Threefold Way for Typical Entanglement

Zongping Gong

Department of Applied Physics, University of Tokyo

Quantum entanglement

Multipartite entanglement in mixed states

$$\rho \neq \sum_{\alpha} p_{\alpha} \rho_1^{(\alpha)} \otimes \rho_2^{(\alpha)} \otimes \cdots \otimes \rho_N^{(\alpha)}$$

- Intrinsically difficult to diagnose in general

Bipartite entanglement in pure states

$$|\psi_{ab}\rangle \neq |\phi_a\rangle|\phi_b\rangle$$

- Fully characterized by entanglement spectrum $\mathrm{eig}\ \rho_{a/b}$

Quantum entanglement

Multipartite entanglement in mixed states

$$\rho \neq \sum_{\alpha} p_{\alpha} \rho_1^{(\alpha)} \otimes \rho_2^{(\alpha)} \otimes \cdots \otimes \rho_N^{(\alpha)}$$

- Intrinsically difficult to diagnose in general

Bipartite entanglement in pure states
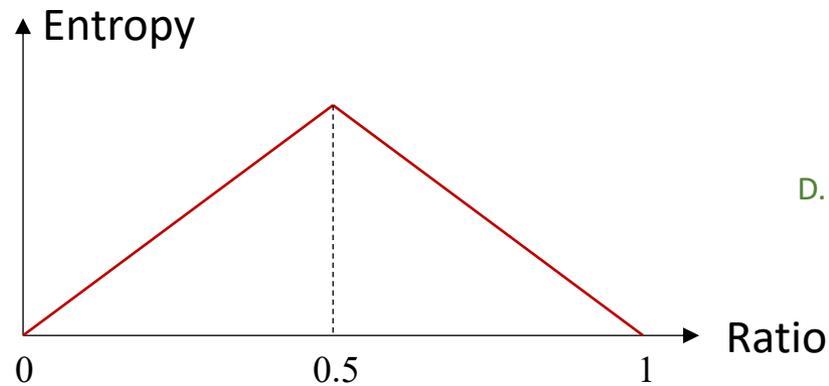
$$|\psi_{ab}\rangle \neq |\phi_a\rangle|\phi_b\rangle$$

- Fully characterized by entanglement spectrum $\mathrm{eig}\ \rho_{a/b}$

Main topic: Entanglement in random symmetric states
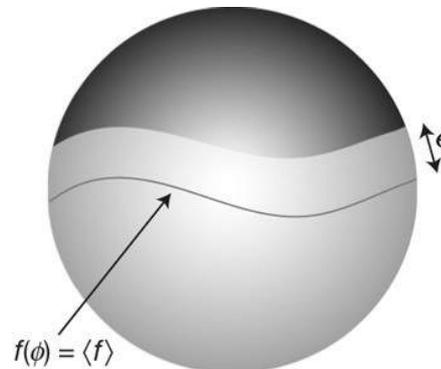
Why random states?

Black hole information paradox

- Page curve

Entropy

Ratio

0   0.5   1

D. Page, PRL **71**, 1291 (1993)

Quantum statistical mechanics

- Canonical typicality

$$\mathrm{Tr}_{\overline{S}}|\Psi\rangle\langle\Psi| \simeq \Omega_S$$

ϵ

$f(\phi) = \langle f \rangle$

S. Goldstein *et al.*, PRL **96**, 050403 (2006)
S. Popescu, A. J. Short, and A. Winter,
Nat. Phys. **2**, 754 (2006)

Why symmetry?

Random matrix theory P. J. Forrester, *Log-Gases and Random Matices* (2010)

- Reduced density matrix as Laguerre ensemble

$$\rho = WW^\dagger, \quad W \in \mathbb{F}^{m \times n}, \ W_{ij} \ \text{i.i.d.}$$

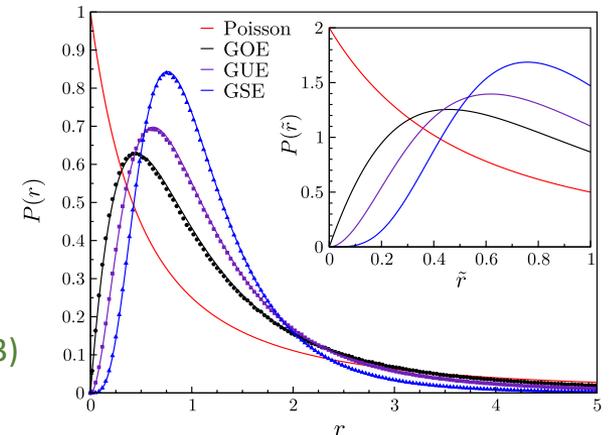|		
$\mathbb{F} = \mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
real/orthogonal	complex/unitary	quaternionic/symplectic
$\mathcal{T}^2 = 1$	No Time-Reversal Sym. (Page curve)	$\mathcal{T}^2 = -1$

- Cf. Hamiltonian as Gaussian ensemble

$$H = H^\dagger \in \mathbb{F}^{m \times m}, \quad H_{ij} = H_{ji}^* \ \text{i.i.d.}$$

Y. Y. Atas *et al.*, PRL **110**, 084101 (2013)

Page curve with TRS?

Laguerre orthogonal ensemble (LOE)

- Random states with normal TRS $\mathcal{T}^2 = 1$ ✓

Laguerre symplectic ensemble (LSE)

- Random states with anomalous TRS $\mathcal{T}^2 = -1$ ✗

  due to the Kramers' theorem: $\langle \psi | \mathcal{T} | \psi \rangle = 0$
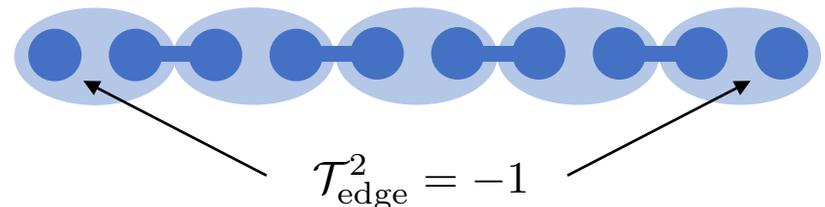
- Solution: ***Fractionalization*** of global ***normal*** TRS

$$\mathcal{T}_a^2 = -1_a \qquad \mathcal{T}_b^2 = -1_b \qquad \mathcal{T}^2 = (\mathcal{T}_a \otimes \mathcal{T}_b)^2 = 1$$

  Cf. SPT (Haldane) phases

  F. D. M. Haldane, PRL **50**, 1153 (1983)
  I. Affleck *et al.*, PRL **59**, 799 (1987)

$$\mathcal{T}_{\text{edge}}^2 = -1$$

LOE <-> LSE

Minimal setup: two qubits

$$|\Psi\rangle = \underset{\text{real}}{x_1}|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle \longrightarrow \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = W$$
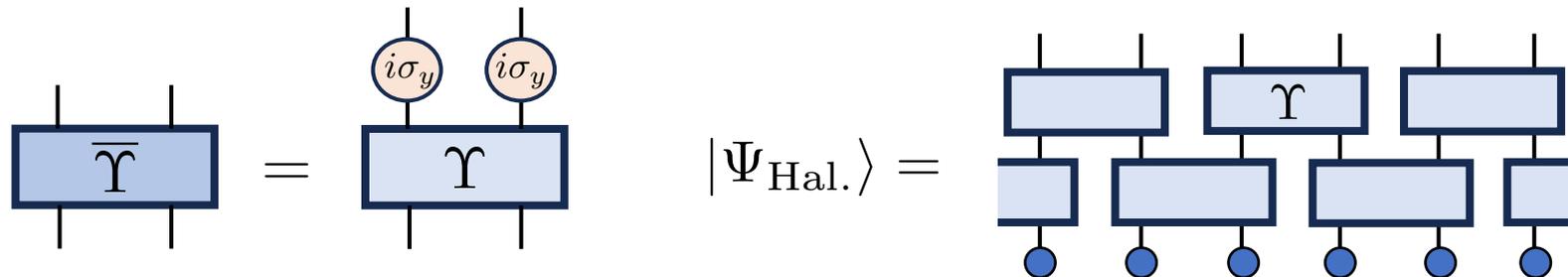
$$\Upsilon = \tfrac{1-i}{2}(1 - i\sigma_y \otimes \sigma_y)$$

$$\Upsilon|\Psi\rangle = \underset{\text{complex}}{q_1}|00\rangle + q_2|01\rangle - \overline{q}_2|10\rangle + \overline{q}_1|11\rangle \longrightarrow \begin{bmatrix} q_1 & q_2 \\ -\overline{q}_2 & \overline{q}_1 \end{bmatrix}$$

quaternion

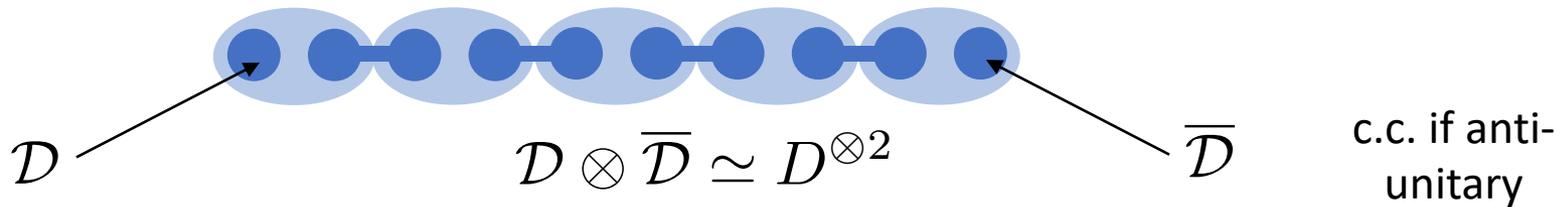$$q_1 = \tfrac{1-i}{2}(x_1 + ix_4) \qquad q_2 = \tfrac{1-i}{2}(x_2 - ix_3)$$

- Cf. Haldane quantum circuit   J. I. Cirac *et al.*, JSM (2017) 083105

Beyond Haldane & LSE?

SPT phases in 1D
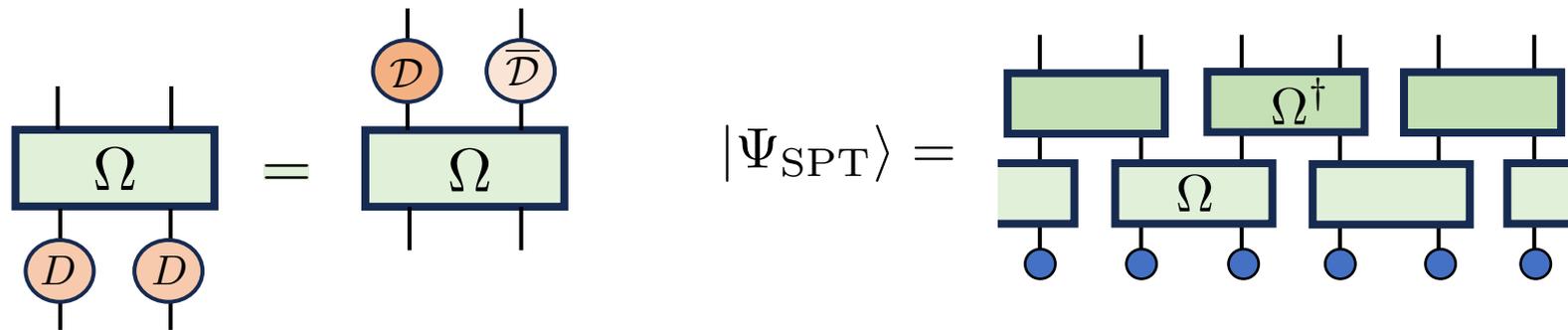F. Pollmann *et al.*, PRB **81**, 064439 (2010); X. Chen *et al.*, PRB **84**, 235128 (2011)

$$\mathcal{D} \otimes \overline{\mathcal{D}} \simeq D^{\otimes 2}$$

$\mathcal{D}$ $\overline{\mathcal{D}}$

c.c. if anti-unitary

Projective rep.:

$$\mathcal{D}(g)\mathcal{D}(h) = \underset{U(1)}{\underline{\omega(g,h)}}\mathcal{D}(gh)$$

Cocycle: $\omega(g,h)\omega(gh,k) = \omega(g,hk)\omega^g(h,k)$

Cohomology: $[\omega]: \ \omega(g,h) \sim \dfrac{\beta(gh)}{\beta(g)\beta^g(h)}\omega(g,h)$

- SPT quantum circuit (unitary symm.)   ZG *et al.*, PRL **124**, 100402 (2020)

$$\Omega = \Omega$$

$$|\Psi_{\mathrm{SPT}}\rangle =$$

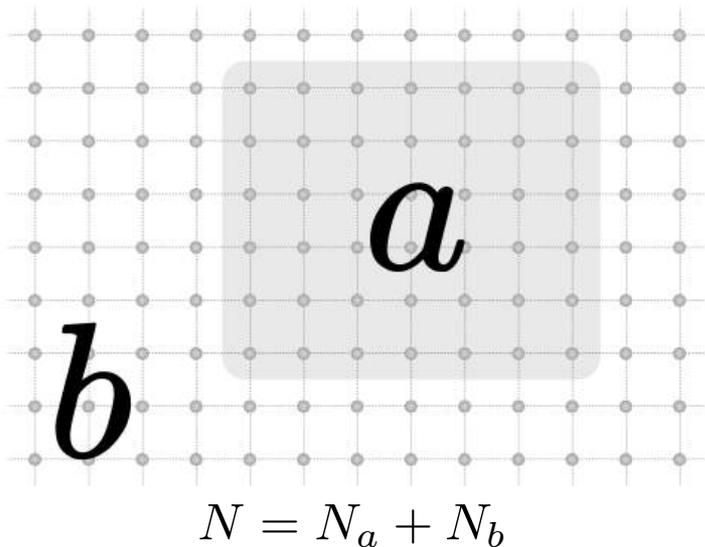**Q: Incorporate general "symm. frac."? New Ensembles?**

Setup

Quantum spin lattice

$$\mathcal{H} = (\mathbb{C}^{|G_0|})^{\otimes N} \qquad \mathbb{C}^{|G_0|} = \mathrm{span}\{|g\rangle : g \in G_0\}$$

Unitary symm. (regular): $D(g)|h\rangle = |gh\rangle$ TRS: $D(t)|g\rangle = |\tilde{g}\rangle$

Symmetric random state

$$D(g)^{\otimes N}|\Psi_{ab}\rangle = |\Psi_{ab}\rangle, \quad g \in G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}} \text{ or } G = G_0$$

$$N = N_a + N_b$$

Random sampling within:

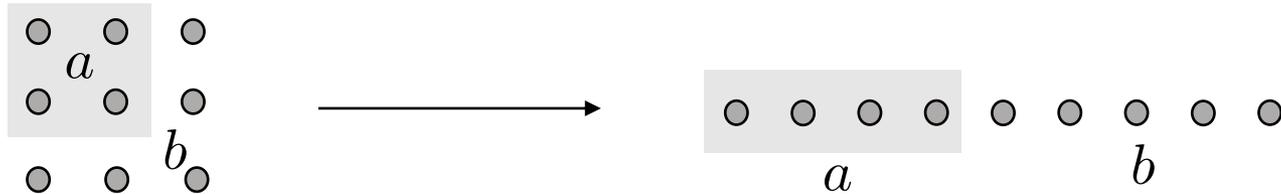$$P = \frac{1}{|G_0|} \sum_{g \in G_0} D(g)^{\otimes N}$$

TRS further requires:
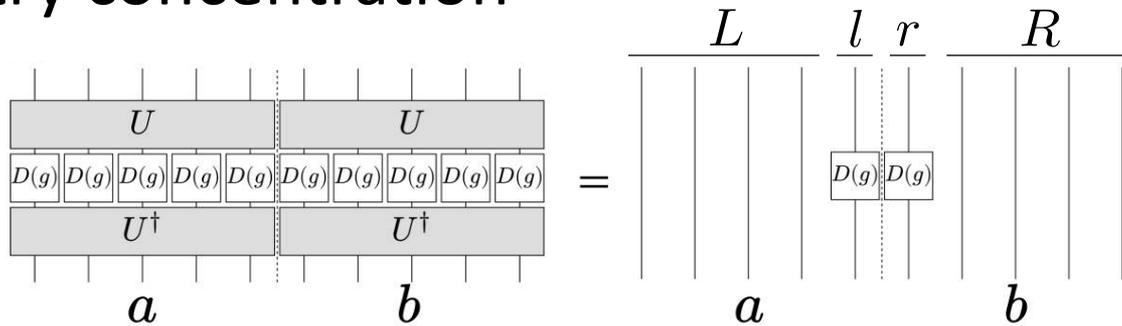
$$D(t)^{\otimes N}|\Psi_{ab}\rangle = |\Psi_{ab}\rangle$$

Target: $\rho_a = \mathrm{Tr}_b|\Psi_{ab}\rangle\langle\Psi_{ab}|$

Simplification

Lattice Reconfiguration

Symmetry concentration

Feasible for regular rep., entanglement spectrum unchanged

Symmetric random state $1_L \otimes D(g)^{\otimes 2} \otimes 1_R |\Psi\rangle = |\Psi\rangle$

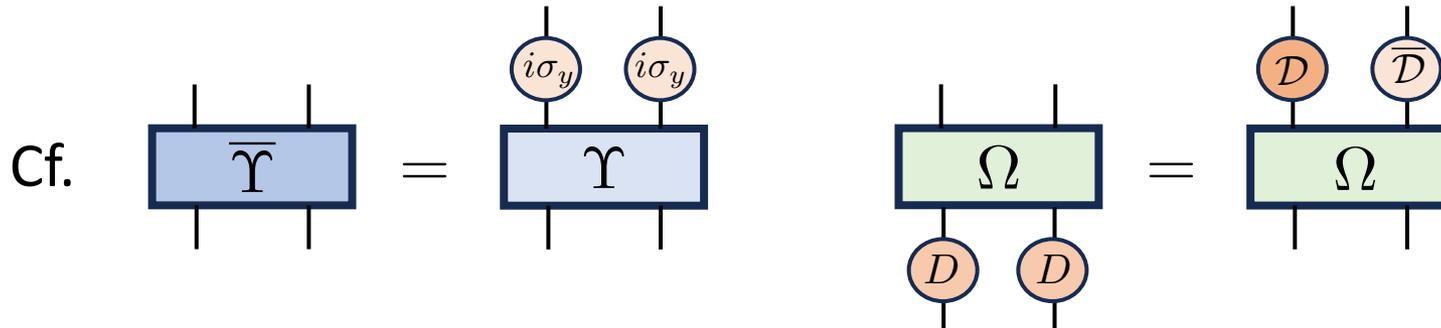$$|\Psi\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{L,g_l,g_r,R} c_{L,g_r^{-1}g_l,R} |Lg_l\rangle |g_r R\rangle \qquad c_{L,\tilde{g},R} = \overline{c_{L,g,R}}$$

i.i.d. $\mathbb{C}$ If TRS imposed

Symmetry Fractionalization

Unitary and anti-unitary parts are decoupled

$$\omega \text{ of } G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}} \quad \Leftrightarrow \quad \begin{cases} \omega \text{ of } G_0 \text{ subject to } \omega(g,h)\omega(\tilde{g},\tilde{h}) = 1 \\ \omega(t,t) = \pm 1 \end{cases}$$
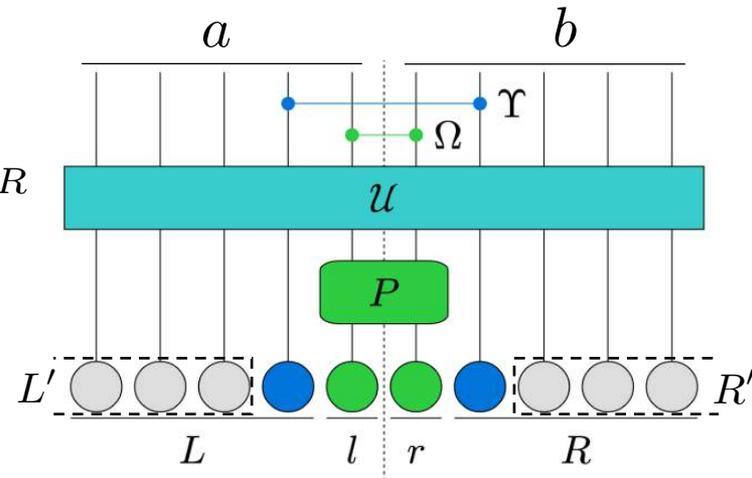
Cf.

Matrix elements

$$\rho_a = WW^\dagger, \quad W_{ab} \propto \omega(g_r, g_r^{-1}g_l)c_{L,g_r^{-1}g_l,R}$$

$$c_{L,\tilde{g},R} = \overline{c_{L,g,R}} \quad \text{if TRS imposed}$$

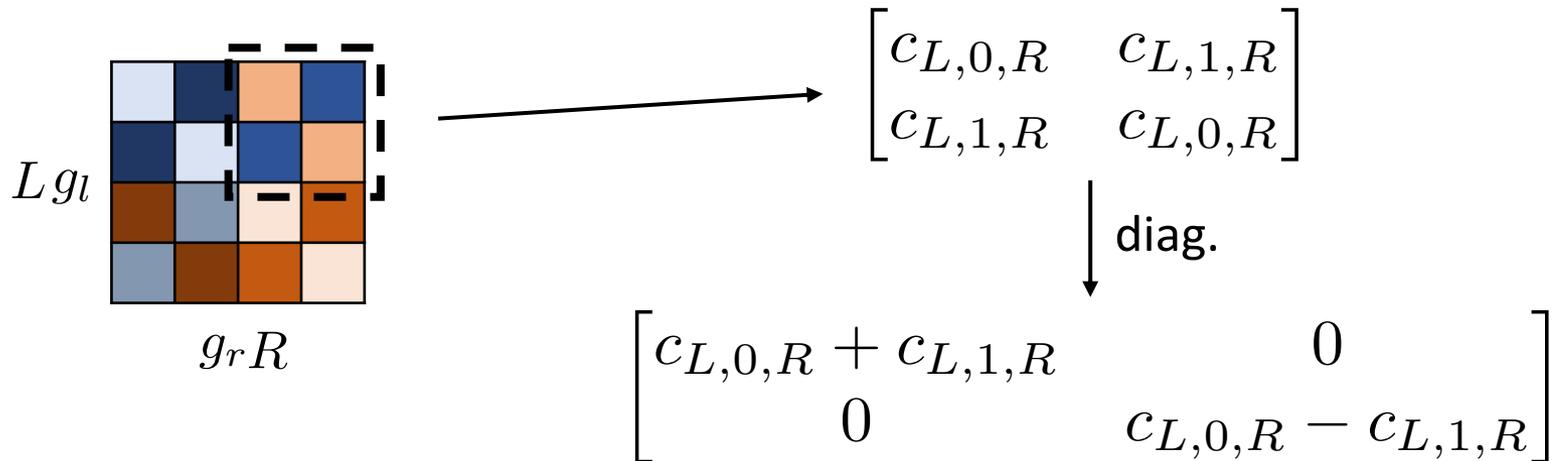$$c_{L',\tilde{g},R'} = \sigma_y \overline{c_{L',g,R'}} \sigma_y \quad \text{if TRS frac.}$$

Minimal Example: $G_0 = Z_2$

Matrix elements (no symm. frac.)

$W_{ab} \propto c_{L,g_r^{-1}g_l,R}$ If TRS imposed: $c_{L,\tilde{g},R} = \overline{c_{L,g,R}}$

Lg_l

$g_r R$

$$\begin{bmatrix} c_{L,0,R} & c_{L,1,R} \\ c_{L,1,R} & c_{L,0,R} \end{bmatrix}$$

$\downarrow$ diag.

$$\begin{bmatrix} c_{L,0,R} + c_{L,1,R} & 0 \\ 0 & c_{L,0,R} - c_{L,1,R} \end{bmatrix}$$

Without TRS: $\mathrm{LUE}_1 \oplus \mathrm{LUE}_2$

With TRS: $\mathrm{LOE}_1 \oplus \mathrm{LOE}_2$ $(\tilde{g} = g$ for $G_0 = \mathbb{Z}_2)$

Matrix elements (with symm. frac., TRS necessary)

$$W_{ab} \propto \boxed{\omega(g_r, g_r^{-1}g_l)}c_{L,g_r^{-1}g_l,R} \qquad \text{If TRS frac.: } c_{L',\tilde{g},R'} = \sigma_y\overline{c_{L',g,R'}}\sigma_y$$

Lg_l

g_rR

$\omega(1,1) = -1$

$$\begin{bmatrix} c_{L,0,R} & -c_{L,1,R} \\ c_{L,1,R} & c_{L,0,R} \end{bmatrix}$$

$\downarrow$ diag.

$$\begin{bmatrix} c_{L,0,R} + ic_{L,1,R} & 0 \\ 0 & c_{L,0,R} - ic_{L,1,R} \end{bmatrix}$$
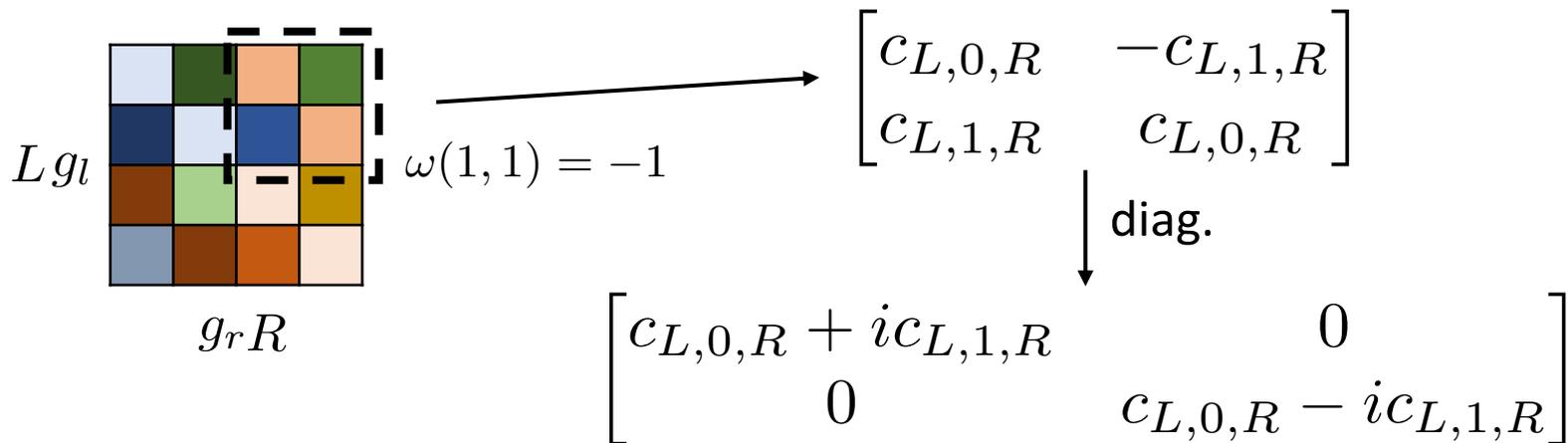
Unitary part fractionalized: $\mathrm{LUE} \oplus \overline{\mathrm{LUE}}$

TRS alone fractionalized: $\mathrm{LSE}_1 \oplus \mathrm{LSE}_2$

General results

Without TRS ($G = G_0$)

$$\rho_a = \bigoplus_\alpha \frac{1_\alpha}{d_\alpha} \otimes \mathrm{LUE}_\alpha$$

Cf. Symmetry-resolved entanglement

$$\mathcal{Z}(q^n, r) = \frac{d_r}{|G|} \sum_{g \in G} \chi_r^*(g) \frac{Z(q^n, g)}{(Z(q))^n}$$

Y. Kusuki *et al.*, JHEP (2023) 216
M. Goldstein and E. Sela, PRL **120**, 200602 (2018)

α labels a (projective) irrep of G_0 with dimension d_α

With TRS ($G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$, $\omega(t, t) = \pm 1$)

$$\rho_a = \bigoplus_{\alpha : \iota_\alpha = 0} \frac{1_\alpha}{d_\alpha} \otimes (\mathrm{LUE}_\alpha \oplus \overline{\mathrm{LUE}_\alpha}) \oplus$$

$$\bigoplus_{\alpha : \iota_\alpha = \pm} \frac{1_\alpha}{d_\alpha} \otimes \mathrm{LOE}_\alpha \oplus \bigoplus_{\alpha : \iota_\alpha = \mp} \frac{1_\alpha}{d_\alpha} \otimes \mathrm{LSE}_\alpha$$

$\iota_\alpha = \dfrac{1}{|G_0|} \displaystyle\sum_{g \in G_0} \omega(g, \tilde{g}) \mathrm{Tr} \mathcal{D}^\alpha(g\tilde{g})$ is the Schur-Frobenius indicator

Cf. Gaussian (GUE/GOE/GSE) version for Hamiltonians

F. J. Dyson, JMP
3, 1199 (1962)

Degeneracy without symm. frac.

Multiple LUE/LOE

$$G = C_{3v} \qquad \mathrm{LUE}_1 \oplus \mathrm{LUE}_2 \oplus \frac{1_2}{2} \otimes \mathrm{LUE}_3$$

$$G = C_{3v} \times \mathbb{Z}_2^{\mathcal{T}} \qquad \mathrm{LOE}_1 \oplus \mathrm{LOE}_2 \oplus \frac{1_2}{2} \otimes \mathrm{LOE}_3$$

LUE conjugate pair

$$G = \mathbb{Z}_3 \times \mathbb{Z}_2^{\mathcal{T}} \qquad \mathrm{LOE} \oplus \mathrm{LUE} \oplus \overline{\mathrm{LUE}}$$

$$\text{Cf. } G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2^{\mathcal{T}} \qquad \mathrm{LOE}_1 \oplus \mathrm{LOE}_2 \oplus \mathrm{LOE}_3$$

LSE

$$G = Q_8 \times \mathbb{Z}_2^{\mathcal{T}} \qquad \bigoplus_{\alpha=1}^{4} \mathrm{LOE}_\alpha \oplus \frac{1_2}{2} \otimes \mathrm{LSE}$$

Take-home message

- LSE arises from TRS fractionalization à la Haldane

$$\mathcal{T}_a^2 = -1_a \qquad \mathcal{T}_b^2 = -1_b \qquad \mathcal{T}^2 = (\mathcal{T}_a \otimes \mathcal{T}_b)^2 = 1$$

$$\mathcal{T}_{\text{edge}}^2 = -1$$
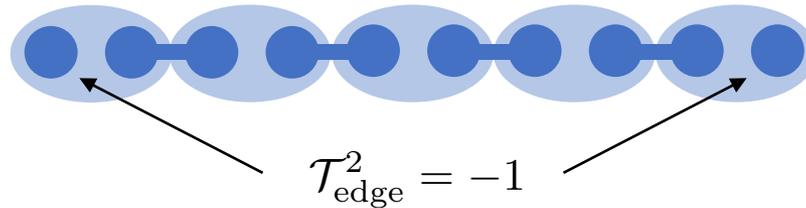
Cf. F. D. M. Haldane, PRL **50**, 1153 (1983)

- Universal reduction to the "threefold way"

$$\rho_a = \bigoplus_{\alpha:\iota_\alpha=0} \frac{1_\alpha}{d_\alpha} \otimes (\text{LUE}_\alpha \oplus \overline{\text{LUE}_\alpha}) \oplus$$

$$\bigoplus_{\alpha:\iota_\alpha=\pm} \frac{1_\alpha}{d_\alpha} \otimes \text{LOE}_\alpha \oplus \bigoplus_{\alpha:\iota_\alpha=\mp} \frac{1_\alpha}{d_\alpha} \otimes \text{LSE}_\alpha$$

Cf. F. J. Dyson, JMP **3**, 1199 (1962)

Beyond Threefold Way?

Symmetry may not be a group

Categorical symmetry: W. Ji and X.-G. Wen, PRR **2**, 033417 (2020)
L. Kong *et al.*, PRR **2**, 043086 (2020)

$$\alpha \cdot \beta = \sum_{\gamma} N_{\alpha,\beta}^{\gamma} \gamma$$

- Tannaka-Krein dual picture of a group
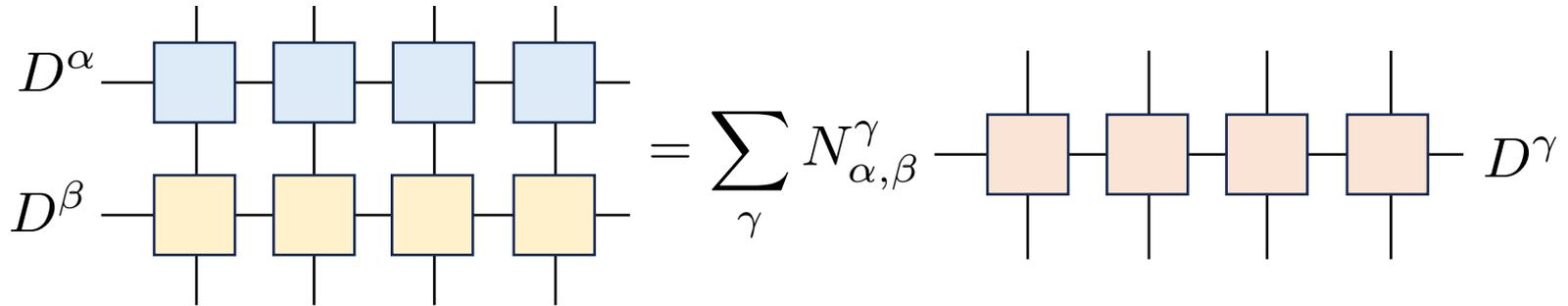- Generally not group rep., e.g., Fibonacci $\tau \cdot \tau = 1 + \tau$

Hilbert space may not be a tensor product

- Fermion: Z_2-graded tensor product L. Fidkowski and A. Kitaev, PRB **83**, 075103 (2011)
- Lattice gauge theory W. Donnelly, PRD **85**, 085004 (2012)

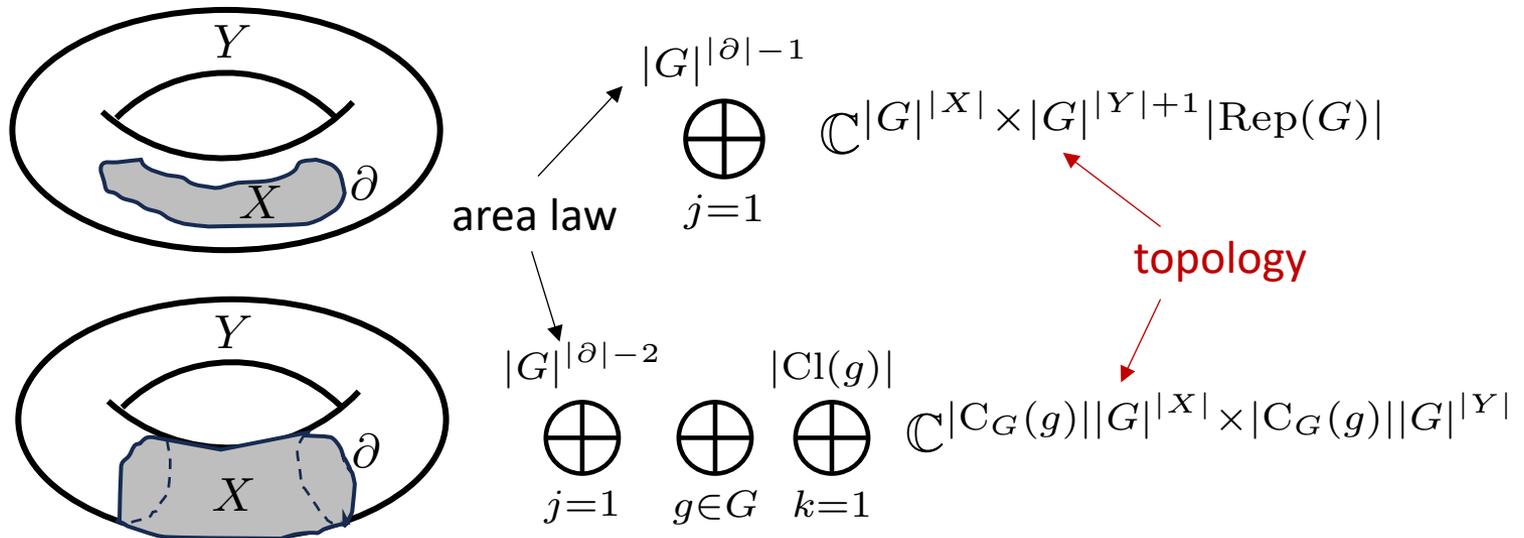# $(d\text{-}1)$-form $\mathrm{Rep}(G)$ Symmetry

MPO symmetry
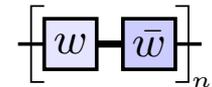
Block structure of W

Beyond Random Matrices?

Local unitary equivalence for pure states

• Bipartite: $|\Psi_{12}\rangle \sim |\Phi_{12}\rangle$ if

$$\exists\, U_1, U_2 \text{ s.t. } |\Phi_{12}\rangle = (U_1 \otimes U_2)|\Psi_{12}\rangle$$

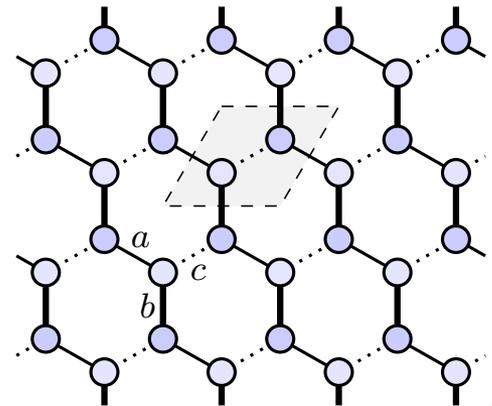LU equivalent = same entanglement spectrum

= same n-Renyi entropy ($n=2,\ldots,\min\{d_1,d_2\}$)

• Multipartite: $|\Psi_{12\ldots N}\rangle \sim |\Phi_{12\ldots N}\rangle$ if

$$\exists\, \{U_j\}_{j=1}^{N} \text{ s.t. } |\Phi_{12\ldots N}\rangle = \bigotimes_{j=1}^{N} U_j |\Psi_{12\ldots N}\rangle$$

LU equivalent = same ***multi-entropies***

M. Grassl *et al.*, PRA **58**, 1833 (1998)
M. S. Leifer *et al.*, PRA **69**, 052304 (2004)
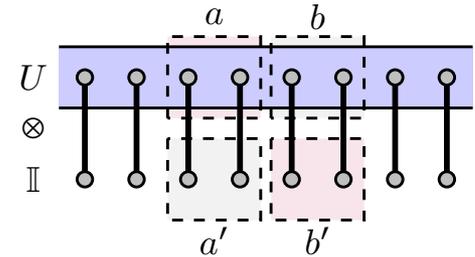A. Gadde *et al.*, PRD **106**, 126001 (2022)

Topological invariants

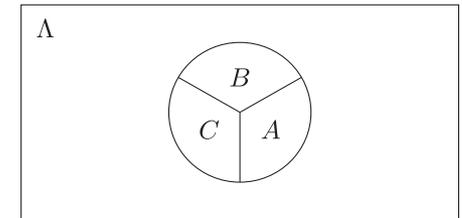## Entanglement reformulation of topological invariants

- 1D unitary: usual (2-)entropy ZG *et al.*, PRL **126**, 160601 (2021)

$$\mathrm{ind} = \frac{1}{2}\langle U|\ln\rho_{a'b} - \ln\rho_{ab'}|U\rangle$$

- 2D state: 3-entropy I. H. Kim *et al.*, PRL **128**, 176402 (2022)

$$c_- = \frac{3i}{\pi}\langle\Psi|[\ln\rho_{AB}, \ln\rho_{BC}]|\Psi\rangle$$

Conjecture: in dD, topological invariant of a unitary (odd d)/ state (even d) is related to $(d+1)$-entropy Cf. B. Liu *et al.*, arXiv:2410.08284

- E.g., index for 3D unitary might be like

$$w \propto \sum_{\sigma\in S_4}(-)^{|\sigma|}\langle U|\ln\rho_{\sigma(a)\sigma(b)'}\ln\rho_{\sigma(b)\sigma(c)'}\ln\rho_{\sigma(c)\sigma(d)'}|U\rangle$$