^c}, \qquad t = 2^{I_{max}(L:L^c)}$$

- Apply a good AGSP:

normalized state

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t\cdot K\sigma_L\otimes\sigma_{L^c}K^\dagger = t\cdot \mathrm{Tr}(K\sigma_L\otimes\sigma_{L^c}K^\dagger)\cdot\tau$$

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t \cdot K\sigma_L \otimes \sigma_{L^c}K^\dagger = t \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau$$

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t \cdot K\sigma_L \otimes \sigma_{L^c}K^\dagger = t \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau$$

Fact I: $\mathrm{SR}(\tau) \leq D^2$ (because $\mathrm{SR}(K) = D$)

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t \cdot K\sigma_L \otimes \sigma_{L^c}K^\dagger = t \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau$$

Fact I: $\mathrm{SR}(\tau) \leq D^2$ (because $\mathrm{SR}(K) = D$)

Fact II: Because of that there exist τ_L, τ_{L^c} s.t., $\tau \preceq D^2 \tau_L \otimes \tau_{L^c}$

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t \cdot K\sigma_L \otimes \sigma_{L^c}K^\dagger = t \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau$$

Fact I: $\mathrm{SR}(\tau) \leq D^2$ (because $\mathrm{SR}(K) = D$)

Fact II: Because of that there exist τ_L, τ_{L^c} s.t., $\tau \preceq D^2 \tau_L \otimes \tau_{L^c}$

$$\Longrightarrow \quad \rho_{max} \preceq t \cdot D^2 \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau_L \otimes \tau_{L^c}$$

$$\rho_{max} = K\rho_{max}K^\dagger \preceq t \cdot K\sigma_L \otimes \sigma_{L^c}K^\dagger = t \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau$$

Fact I: $\mathrm{SR}(\tau) \leq D^2$ (because $\mathrm{SR}(K) = D$)

Fact II: Because of that there exist τ_L, τ_{L^c} s.t., $\tau \preceq D^2 \tau_L \otimes \tau_{L^c}$

$$\Longrightarrow \quad \rho_{max} \preceq t \cdot D^2 \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger) \cdot \tau_L \otimes \tau_{L^c}$$

- Since t was the **optimal** bound $\Longrightarrow$ $t \leq t \cdot D^2 \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger)$

$$\Longrightarrow \quad 1 \leq D^2 \cdot \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger)$$

$$\Longrightarrow \quad \boxed{\frac{1}{D^2} \leq \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c}K^\dagger)}$$

$$\frac{1}{D^2} \leq \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c} K^\dagger)$$

Expand: $\sigma_L \otimes \sigma_{L^c} = \Pi_0 \sigma_L \otimes \sigma_{L^c} + (\mathbb{1} - \Pi_0)\sigma_L \otimes \sigma_{L^c}$

$$\implies \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c} K^\dagger) = \mathrm{Tr}(K\Pi_0\sigma_L \otimes \sigma_{L^c} K^\dagger) + \underbrace{\mathrm{Tr}\left(K(\mathbb{1} - \Pi_0)\sigma_L \otimes \sigma_{L^c} K^\dagger\right)}_{\leq \Delta}$$

$$\leq \mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) + \Delta$$

$$\implies 1 \leq D^2 \cdot \left(\mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) + \Delta\right)$$

$D^2 \cdot \Delta \leq 1/2$

$$\implies \mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{D^2}(1 - D^2 \cdot \Delta) \geq \frac{1}{2D^2}$$

$$\frac{1}{D^2} \leq \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c} K^\dagger)$$

Expand: $\sigma_L \otimes \sigma_{L^c} = \Pi_0 \sigma_L \otimes \sigma_{L^c} + (\mathbb{1} - \Pi_0)\sigma_L \otimes \sigma_{L^c}$

$$\Longrightarrow \quad \mathrm{Tr}(K\sigma_L \otimes \sigma_{L^c} K^\dagger) = \mathrm{Tr}(K\Pi_0\sigma_L \otimes \sigma_{L^c} K^\dagger) + \mathrm{Tr}\left(\overbrace{K(\mathbb{1} - \Pi_0)\sigma_L \otimes \sigma_{L^c} K^\dagger}^{\leq \Delta}\right)$$

$$\leq \mathrm{Tr}(\Pi_0\sigma_L \otimes \sigma_{L^c}) + \Delta$$

$$\Longrightarrow \quad 1 \leq D^2 \cdot \left(\mathrm{Tr}(\Pi_0\sigma_L \otimes \sigma_{L^c}) + \Delta\right)$$

$$D^2 \cdot \Delta \leq 1/2$$

$$\Longrightarrow \quad \mathrm{Tr}(\Pi_0\sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{D^2}(1 - D^2 \cdot \Delta) \geq \frac{1}{2D^2}$$

We got:

$$\mathrm{Tr}(\Pi_0\sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{2D^2}, \qquad \text{where,} \qquad \Omega \preceq t \cdot \sigma_L \otimes \sigma_{L^c}, \qquad t = 2^{I(L:L^c)_\Omega}$$

We found a product state with a large overlap with the groundspace:

$$\mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{2D^2}$$

$$\rho_{max} = \frac{1}{\mathrm{Tr}(\Pi_0)} \Pi_0, \qquad \rho_{max} \preceq t \cdot \sigma_L \otimes \sigma_{L^c}$$

We found a product state with a large overlap with the groundspace:

$$\mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{2D^2}$$

$$\rho_{max} = \frac{1}{\mathrm{Tr}(\Pi_0)} \Pi_0, \qquad \rho_{max} \preceq t \cdot \sigma_L \otimes \sigma_{L^c}$$

But this is not good enough:

Applying K on $\sigma_L \otimes \sigma_{L^c}$ will **not** necessarily take us to ρ_{max} !

$$\sigma_L \otimes \sigma_{L^c} \;\rightarrow\; K \sigma_L \otimes \sigma_{L^c} K^\dagger \;\rightarrow\; K^2 \sigma_L \otimes \sigma_{L^c} (K^\dagger)^2 \;\rightarrow\; \ldots \;\rightarrow\; \text{?}$$

We found a product state with a large overlap with the groundspace:

$$\boxed{\mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{2D^2}} \qquad \rho_{max} = \frac{1}{\mathrm{Tr}(\Pi_0)} \Pi_0, \qquad \rho_{max} \preceq t \cdot \sigma_L \otimes \sigma_{L^c}$$

But this is not good enough:

Applying K on $\sigma_L \otimes \sigma_{L^c}$ will **not** necessarily take us to ρ_{max} !

$$\sigma_L \otimes \sigma_{L^c} \rightarrow K \sigma_L \otimes \sigma_{L^c} K^\dagger \rightarrow K^2 \sigma_L \otimes \sigma_{L^c} (K^\dagger)^2 \rightarrow \ldots \rightarrow \text{?}$$

In the language of entropies:

We found an upperbound $I_{min}(L : L_c) \leq \log(2D^2) \simeq \log(D)$, but we wanted to upperbound $I_{max}(L : L_c)$

We found a product state with a large overlap with the groundspace:

$$\boxed{\mathrm{Tr}(\Pi_0 \sigma_L \otimes \sigma_{L^c}) \geq \frac{1}{2D^2}} \qquad \rho_{max} = \frac{1}{\mathrm{Tr}(\Pi_0)} \Pi_0, \qquad \rho_{max} \preceq t \cdot \sigma_L \otimes \sigma_{L^c}$$

B

A

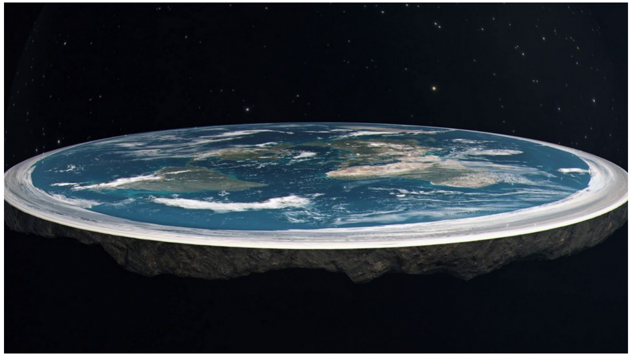
If only we could live in world where $I_{min}(L:L^c) = I_{max}(L:L^c)$ ...

$$\sigma_L \otimes \sigma_{L^c} \quad \rightarrow \quad K \sigma_L \otimes \sigma_{L^c} K^\dagger \quad \rightarrow \quad K^2 \sigma_L \otimes \sigma_{L^c} (K^\dagger)^2 \quad \rightarrow \quad \ldots \quad \rightarrow \quad ?$$

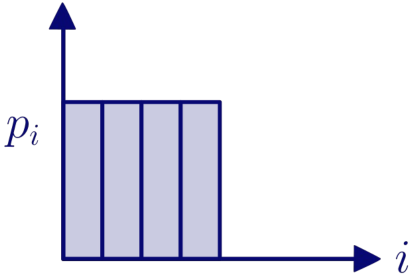
In the language of entropies:

We found an upperbound $I_{min}(L:L_c) \leq \log(2D^2) \simeq \log(D)$, but we wanted to upperbound $I_{max}(L:L_c)$

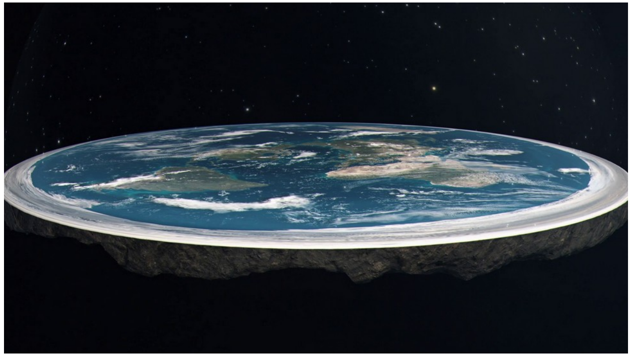
Flat states



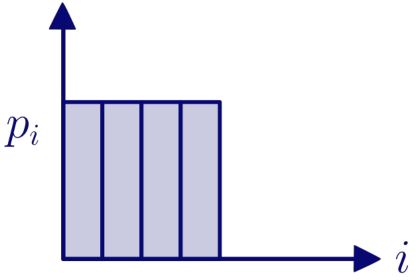
$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is a **flat state** when $p_i = const$



Flat states



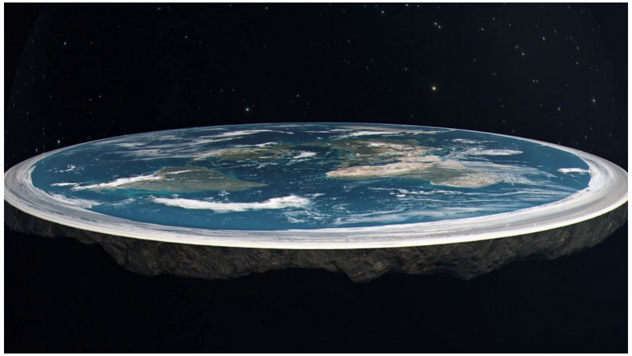
$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is a **flat state** when $p_i = const$



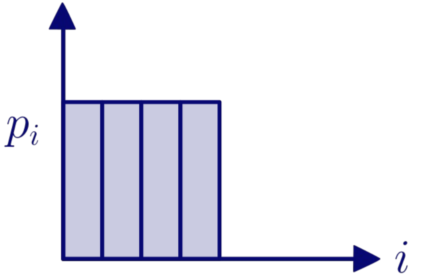
Properties:

- $\rho \propto \sqrt{\rho}$
- If ρ, σ are flat and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ then $D_{min}(\rho||\sigma) = D_{max}(\rho||\sigma) = \log(d_\sigma / d_\rho)$

Flat states



$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is a **flat state** when $p_i = const$



Properties:

- $\rho \propto \sqrt{\rho}$
- If ρ, σ are flat and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ then $D_{min}(\rho||\sigma) = D_{max}(\rho||\sigma) = \log(d_\sigma/d_\rho)$

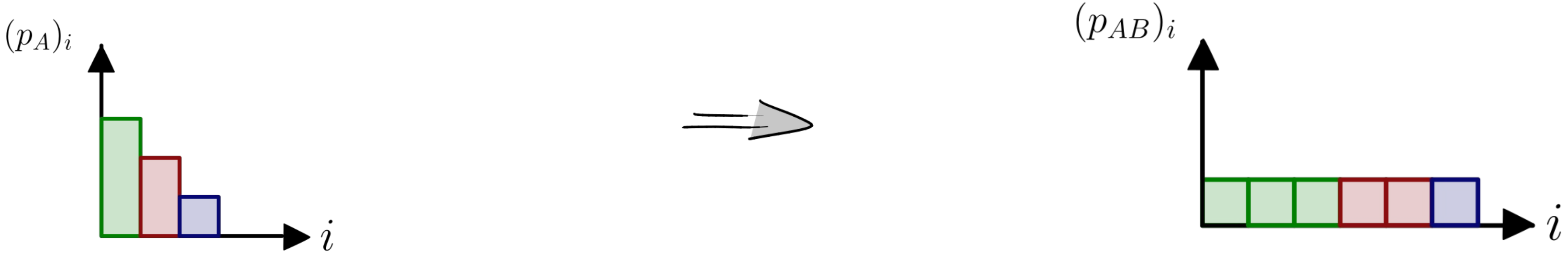
⟹ If ρ_{max} was flat, we would get:

$$I_{min}(L : L^c) = I(L : L^c) = I_{max}(L : L^c) = O(\log D)$$

Proof idea II:

The brothers extension

σ_L, σ_{L^c} are not flat, but we can **extend** them to another space where they **are** flat. This is done using the **Brothers Extension** (Anshu et al. ’16).

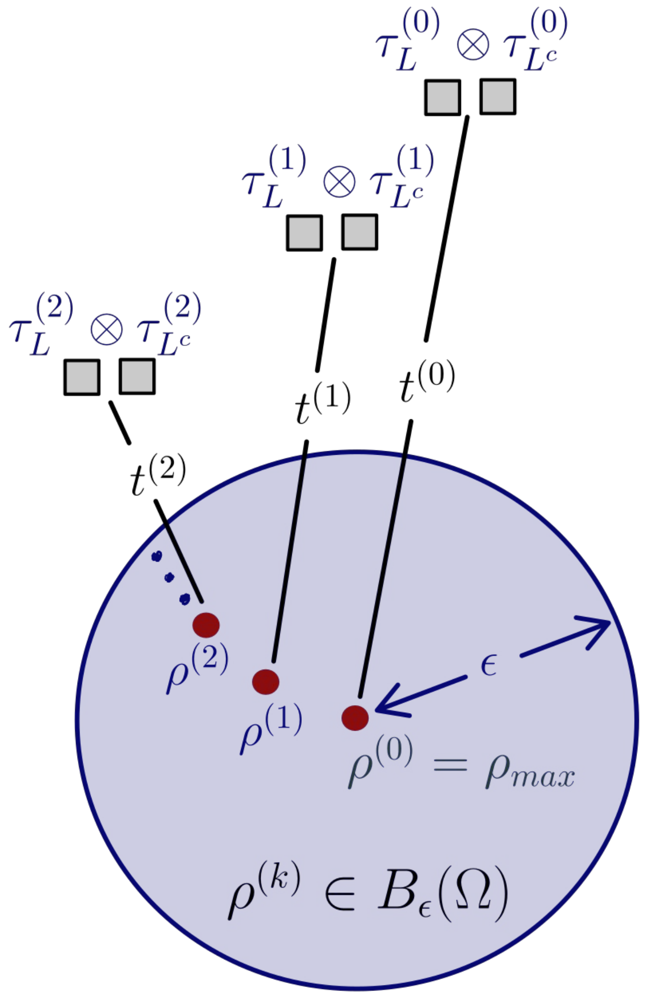


$$\rho_A = \frac{1}{6}\big(3|\psi_1\rangle\langle\psi_1| + 2|\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_1|\big)$$

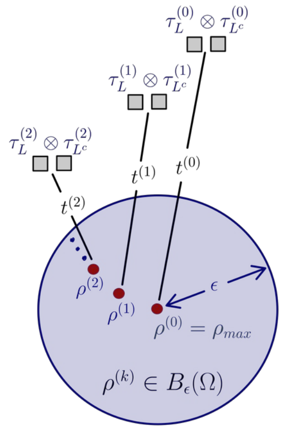
$$\rho_{AB} = \frac{1}{6}\Big[|\psi_1\rangle\langle\psi_1| \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|) + |\psi_2\rangle\langle\psi_2| \otimes (|1\rangle\langle 1| + |2\rangle\langle 2|) + |\psi_3\rangle\langle\psi_3| \otimes |1\rangle\langle 1|\Big]$$

Proof idea:

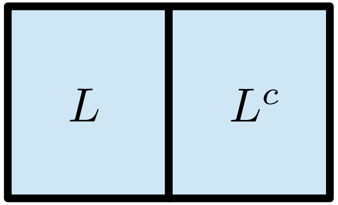
Use the brothers space extension to construct, a sequence of states in the vicinity of ρ_{max}, and for one of them find an upperbound to I_{max}



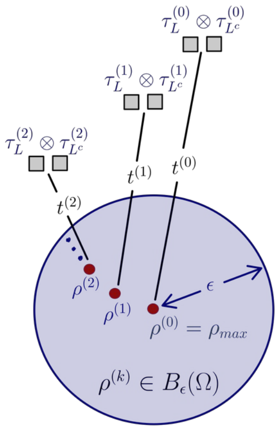
- $\rho_{max} = \rho^{(0)} \to \rho^{(1)} \to \rho^{(2)} \to \dots$
- $\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$ $\qquad t^{(0)} = 2^{I_{max}(L:L^c)_{\rho_{max}}} \leq 2^{2|L|}$



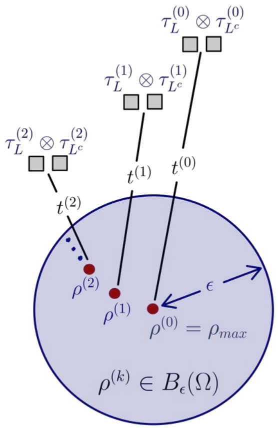
- $\rho_{max} = \rho^{(0)} \to \rho^{(1)} \to \rho^{(2)} \to \dots$
- $\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$ $\qquad t^{(0)} = 2^{I_{max}(L:L^c)_{\rho_{max}}} \leq 2^{2|L|}$

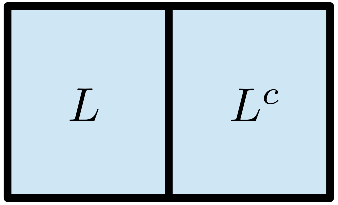


$$\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$$



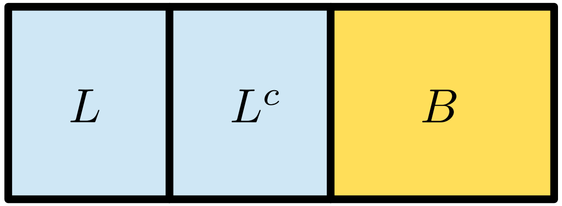
- $\rho_{max} = \rho^{(0)} \to \rho^{(1)} \to \rho^{(2)} \to \dots$
- $\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$ $\qquad t^{(0)} = 2^{I_{max}(L:L^c)_{\rho_{max}}} \leq 2^{2|L|}$





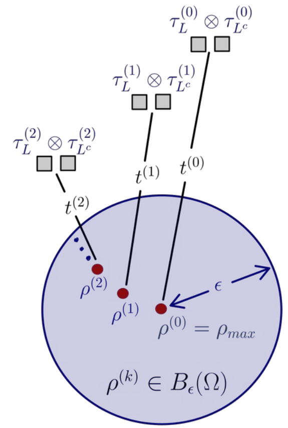
$$\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$$

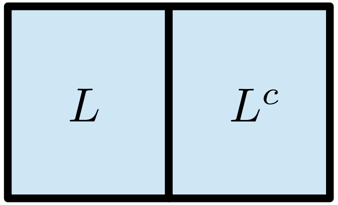
Brother extension



$$\rho_{LL^cB}^{(k)} \preceq t^{(k)} \cdot O(1/\delta^2) \cdot \tau_{LL^cB}^{(k)}$$

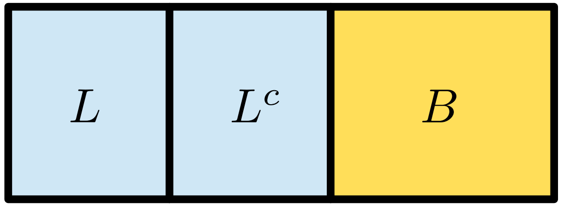
- $\rho_{max} = \rho^{(0)} \to \rho^{(1)} \to \rho^{(2)} \to \dots$
- $\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$ $\quad t^{(0)} = 2^{I_{max}(L:L^c)_{\rho_{max}}} \leq 2^{2|L|}$





$$\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$$

Brother extension

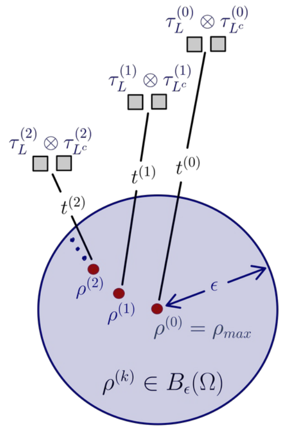


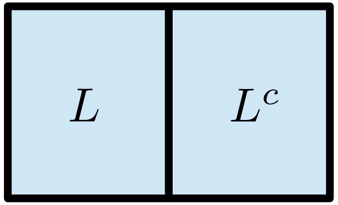
flat states

$$\rho_{LL^cB}^{(k)} \preceq t^{(k)} \cdot O(1/\delta^2) \cdot \tau_{LL^cB}^{(k)}$$

- τ_{LL^cB} has low SR
- δ is small param'

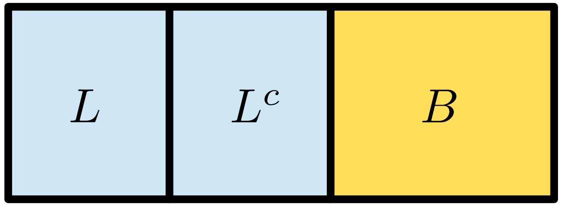
- $\rho_{max} = \rho^{(0)} \to \rho^{(1)} \to \rho^{(2)} \to \dots$
- $\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$ $\qquad t^{(0)} = 2^{I_{max}(L:L^c)_{\rho_{max}}} \leq 2^{2|L|}$





$$\rho^{(k)} \preceq t^{(k)} \cdot \tau_L^{(k)} \otimes \tau_{L^c}^{(k)}$$

Brother extension



flat states

$$\rho_{LL^cB}^{(k)} \preceq t^{(k)} \cdot O(1/\delta^2) \cdot \tau_{LL^cB}^{(k)}$$

- τ_{LL^cB} has low SR
- δ is small param'

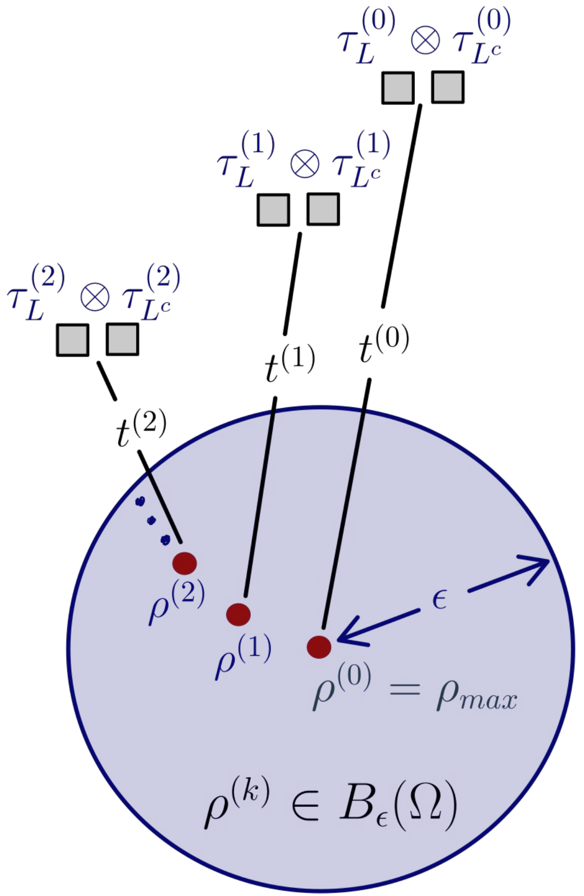
⟹ Extend the AGSP $K \to K_{LL^cB}$ and apply on both sides

⟹ Use the same argument as before (now we are flat so $I_{max} = I_{min}$)

⟹ Trace out B and obtain $\rho^{(k+1)} \preceq t^{(k+1)} \cdot \tau_L^{(k+1)} \otimes \tau_{L^c}^{(k+1)}$

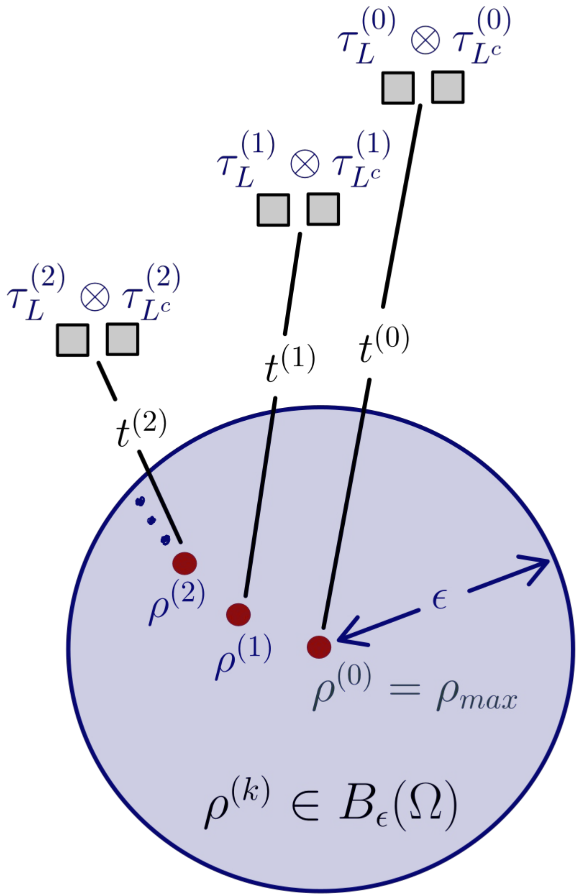
ρ' is inside B_ϵ

$$t^{(k+1)} = t^{(k)} \cdot \left(\frac{1}{4} + 2^{-I_{max}(L:L^c)_{\rho'}} f(\epsilon, D, |L|) \right)$$



ρ' is inside B_ϵ

$$t^{(k+1)} = t^{(k)} \cdot \left(\frac{1}{4} + 2^{-I_{max}(L:L^c)_{\rho'}} f(\epsilon, D, |L|)\right)$$



Analysis:

$$t^{(0)} = 2^{I_{max}(L:L^c)} \leq 2^{2|L|}$$

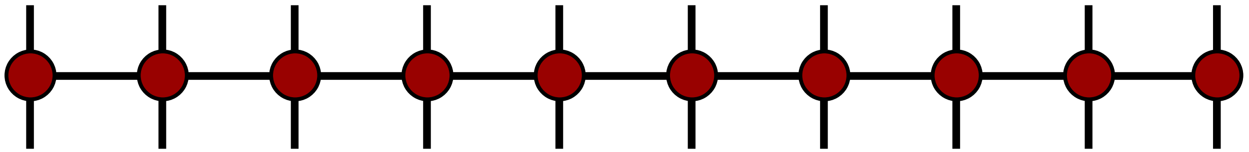
$\implies$ There must be $k \leq 2|L|$ for which $t^{(k+1)} \geq t^k/2$ (otherwize $t^{(k)} < 1$)

$\implies$ $t^{(k+1)} \geq t^k/2$

$\implies$ $$I_{max}(L : L_c)_{\rho'} = 2\log D + 12\log(|L|/\epsilon) + O(1)$$

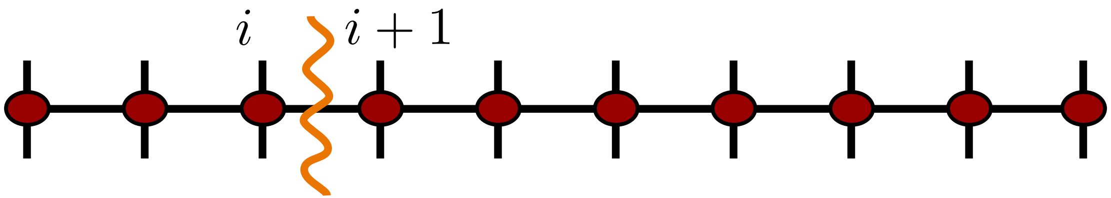
Proof of the MPO result

Ψ



$\|\rho_{max} - \Psi\|_1 \leq \epsilon \qquad \mathrm{SR}(\Psi) = \mathrm{poly}(n/\epsilon)$

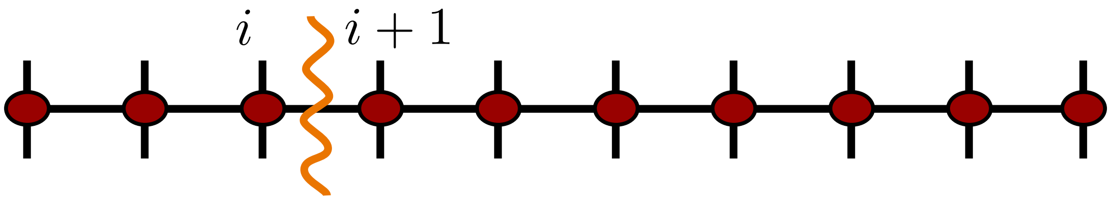
Outline:



1. Show a low SR approx' at every cut $i, i+1$:

$$\forall i, \quad \exists \rho_i \quad \text{s.t.,} \quad \|\rho_{max} - \rho_i\|_1 \leq \delta \quad \text{and} \quad \mathrm{SR}_{i,i+1}(\rho_i) \leq R$$

Outline:



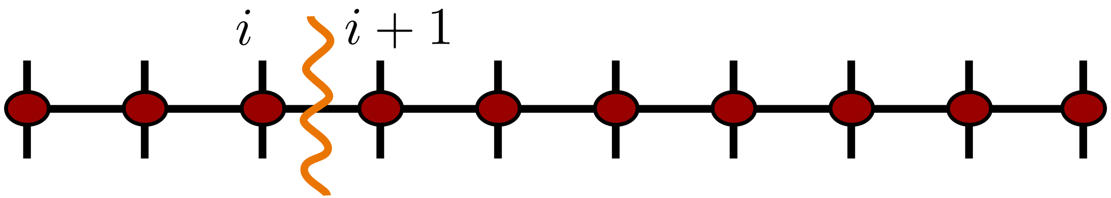
1. Show a low SR approx' at every cut $i, i+1$:

$$\forall i, \quad \exists \rho_i \quad \text{s.t.,} \quad \|\rho_{max} - \rho_i\|_1 \leq \delta \quad \text{and} \quad \mathrm{SR}_{i,i+1}(\rho_i) \leq R$$

2. Combine all these approx to a single MPO σ s.t.,

$$\mathrm{SR}_{i,i+1}(\sigma) \leq R, \qquad \|\rho_{max} - \sigma\|_1 \leq f(\delta, R, n)$$

Outline:



1. Show a low SR approx' at every cut $i, i+1$:

$$\forall i, \quad \exists \rho_i \quad \text{s.t.,} \quad \|\rho_{max} - \rho_i\|_1 \le \delta \quad \text{and} \quad \mathrm{SR}_{i,i+1}(\rho_i) \le R$$

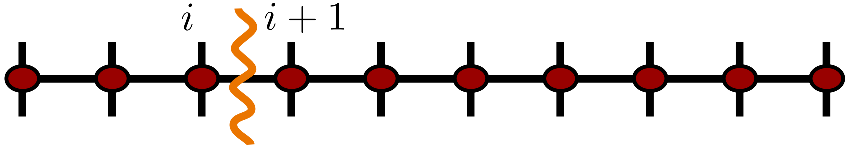
2. Combine all these approx to a single MPO σ s.t.,

$$\mathrm{SR}_{i,i+1}(\sigma) \le R, \qquad \|\rho_{max} - \sigma\|_1 \le f(\delta, R, n)$$

Problem:

Standard derivations use L_2 approximation, but we need an L_1 approximation.
This makes it non-trivial to combine the different cuts

Previous results:



Jarkovský et al., PRX 2020:

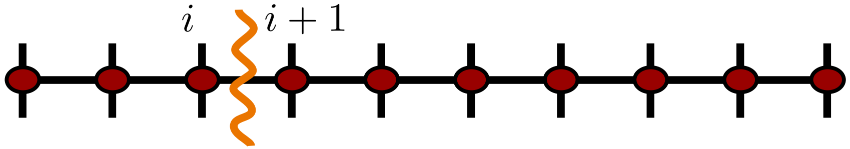
For any state ρ, if at every cut there's an approximation ρ_i such that:

$$\|\rho - \rho_i\| \leq \delta, \qquad \mathrm{SR}_{i,i+1}\,\rho_i \leq R$$

then these can be combined to an MPO σ with bond-dim R and

$$\|\rho - \sigma\|_1 \leq (2R+1)^{\log n}\delta$$

Previous results:



Jarkovský et al., PRX 2020:

For any state ρ, if at every cut there's an approximation ρ_i such that:

$$\|\rho - \rho_i\| \leq \delta, \qquad \mathrm{SR}_{i,i+1}\, \rho_i \leq R$$

then these can be combined to an MPO σ with bond-dim R and

$$\|\rho - \sigma\|_1 \leq (2R+1)^{\log n} \delta$$

$\Longrightarrow$ To use their result we need $\delta \ll (2R+1)^{\log n}$, which is probably too strong to show using our AGSP techniques.

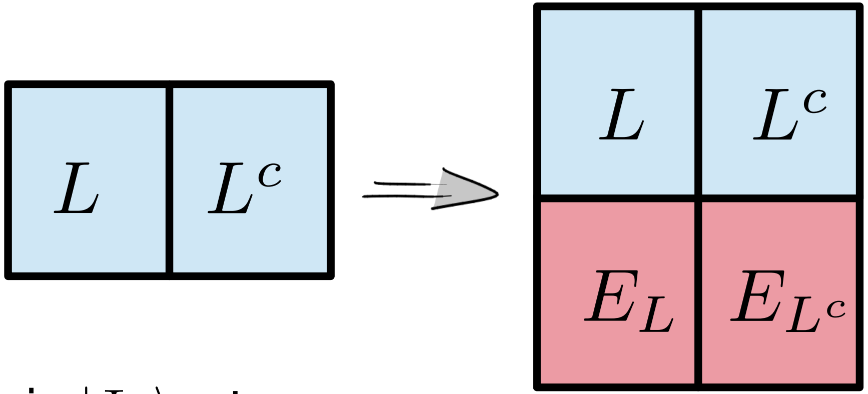
Our approach:

Find a **single** purification $|\Phi\rangle$ of ρ_{max} s.t.:

- $\operatorname{Tr}_E |\Phi\rangle\langle\Phi| = \rho_{max}$
- E can also be partitioned to n sequential subsystems and for every cut $(i, i+1)$ there is $|\Psi_i\rangle$ s.t.:

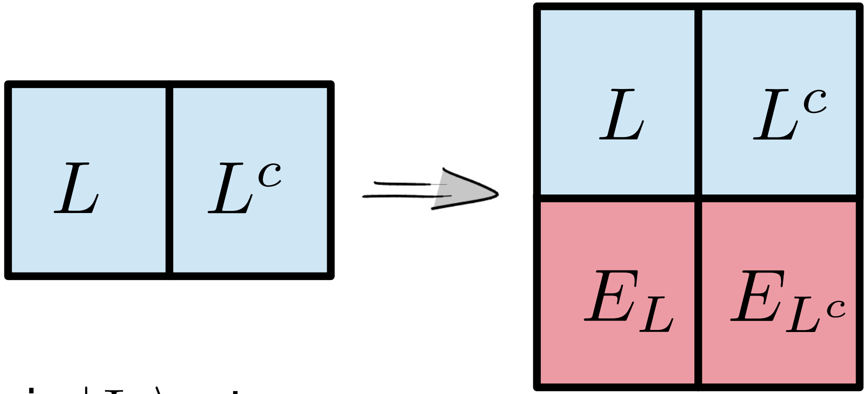
$$\| |\Phi\rangle - |\Psi_i\rangle \| \le \delta$$

$$\mathrm{SR}_{LE_L:L^cE_{L^c}}(|\Psi_i\rangle) = O\Big((n/\delta)^\kappa\Big) \qquad \kappa = O(1)$$



Our approach:

Find a **single** purification $|\Phi\rangle$ of ρ_{max} s.t.:



- $\operatorname{Tr}_E |\Phi\rangle\langle\Phi| = \rho_{max}$

- E can also be partitioned to n sequential subsystems and for every cut $(i, i+1)$ there is $|\Psi_i\rangle$ s.t.:

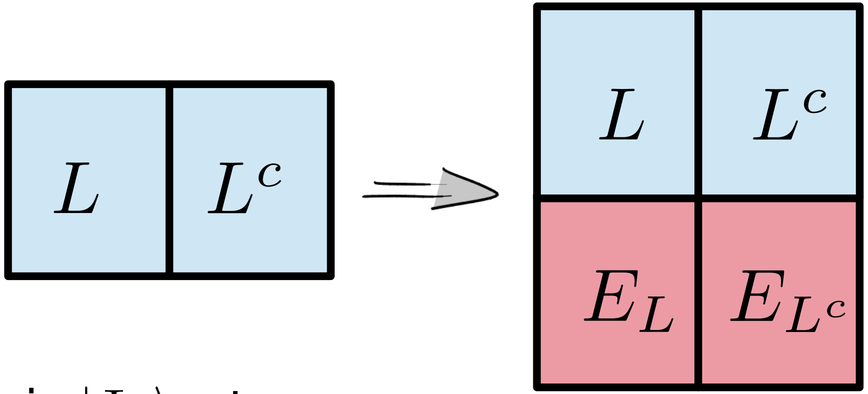
$$\| |\Phi\rangle - |\Psi_i\rangle \| \leq \delta$$

$$\mathrm{SR}_{LE_L:L^cE_{L^c}}(|\Psi_i\rangle) = O\Big((n/\delta)^\kappa\Big) \qquad \kappa = O(1)$$

⟹ We then use the regular MPS construcation for $|\Phi\rangle$ and trace out E to get the desired MPO result

Our approach:

Find a **single** purification $|\Phi\rangle$ of ρ_{max} s.t.:



- $\operatorname{Tr}_E |\Phi\rangle\langle\Phi| = \rho_{max}$

- E can also be partitioned to n sequential subsystems and for every cut $(i, i+1)$ there is $|\Psi_i\rangle$ s.t.:

$$\| |\Phi\rangle - |\Psi_i\rangle \| \leq \delta$$

$$\mathrm{SR}_{LE_L:L^cE_{L^c}}(|\Psi_i\rangle) = O\Big((n/\delta)^\kappa\Big) \qquad \kappa = O(1)$$

⟹ We then use the regular MPS construcation for $|\Phi\rangle$ and trace out E to get the desired MPO result

⟹ We get total error $n\delta$ for local $\mathrm{SR} \leq \Big((n/\delta)^\kappa\Big)$

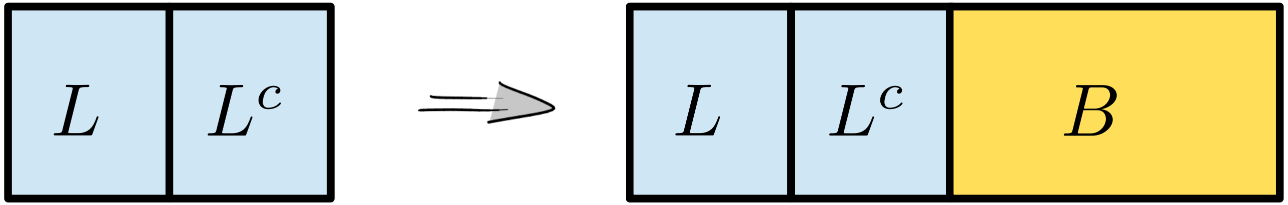
Some ideas in the proof:

- ρ_{max} is flat, and therefore $\rho_{max} \propto \sqrt{\rho_{max}}$

Some ideas in the proof:

- ρ_{max} is flat, and therefore $\rho_{max} \propto \sqrt{\rho_{max}}$

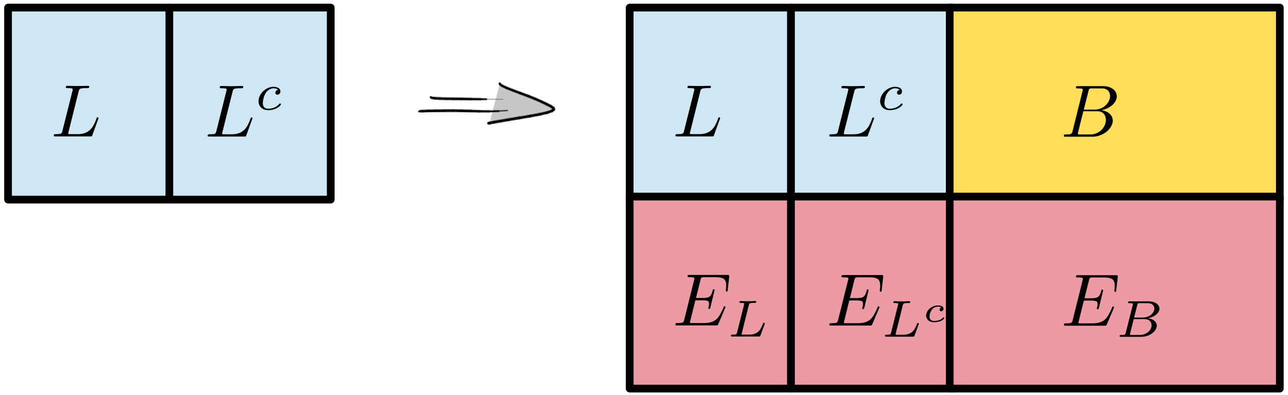
- We can repeat the AL proof, but now for $\sqrt{\rho_{max}}$



Some ideas in the proof:

- ρ_{max} is flat, and therefore $\rho_{max} \propto \sqrt{\rho_{max}}$

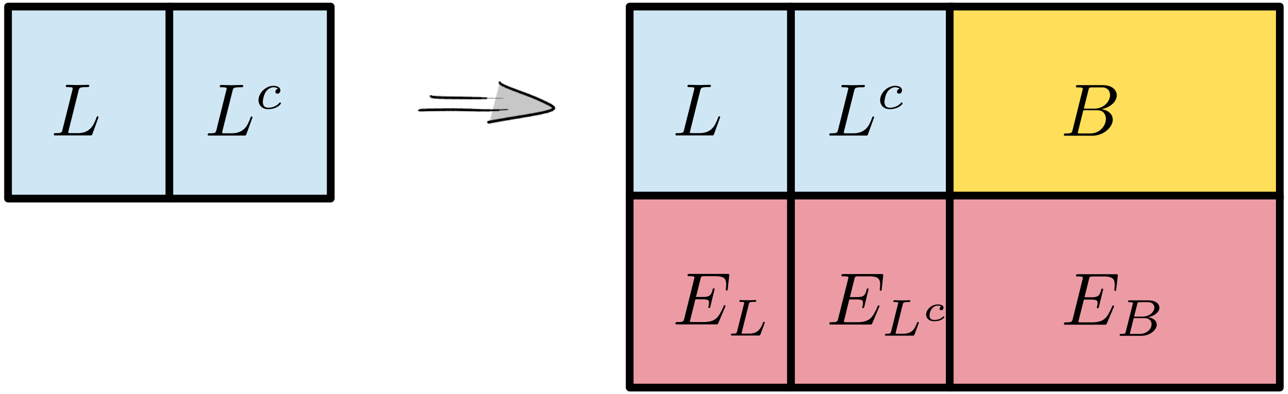
- We can repeat the AL proof, but now for $\sqrt{\rho_{max}}$



- **Vectorize** the brother-extended $\sqrt{\rho_{max}}$, $|i\rangle\langle j| \to |i\rangle \otimes |j\rangle$

Some ideas in the proof:

- ρ_{max} is flat, and therefore $\rho_{max} \propto \sqrt{\rho_{max}}$
- We can repeat the AL proof, but now for $\sqrt{\rho_{max}}$



- **Vectorize** the brother-extended $\sqrt{\rho_{max}}$, $\quad |i\rangle\langle j| \to |i\rangle \otimes |j\rangle$
- Use the fact that for every state ρ, the **vectorization** $|\sqrt{\rho}\rangle$ is a **purification** of ρ that preserves the SR structure

Summary

- Proved a **bootstrapping lemma** for showing an area-law for the maximally-mixed ground state, which is **independent** of the underlying degeneracy:

  Good AGSP $\Longrightarrow$ Area-law in $I(L : L^c)$

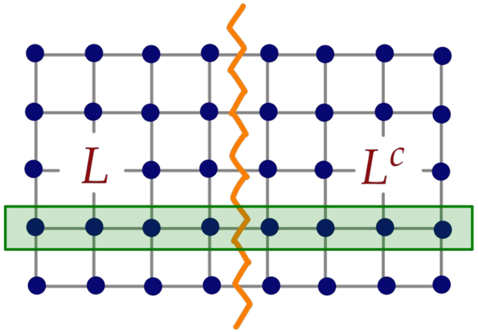
- Using existing AGSP constructions, proves an AL for gapped 1D systems and locally gapped, 2D, F.F. systems

- The proof also guarantees the existence of an efficient MPO description of ρ_{max} in 1D

- Uses rather involved tools from quantum information theory — smooth min/max entropies + brothers extension

Open questions

- Is there a simpler, combinatoric proof ?
- Can it be generalized to the case of a low, high density spectrum?
- Can it be used to show an area-law in $I(L : L_c)$ for Gibbs states at **any** temperature for gapped systems?
- Can it teach something non-trivial on Π_0? Can it be used to prove AL for higher dims?



Thank you!