# Extracting average properties of random spin chains with translationally invariant tensor networks

YITP workshop on tensor networks, Kyoto, August 2025

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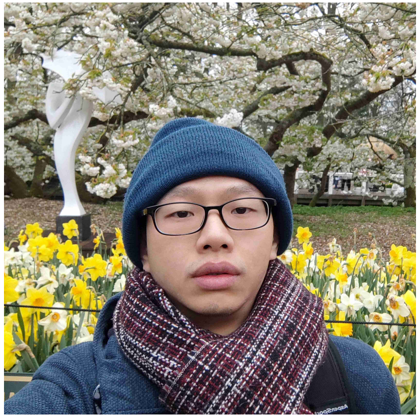

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This talk is about spin chains with quenched randomness, for example:

$$H = -\sum_n J_n \sigma_n^z \sigma_{n+1}^z + \sum_n h_n \sigma_n^x$$

$J_n, h_n$ are uncorrelated random variables with identical distributions on every site (statistical translation invariance)

We want to calculate $\left[ \langle \hat{O} \rangle_\beta \right]$

Disorder average

Thermal average

# Theory of the Random Transverse-Field Ising model (RTFIM)

PHYSICAL REVIEW VOLUME 176, NUMBER 2 10 DECEMBER 1968

**Theory of a Two-Dimensional Ising Model with Random Impurities. I. Thermodynamics**

BARRY M. MCCOY
*Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790*
AND
TAI TSUN WU*
*Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138*
(Received 9 July 1968)

Exact solution of 2D classical model with fixed disorder in one direction

VOLUME 69, NUMBER 3 PHYSICAL REVIEW LETTERS 20 JULY 1992

**Random Transverse Field Ising Spin Chains**

Daniel S. Fisher
*Physics Department, Harvard University, Cambridge, Massachusetts 02138*
(Received 30 April 1992)

A renormalization-group analysis of the spin-$\frac{1}{2}$ transverse field Ising model with quenched randomness is presented; it becomes *exact* asymptotically near the zero temperature ferromagnetic phase transition. The spontaneous magnetization is found to vanish with an exponent $\beta = \frac{1}{2}(3-\sqrt{5})$, while in the disordered phase the typical and average spin correlations are found to decay with different correlation lengths, which diverge with exponents $\tilde{\nu}=1$ and $\nu=2$, respectively.

PACS numbers: 75.10.Hk

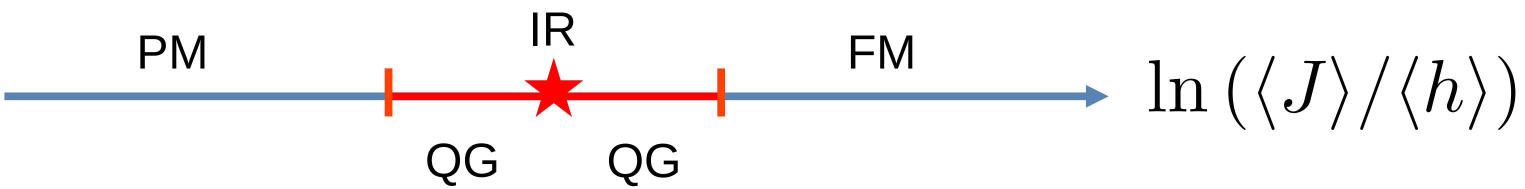
Strong disorder RG

## Theory of the Random Transverse-Field Ising model (RTFIM)

Main results:

1) Disorder is relevant at the critical point $\rightarrow$ Flow to Infinite Randomness (IR) fixed point

2) Rare regions are important $\rightarrow$ Quantum Griffiths (QG) physics




$$\chi \sim \frac{1}{T^{\alpha}}$$

**Main idea of our approach:**

$$\rho = \sum_{\{R_n\}} P[\{R_n\}]\rho_G[\{R_n\}]$$

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Sum over disorder samples

Gibbs state in fixed disorder sample

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$$\rho_G[\{R_n\}] = \frac{1}{Z[\{R_n\}]} e^{-\beta H[\{R_n\}]}$$

$$H[\{R_n\}] = \sum_n \tilde{h}_n[R_n]$$

Hamiltonian for fixed disorder sample

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Disorder-average thermal exp values: $\left[\langle \hat{O} \rangle_\beta\right] = tr\left(\hat{O}\rho\right)$

**Main idea of our approach:**

Sample probability

$$\rho = \sum_{\{R_n\}} P[\{R_n\}]\rho_G[\{R_n\}] \approx \rho_{MPO}$$

Sum over disorder samples

Gibbs state in fixed disorder sample

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Hamiltonian for fixed disorder sample

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**How we do it in practice:**

Consider discrete disorder, introduce ancilla disorder qudits $|R_n\rangle$ and define

$$H = \sum_n h_n \,, \qquad h_n = \sum_{R_n} \tilde{h}[R_n] \otimes |R_n\rangle\langle R_n|$$

This is a translationally invariant Hamiltonian, due to statistical translation invariance.

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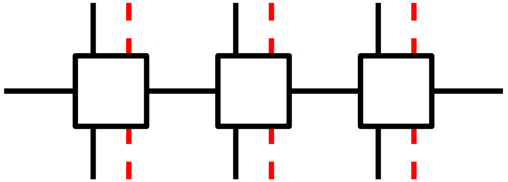
Our numerical algorithm works with

$$\tilde{\rho}(\tau) = N(\tau) e^{-\tau H}$$

diagonal normalization operator:

$$tr_\sigma\left(\tilde{\rho}(\tau)\right) = \mathbb{1}_R$$

$\tilde{\rho}(\tau) = \cdots$  $\cdots$

## Construction of $\tilde{\rho}(\tau)$

Step 1: Imaginary-time evolution with TEBD

$$\tilde{\rho}(\tau) = N(\tau)e^{-\tau H} \rightarrow N(\tau)e^{-(\tau+\delta\tau)H}$$

Standard, same as for clean systems due to the use of ancilla disorder qudits

**Construction of** $\tilde{\rho}(\tau)$

Step 2: Adjust normalization

$$N(\tau)e^{-(\tau+\delta\tau)H} \to N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H}$$

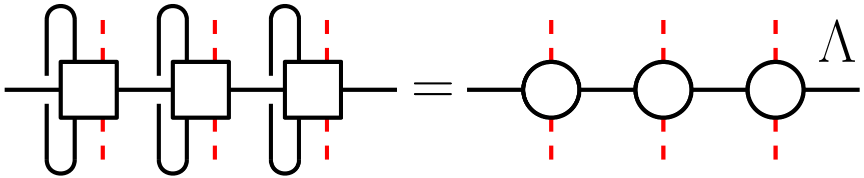
This we do approximately.

Step 2: Adjust normalization

$$N(\tau)e^{-(\tau+\delta\tau)H} \rightarrow N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H}$$

First define the diagonal MPO

$$\Lambda = tr_\sigma\left(N(\tau)e^{-(\tau+\delta\tau)H}\right)$$



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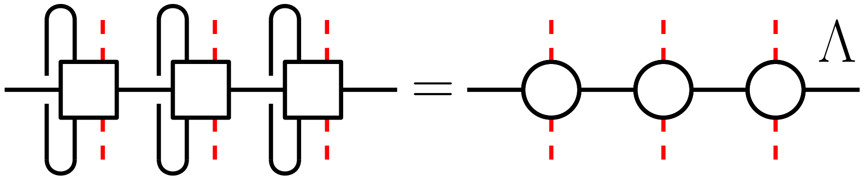


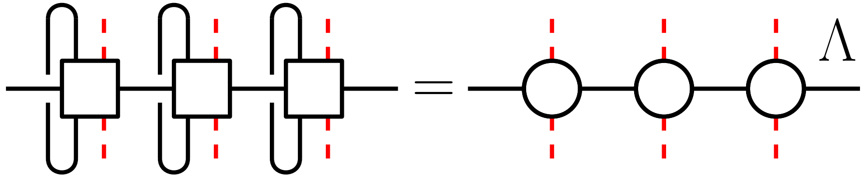

We want to find the inverse of this MPO:

$$\Lambda^{-1} = N(\tau+\delta\tau)N^{-1}(\tau)$$

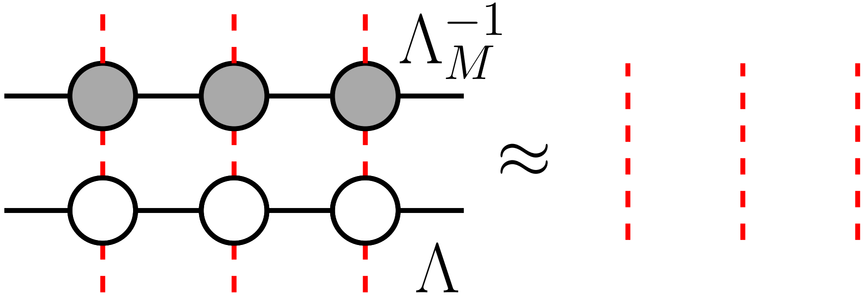
For small imaginary-time steps this diagonal operator is close to the identity

Step 2: Adjust normalization

$$N(\tau)e^{-(\tau+\delta\tau)H} \to N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H}$$

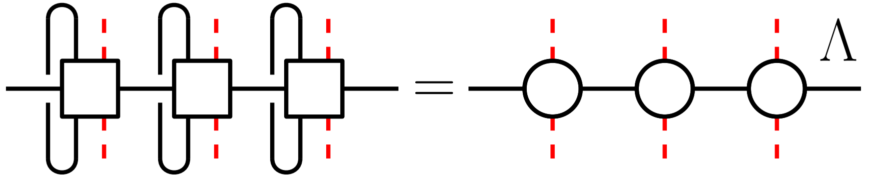



We will approximate the inverse as another MPO:

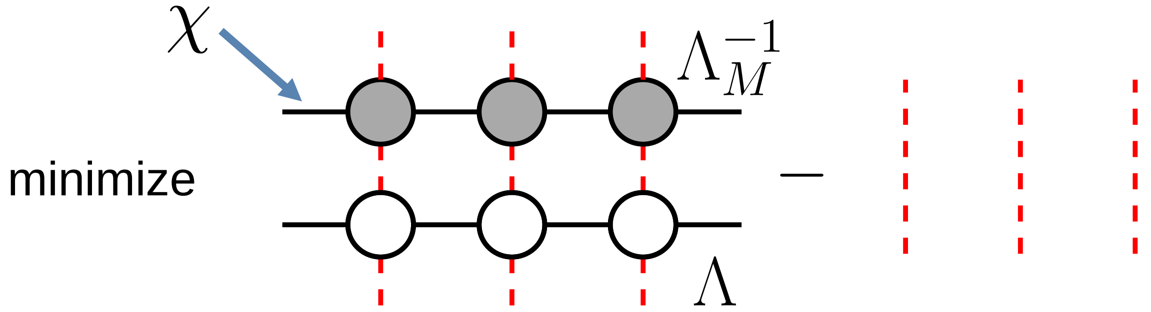


Step 2: Adjust normalization

$$N(\tau)e^{-(\tau+\delta\tau)H} \rightarrow N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H}$$




We can do this via a variational procedure:




Generalization of the VOMPS algorithm

Step 2: Adjust normalization

$$N(\tau)e^{-(\tau+\delta\tau)H} \to N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H}$$

From approximate inverse, obtain updated $\tilde{\rho}(\tau)$

$$\tilde{\rho}(\tau+\delta\tau) = N(\tau+\delta\tau)e^{-(\tau+\delta\tau)H} \approx \underbrace{\Lambda_M^{-1}}_{\text{MPO}}\underbrace{N(\tau)e^{-(\tau+\delta\tau)H}}_{\text{MPO}}$$

**Construction of** $\tilde{\rho}(\tau)$

Step 3: Truncation

After steps 1 and 2, the bond dimension of the MPO is $D \times D_T \times \chi$

- $D$: Original bond dimension
- $D_T$: Bond dimension of Trotter gates (TEBD step)
- $\chi$: Bond dimension of $\Lambda_M^{-1}$

We do a standard MPS-type truncation of the infinite-half Schmidt spectrum to bring this back to $D$

## The complete algorithm

Construct $\tilde{\rho}(\tau)$ via imaginary time-evolution:

1) TEBD
2) Adjust normalization
3) Truncate

When $\tau = \beta = T^{-1}$, trace out the ancilla disorder qudits

$$tr_R\left(\prod_n P[R_n]\tilde{\rho}(\beta)\right) \approx \rho = \sum_{\{R_n\}} P[\{R_n\}]\rho_G[\{R_n\}]$$

$$\left[\langle\hat{O}\rangle_\beta\right] = tr\left(\hat{O}\rho\right)$$

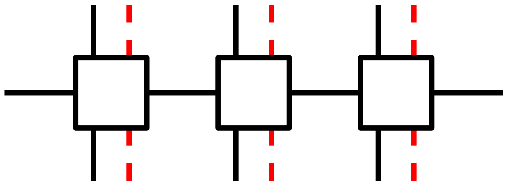
Benchmark results on RTFIM:

$$H = -\sum_n J_n \sigma_n^z \sigma_{n+1}^z + \sum_n h_n \sigma_n^x$$

We take a uniform distribution for both J and h (with three values each)

Near critical point at

$$\delta = \frac{\ln\langle h_n \rangle - \ln\langle J_n \rangle}{Var(\ln h_n) + Var(\ln J_n)} = 0$$

$\tilde{\rho}(\tau) = \cdots$  $\cdots$

$$\tilde{\rho}(\tau) = \sum_{\{\sigma\},\{\sigma'\}} \sum_{\{R\}} tr\left(\cdots A_{R_n}^{\sigma_n,\sigma'_n} A_{R_{n+1}}^{\sigma_{n+1},\sigma'_{n+1}} \cdots\right) \cdots |\sigma_n, R_n\rangle\langle\sigma'_n, R_n| \otimes |\sigma_{n+1}, R_{n+1}\rangle\langle\sigma'_{n+1}, R_{n+1}| \cdots$$

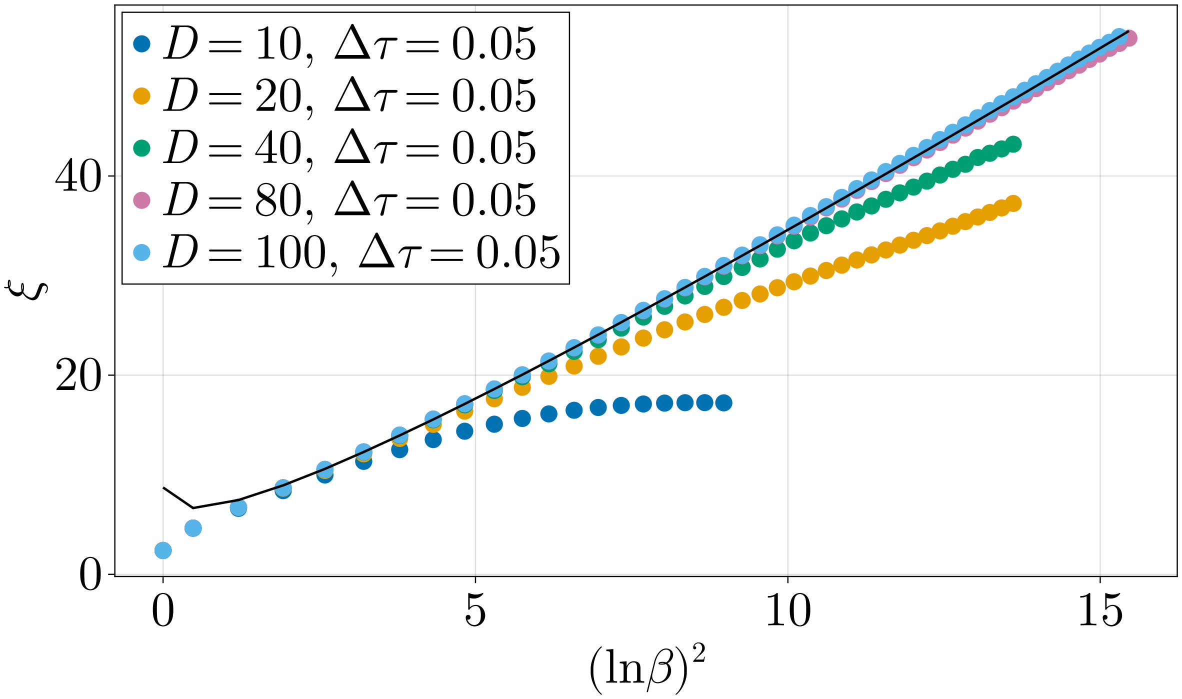
Definition of transfer matrix:

$$T = \sum_R \sum_\sigma P(R) A_R^{\sigma,\sigma}$$

Leading eigenvalues of transfer matrix determine correlation length:

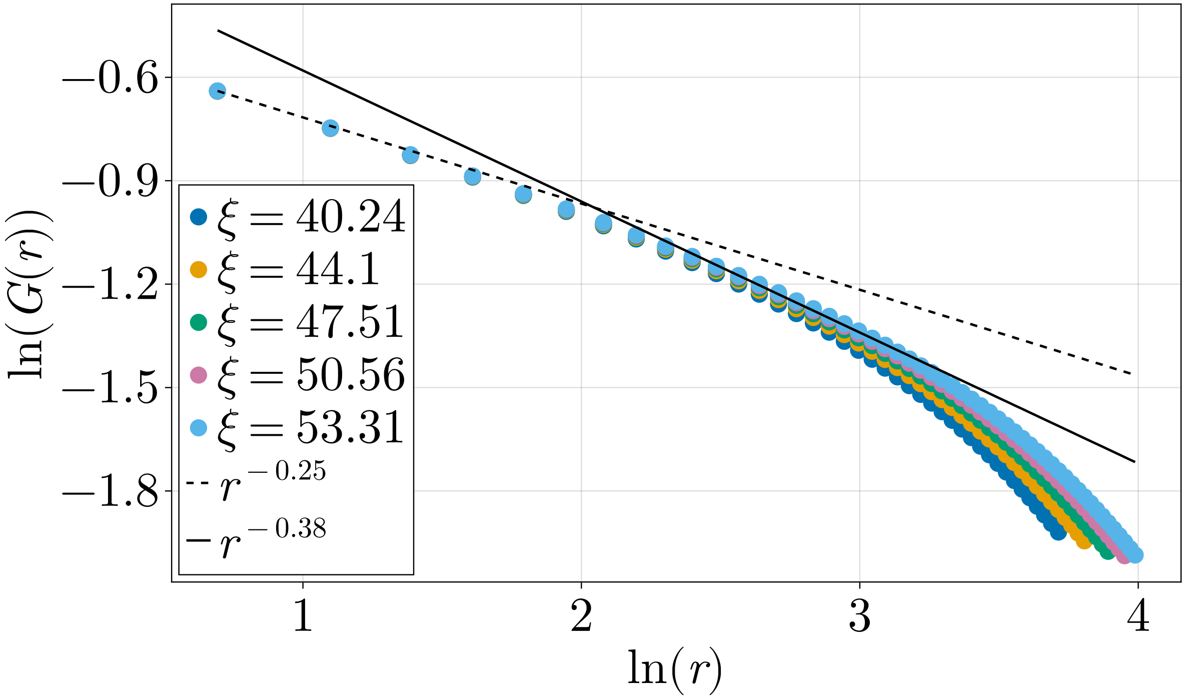
$$\xi = -\frac{1}{\ln(\lambda_2/\lambda_1)}$$

# Correlation length as function of temperature:



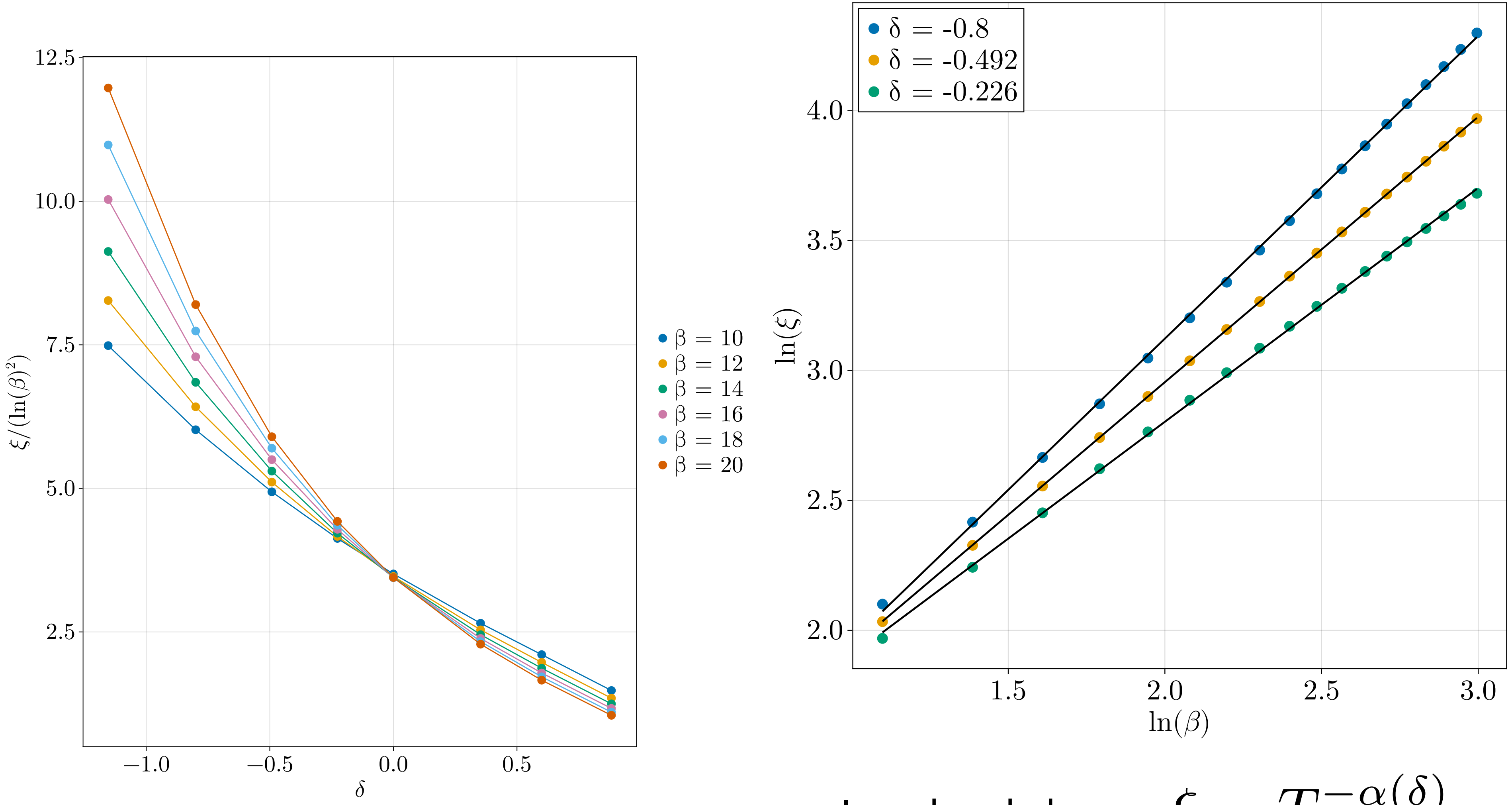

Infinite randomness fixed point: *Activated scaling* $\xi \sim [\ln(\beta)]^2$

Disorder-averaged spin-spin correlation function at criticality $G(r) = [\langle \sigma_r^z \sigma_0^z \rangle_\beta]$




Clean case: $G(r) \sim \frac{1}{r^{1/4}}$

Infinite randomness case: $G(r) \sim \frac{1}{r^{2-\phi}} \approx \frac{1}{r^{0.38}}$




In ordered phase: $\xi \sim T^{-\alpha(\delta)}$

# Future work

- Improve efficiency of the algorithm
- Explore other test cases (e.g. random singlet physics in disordered Heisenberg models)
- Construct zero temperature version
- Dynamics