

# Exact renormalization flow for matrix product density operators



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# Outline

1. **Classification of Gapped quantum phases and MPS**
  2. **Quantum mixed-state phases and MPDOs**
  3. **Main results: exact RG-flows for MPDOs**
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# Classification of gapped quantum phases and MPSs

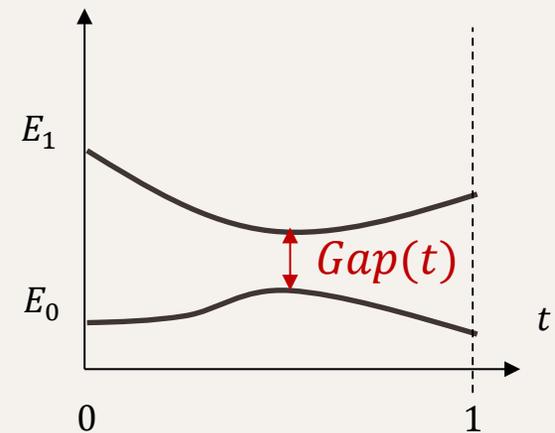
## Gapped quantum phases

- A class of ground states of local Hamiltonians that can be connected without closing the energy gap (i.e., undergoing a phase transition).

$|\psi_k^i\rangle$ : ground state of a *gaped local* Hamiltonian  $H_i$  ( $k = 1, \dots, m$ )

$$|\psi_k^1\rangle \sim |\psi_k^2\rangle, \quad \forall k = 1, \dots, m$$

$$\Leftrightarrow \exists H(t) = - \sum_X h_X(t) \quad \begin{cases} H(0) = H_1 \\ H(1) = H_2 \end{cases}$$
$$\text{Gap}(t) > 0, \quad \forall t \in [0,1].$$



## Classification by short-depth unitary circuit

- Two ground states are in the same if they are (approximately) connected by a short-depth unitary circuit

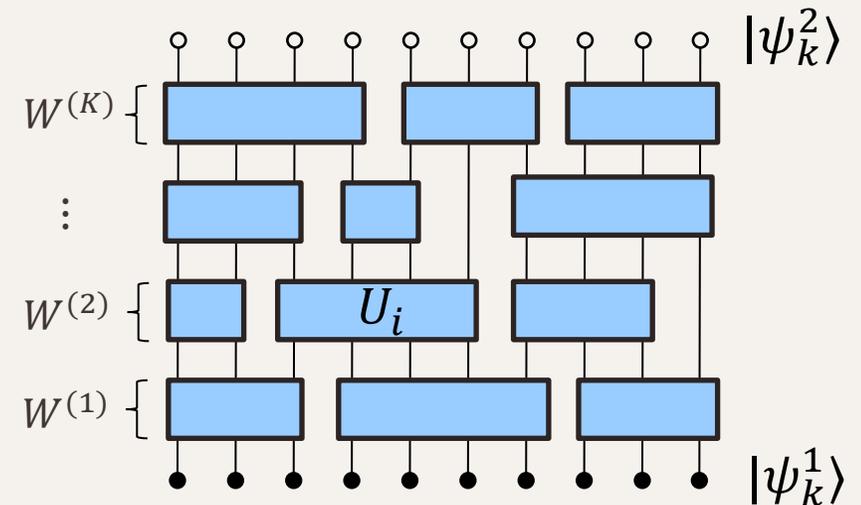
$|\psi_k^i\rangle$ : ground state of a *gaped local* Hamiltonian  $H_i$  ( $k = 1, \dots, m$ )

$$|\psi_k^1\rangle \sim |\psi_k^2\rangle, \quad \forall k = 1, \dots, m$$

$\exists K = O(\log N)$ ,  $W^{(i)}$ : product of local unitaries

$$\Leftrightarrow |\psi_k^2\rangle = W^{(K)} \dots W^{(2)} W^{(1)} |\psi_k^1\rangle$$

$$\forall k = 1, \dots, m$$



# Matrix product states

- 1D gapped ground states  $\cong$  matrix product states (MPS)

**MPS**  $|\psi\rangle = \sum_{i_1, \dots, i_N} \text{Tr}(A^{i_1} A^{i_2} \dots A^{i_N}) |i_1 i_2 \dots i_N\rangle$   $A^{i_k}: D \times D$  matrix (for each  $i_k \in [d]$ )

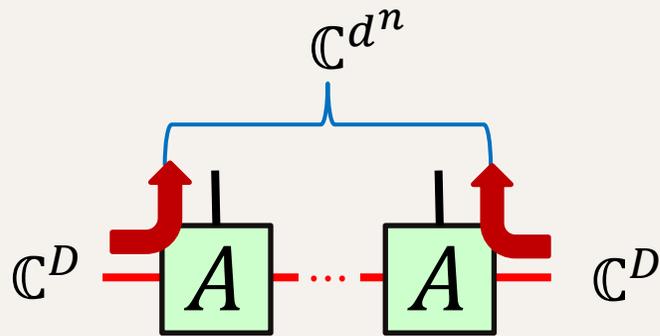
$$(A^i)_{\alpha\beta} \in \mathbb{C} \Leftrightarrow \alpha \text{ --- } \boxed{A} \text{ --- } \beta$$

$$A := \sum_{i=1}^d |i\rangle \otimes A^i \in \mathbb{C}^d \otimes \mathcal{B}(\mathbb{C}^D)$$

$$\text{--- } \boxed{A} \text{ --- } \boxed{A} \text{ ---} \Leftrightarrow AA = \sum_{i,j=1}^d |ij\rangle \otimes A^i A^j \in (\mathbb{C}^d)^{\otimes 2} \otimes \mathcal{B}(\mathbb{C}^D)$$

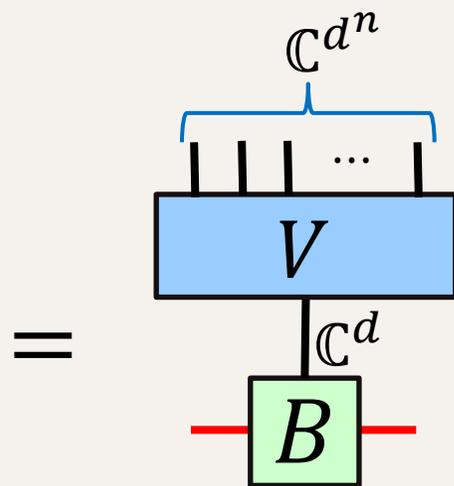
# Exact renormalization flow for MPS

- Any MPS has a well-defined renormalization flow.



By regarding few neighboring tensors together, it can be regarded as a  $d^n \times D^2$  matrix.

w.l.o.g assume  $d$  to be  $d > D^2$  by coarse-graining few sites.



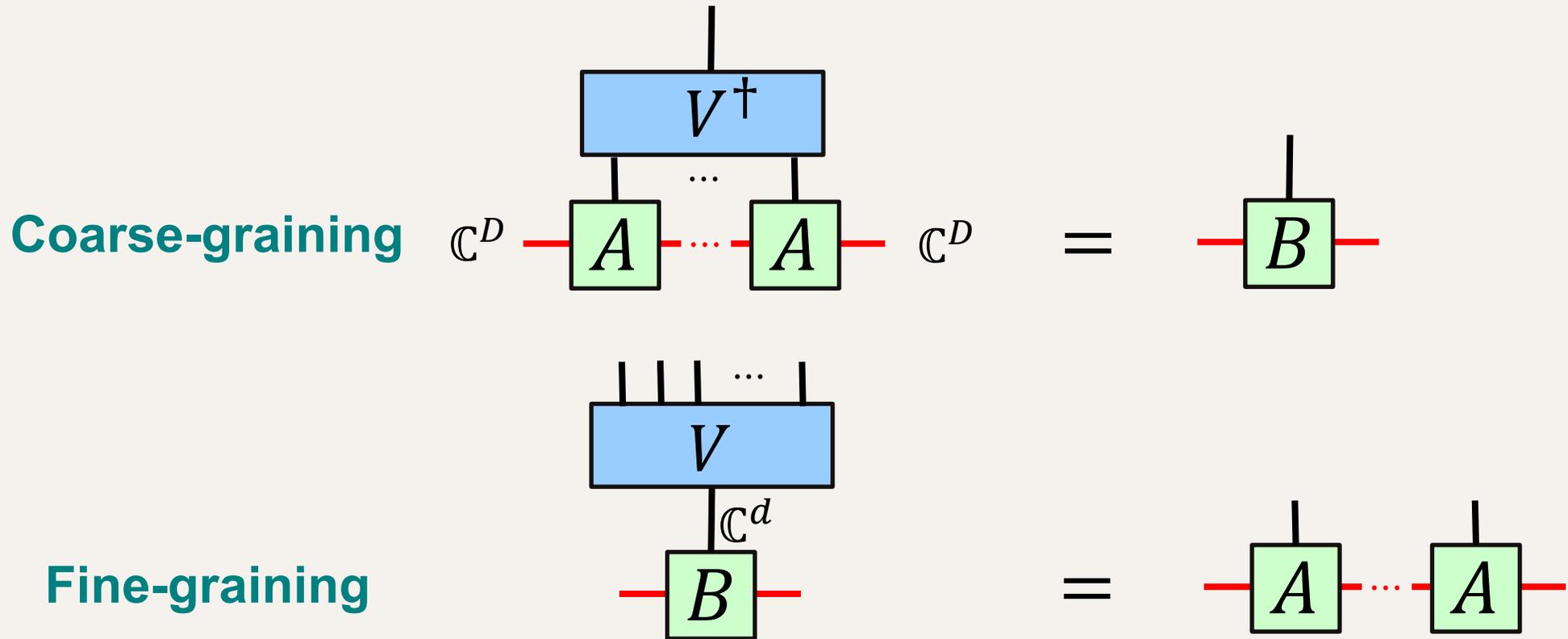
The polar decomposition: there is an isometry  $V$  s. t.

$$A \dots A = VB,$$

where  $B$  is tensor on ***a single physical site***.

# Exact renormalization flow for MPS

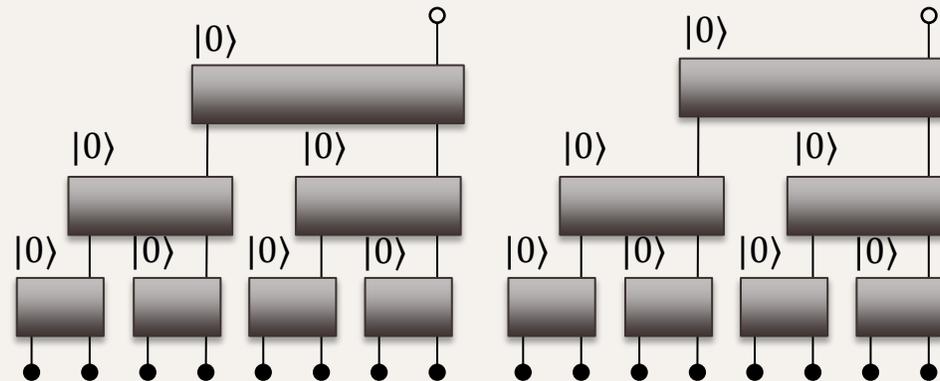
- Since  $V$  is an isometry, this RG-flow is **exactly reversible**.



# Classification of quantum phases via RG-fixed points

- Two MPS connected by the exact RG-flow are in the same phase.

$$|\psi\rangle \xrightarrow{\text{exact RG}} |\phi\rangle \Leftrightarrow |\psi\rangle \xleftarrow{\text{short-depth QC}} |\phi\rangle \otimes |0\rangle^{\otimes m}$$

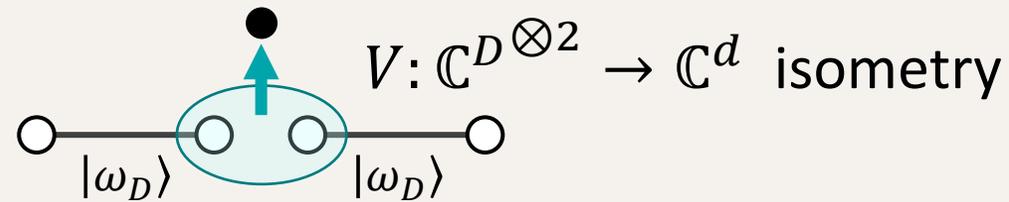


Thus, it suffices to classify the RG-fixed points.

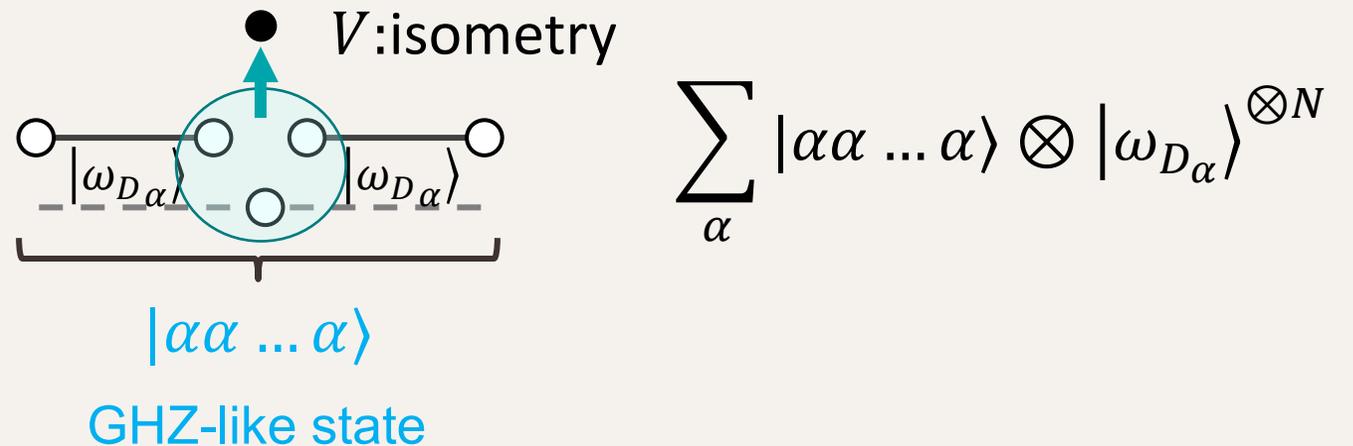
## RG-fixed points of MPS

- The RG-fixed points of MPS are isometric MPSs.

Finite correlation length →  
(injective MPS)



Long-range correlation →  
(non-injective MPS)



## Gapped quantum phases in 1D

- Interesting gapped phases exist in 2 or higher-dimensions, such as topologically ordered phases and fracton phases.
- It is known that quantum phases in 1D are classified by GSD only, when no symmetry is imposed.

$\text{GSD} = 1 \rightarrow$  Trivial phase (connected to 1-body Hamiltonian with product g.s.)

$\text{GSD} \geq 2 \rightarrow$  Dimer ground states  $\times$  GHZ-like state

↑  
Not stable without protection from symmetries

Only one stable gapped phase: trivial phase

# Quantum mixed-state phases and MPDOs

# Quantum mixed-state phases

- New classification of phases: A class of ***mixed*** states that can be connected by short-depth circuits of ***quantum channels***.

Quantum channel = Completely-Positive Trace-Preserving (CPTP) map  
= Isometry + partial trace

$\rho^1, \rho^2$ : “physically reasonable” mixed states

$$\rho^1 \sim \rho^2$$

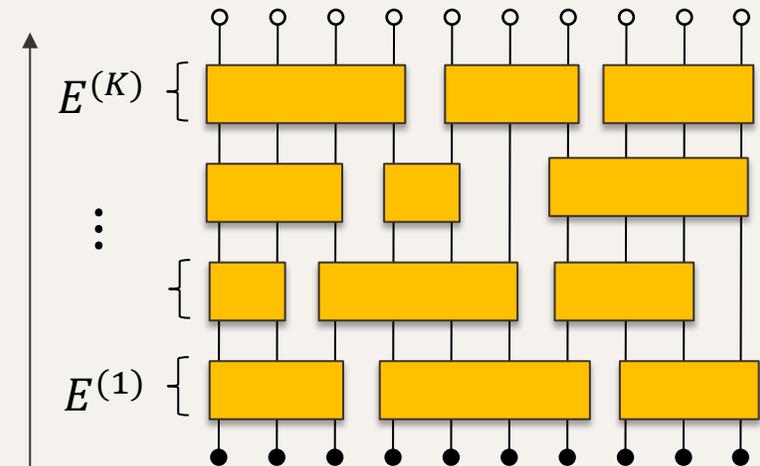
∃ a short-depth circuit of quantum channels

$$\rho^2 = E^{(M)} \circ \dots \circ E^{(1)}(\rho^1)$$



s. t. there is another local circuit of quantum channels

$$\rho^1 = F^{(M)} \circ \dots \circ F^{(1)}(\rho^2)$$



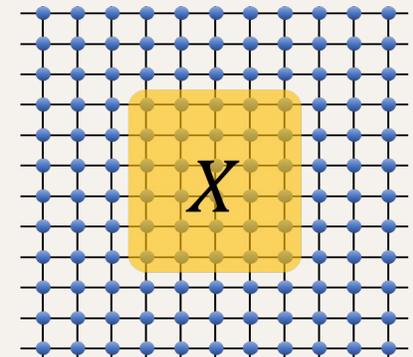
## Physically reasonable mixed states?

- We want to study the mixed-state analog of “gapped” ground states.
- Gapped ground states satisfy area law of entanglement entropy:

Entanglement entropy:  $S(X)_\rho := -\text{Tr} \rho_X \log_2 \rho_X$

$$\rho_X := \text{Tr}_{X^c} |\psi\rangle\langle\psi|$$

**Area law** :  $S(X)_\rho \leq c|\partial X|$ .



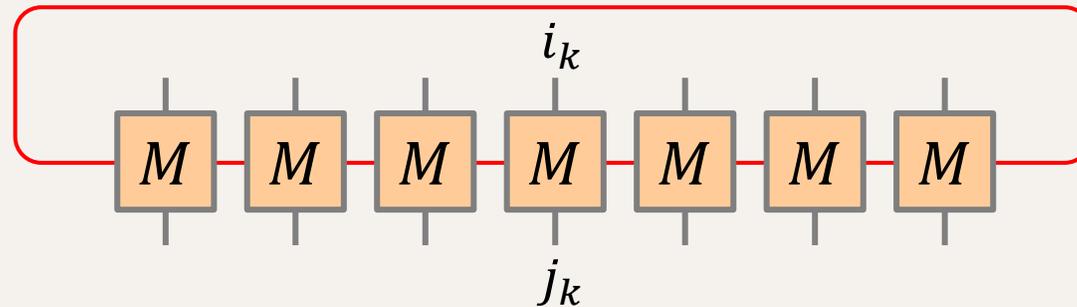
- Mixed-states → the mutual information quantifies the total correlation.

Mutual information:  $I(X:X^c)_\rho := S(X) + S(X^c) - S(XX^c)$

**Area law** :  $I(X:X^c)_\rho \leq c|\partial X|$ .

# Matrix product density operators (MPDOs)

- A natural generalization of MPS to 1D mixed states.



$$\rho_{MPDO} = \sum_{i,j} \text{Tr}(M^{i_1 j_1} M^{i_2 j_2} \dots M^{i_N j_N}) |i_1 i_2 \dots i_N\rangle \langle j_1 j_2 \dots j_N|$$

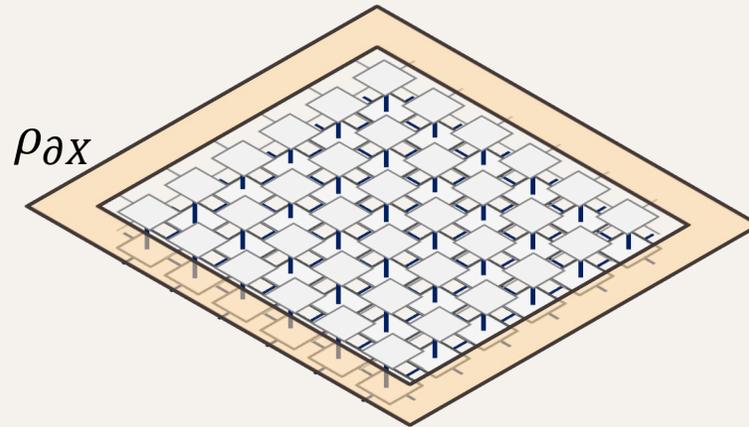
$M^{i_k j_k}$ :  $D \times D$  matrix (for each  $i_k, j_k$ )

- Good ansatz for **quantum Gibbs states** (thermal equilibrium).
- Always satisfy the area law of mutual information.

# MPDOs and boundary theory of PEPS

- MPDOs can describe more than Gibbs states.

PEPS  
(2D pure states)



Typically, MPDOs



$D$ -dimensional *pure* states  $\leftrightarrow$   $(D - 1)$ -dimensional *mixed* states

- The boundary of a 2D topological order can be a non-thermal MPDO.

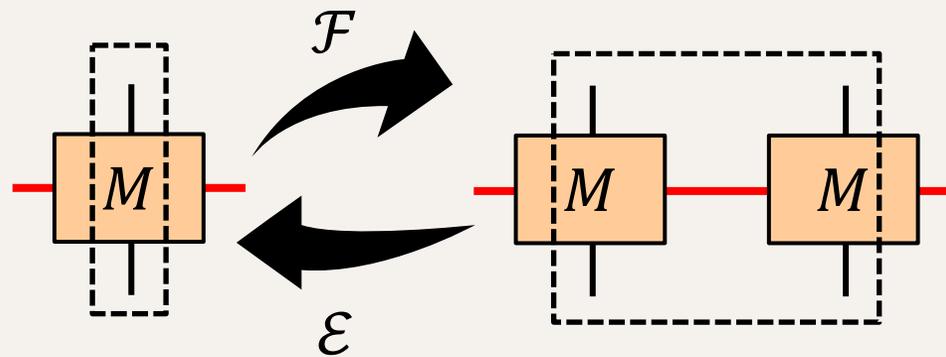
Boundary of toric code:  $\rho_{\text{MPDO}} = \frac{1}{2^N} (I^{\otimes N} + Z^{\otimes N}) \neq \frac{1}{Z} e^{-\beta \sum_i h_{i,i+1}}$

# Classifying 1D mixed-state phases via MPDO

- A similar classification strategy to that used for MPS and gapped ground states might be expected to apply here.

## Renormalization fixed-points for MPDO [Cirac, et al., '18]:

A MPDO is called a fixed-point MPDO if there is a pair of **CPTP-maps**  $\mathcal{E}, \mathcal{F}$  s. t.



### Coarse-graining

$$\mathcal{E}: \mathcal{B}(\mathbb{C}^{d^2}) \rightarrow \mathcal{B}(\mathbb{C}^d)$$

### Fine-graining

$$\mathcal{F}: \mathcal{B}(\mathbb{C}^d) \rightarrow \mathcal{B}(\mathbb{C}^{d^2})$$

# Renormalization fixed-point (RFP) MPDOs

Theorem [Cirac, et al., '18]

If  $\rho$  is a **fixed-point MPDO**, then  $\rho$  is a “**global MPO**”  $\times$  a **commuting Gibbs state**.

$$\rho_{\text{MPDO}} = \bigoplus_{i=1}^n \lambda_i P_i e^{-\sum_k h_{k,k+1}} \quad \left[ P_i, \sum_k h_{k,k+1} \right] = [h_{k,k+1}, h_{l,l+1}] = 0.$$

What are these global MPOs?

Theorem [Ruiz-de-Alarcón, et al., '24]

Any  $C^*$ -**weak-Hopf algebra**  $\Rightarrow$  RFP MPDOs

$C^*$ -weak-Hopf algebra  $\Leftrightarrow$  (multi-) fusion category: **classifying 2D topological orders!**

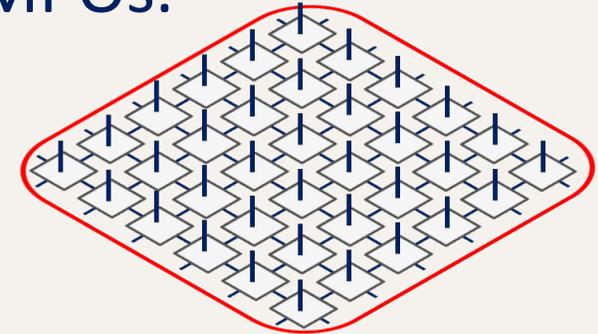
$H$

$\text{Rep}(H)$

## Boundary non-invertible symmetry = MPO algebra

- 2D PEPS in a topological order obey a non-invertible symmetry acting on the boundary (virtual legs).
- The symmetry action forms an algebra generated by MPOs.

For topological orders, the MPO algebra is a rep. of the corresponding  $C^*$ -weak Hopf algebra.



Is the 1D classification fully captured by  $C^*$ -weak Hopf algebras as well?

RFP MPDOs constructed from  $C^*$ -Hopf algebra (subclass of weak HA) are in the trivial mixed state phase [Ruiz-de-Alarcón, et al., '24].

**Whether there exists a non-trivial mixed-state phase in 1D is an open problem.**

**Main results: exact RG-flows for MPDOs**  
**[arXiv:2410.22696, will be updated]**

## Motivation

- RFP MPDOs are partly characterized by  $C^*$ -weak Hopf algebra.

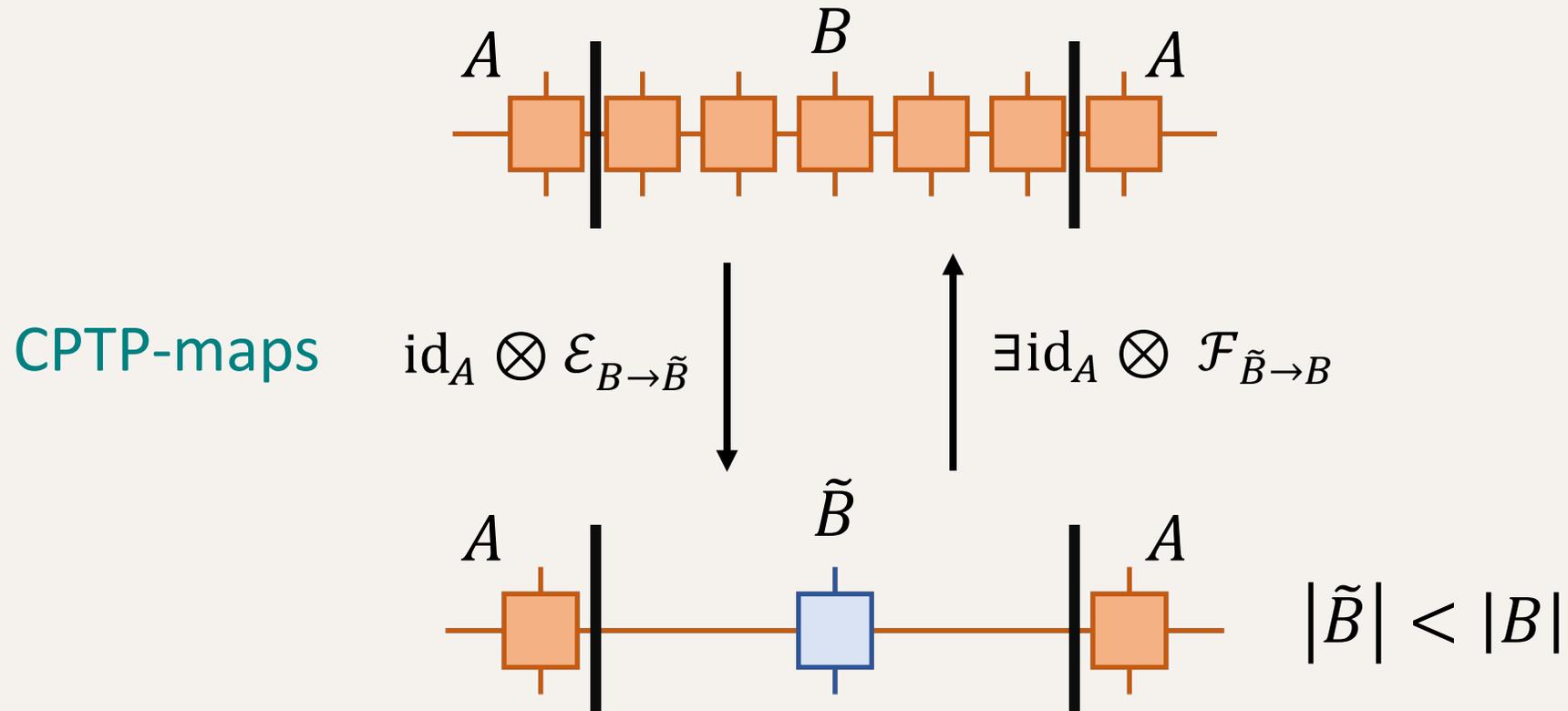
Is this classification complete? To answer that, we need to go in the reverse direction.

- Any MPS admit the exact RG-flow leading to RFP.

How about MPDOs? Most of CPTP-maps are not invertible unlike isometries...

**This work:** establishes exact RG-flow for MPDOs and extract algebraic information.

## Exact renormalization flow for MPDOs?



If  $|\tilde{B}|$  can be chosen to be independent of  $|B|$ , then we obtain the desired RG-flow.

Does such a pair  $(\mathcal{E}, \mathcal{F})$  always exist?

## Canonical forms of MPDOs

- The 4-leg tensor of MPDO have two direct sum decompositions.

$$M = \sum_{i,j=1}^d |i\rangle\langle j| \otimes M^{ij} = \sum_{\alpha,\beta=1}^D W^{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$

### Horizontal canonical form

$$X M X^{-1} = \bigoplus_k M_k \otimes N_k$$

$M_k$ : injective block (normal tensor)

### Vertical canonical form

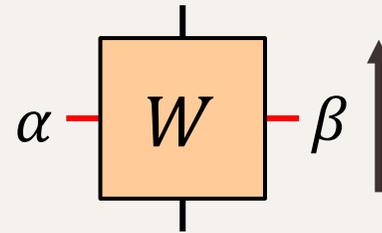
$$U W_k U^\dagger = \bigoplus_a W_{k,a} \otimes \Omega_a$$

We will omit k for simplicity

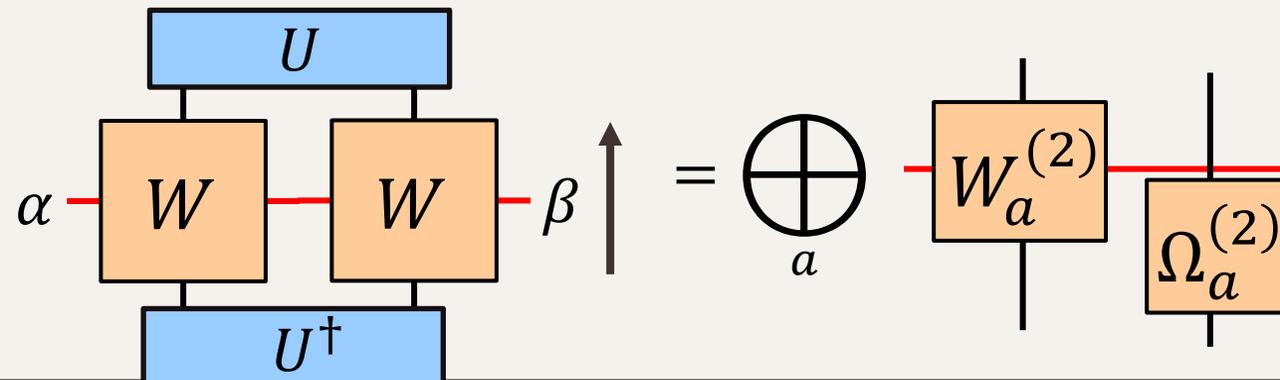
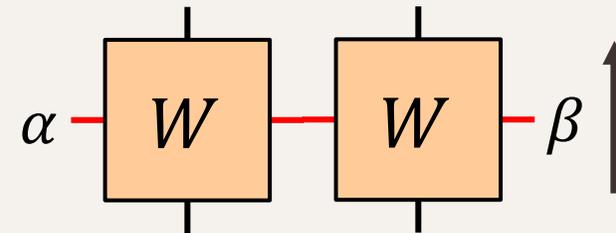
## Vertical canonical form (v-CF) changes under grouping

- Unlike h-CF, v-CF changes when multiple tensors are considered.

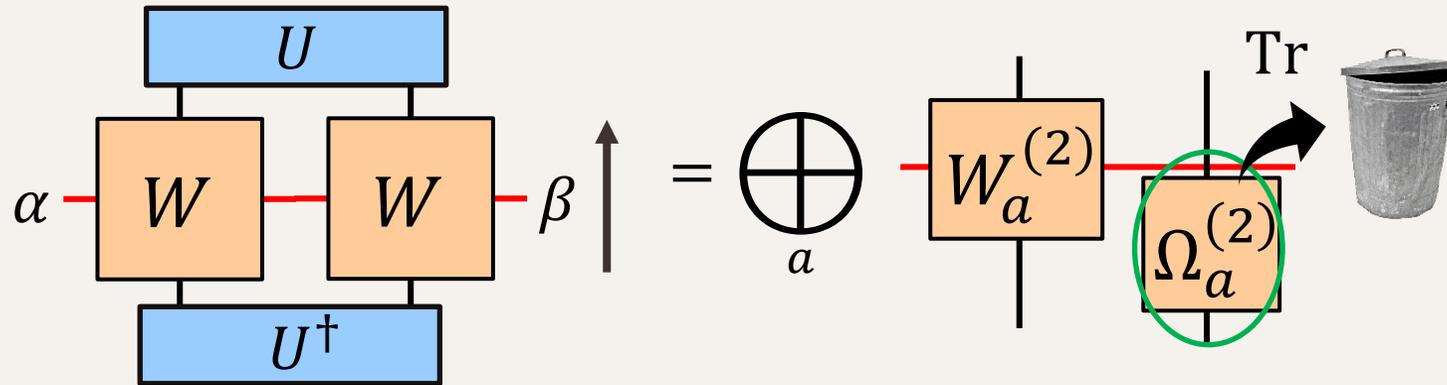
$$M = \sum_{\alpha, \beta=1}^D W^{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$



$$MM = \sum_{\alpha, \beta=1}^D \left( \sum_{\gamma} W^{\alpha\gamma} \otimes W^{\gamma\beta} \right) \otimes |\alpha\rangle\langle\beta|$$



# Compression bound from quantum information theory



## Theorem [Kato, '24]

This is the optimal, exactly reversible dimension reduction among all CPTP maps.

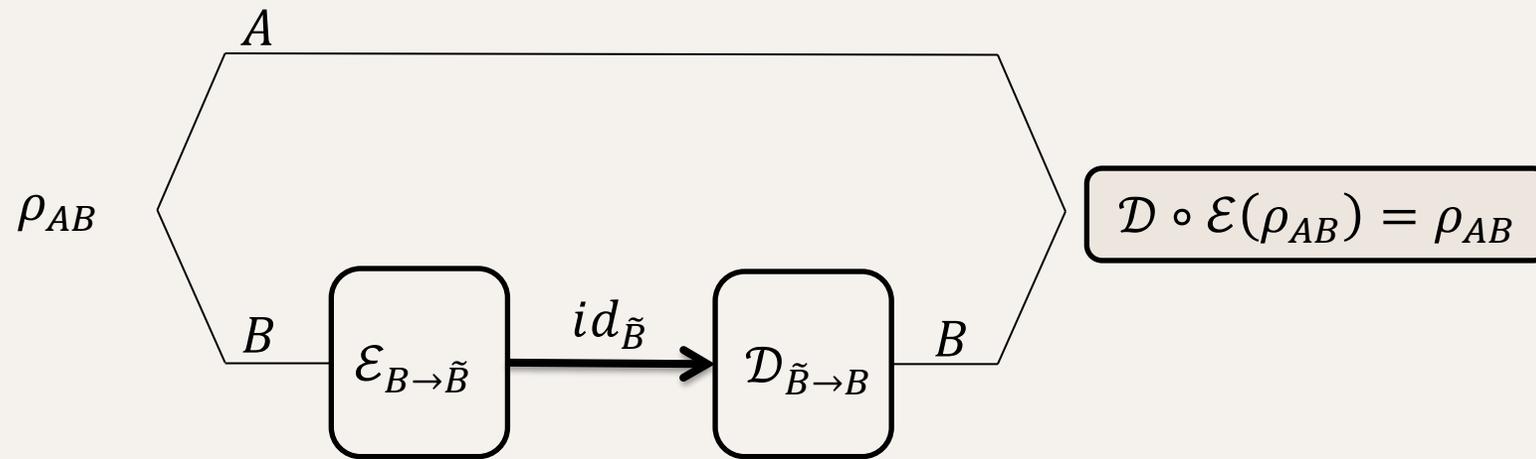
Equivalent to the Koashi-Imoto decomposition for bipartite quantum states.

[Koashi, Imoto, '02; Hayden, et al., '04]

$$\rho_{AB} = \bigoplus_i p_i \rho_{AB_i^L} \otimes \omega_{B_i^R} .$$

# Exact compression of general bipartite states (Technical)

- One-shot, exact compression of mixed bipartite state



## Question

What is the minimum dimension of  $\mathcal{H}_{\tilde{B}}$ ?

# Minimal sufficient subalgebra (Technical)

The condition  $\mathcal{D}_{\tilde{B} \rightarrow B} \circ \mathcal{E}_{B \rightarrow \tilde{B}}(\rho_{AB}) = \rho_{AB}$  is equivalent to the following [Hayden, et al., '04]

$$\mathcal{D}_{\tilde{B} \rightarrow B} \circ \mathcal{E}_{B \rightarrow \tilde{B}}(\mu_B) = \mu_B, \forall \mu_B \in \mathcal{S}, \quad \mathcal{S} := \left\{ \mu_B = \frac{\text{tr}_A(O_A \rho_{AB})}{\text{tr}(O_A \rho_A)} \mid 0 \leq O_A \leq I_A \right\}.$$

- The minimal dimension of  $\tilde{B}$  is then derived from the min. sufficient subalgebra of  $\mathcal{S}$ .  
[Petz, '86, '88; Jenčová&Petz, '06]

$$\mathcal{M}_B^{\mathcal{S}} := \text{Alg}\{\mu_B^{it} \rho_B^{-it}, \mu \in \mathcal{S}, t \in \mathbb{R}\} \subset \mathcal{B}(\mathcal{H}_B)$$

$\mathcal{M}_B^{\mathcal{S}}$  is a finite-dimensional  $C^*$ -algebra, thus there is a decomposition

$$\mathcal{H}_B \cong \bigoplus_i \mathcal{H}_{B_i^L} \otimes \mathcal{H}_{B_i^R} \quad \text{s.t.} \quad \mathcal{M}_B^{\mathcal{S}} \cong \bigoplus_i \text{Mat}(\mathcal{H}_{B_i^L}) \otimes I_{B_i^R}.$$

# Minimal sufficient subalgebra (cont.) (Technical)

➤ For any bipartite state  $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , s.t.,  $\rho_B > 0$ ,

$$\mathcal{H}_B \cong \bigoplus_i \mathcal{H}_{B_i^L} \otimes \mathcal{H}_{B_i^R} \quad \text{s.t.} \quad \mathcal{M}_B^S \cong \bigoplus_i \text{Mat}(\mathcal{H}_{B_i^L}) \otimes I_{B_i^R}$$

and

$$\rho_{AB} = \bigoplus_i p_i \underbrace{\rho_{AB_i^L}}_{\text{Quantumly correlated}} \otimes \underbrace{\omega_{B_i^R}}_{\text{Classically correlated}}.$$

called “**Koashi-Imoto decomposition**”. [Koashi, Imoto, '02; Hayden, et al., '04]

➤ The minimal exact compression is then given by

$$\mathcal{E}_{B \rightarrow \tilde{B}}: \rho_{AB} \mapsto \rho_{A\tilde{B}} := \bigoplus_i p_i \rho_{AB_i^L}.$$

## Diverging exact RG-flow

- We show that not all MPDOs admit a converging exact RG flow.

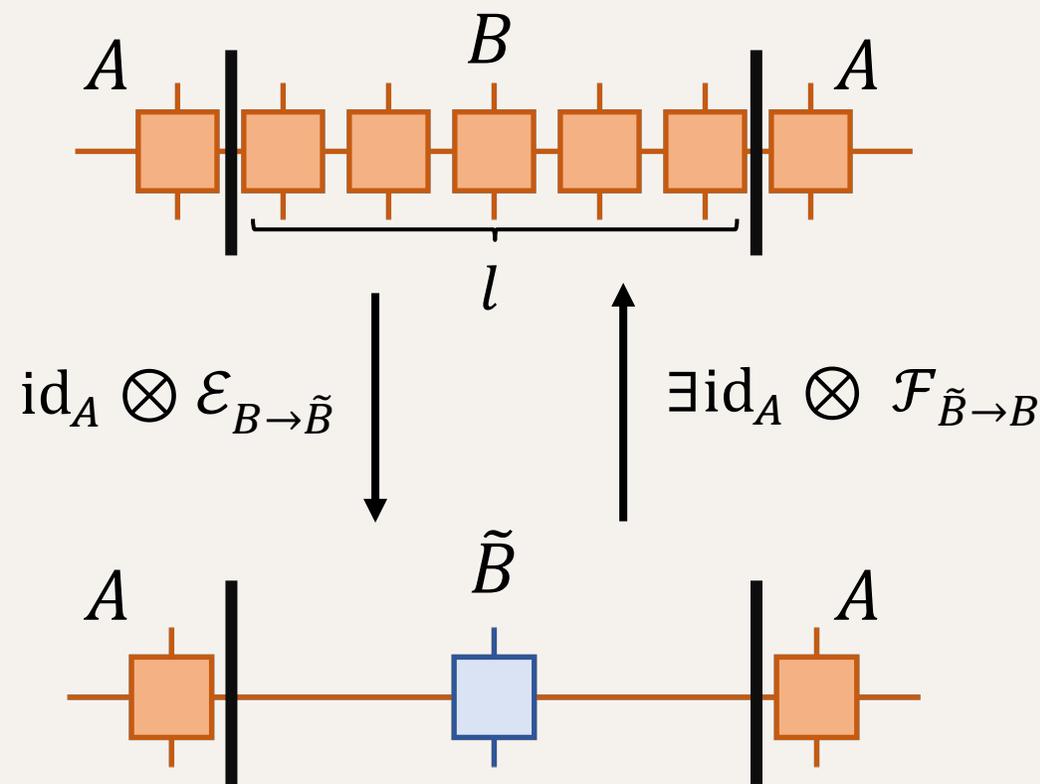
$$\rho^{(L)} := \frac{1}{3^L} (I^{\otimes L} + \Lambda^{\otimes L})$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\text{tr}\Lambda = 0, \|\Lambda\| \leq 1,$$

$$\lambda_1 \neq \lambda_2 \neq \lambda_3.$$

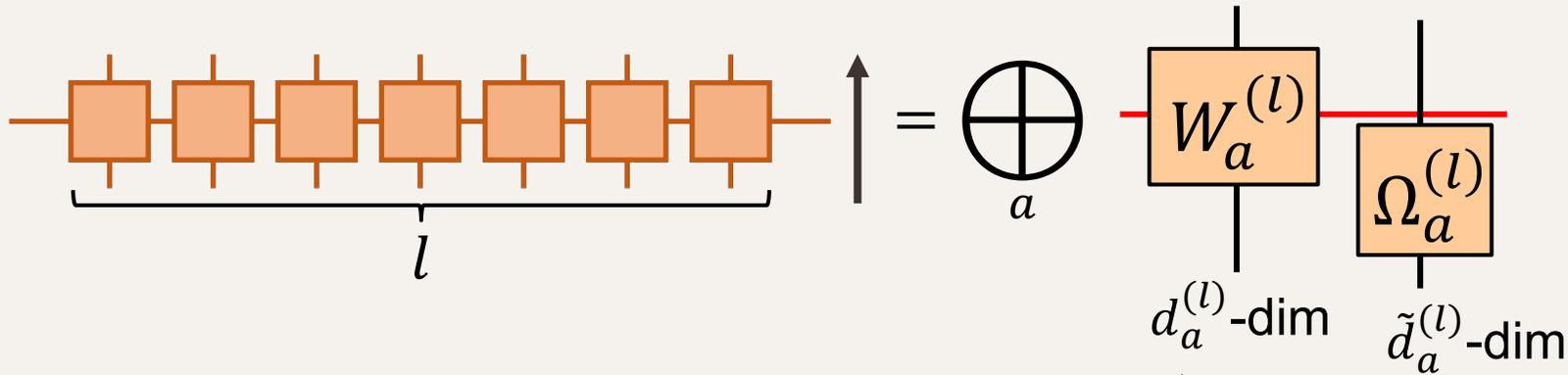
$$\dim(\tilde{B}) \cong \mathbb{C}^{\text{poly}(l)}$$



Exact RG-flow must diverge!

## Conditions for well-defined exact RG-flow

- We thus consider a subclass of MPDOs with an exact RG-flow.



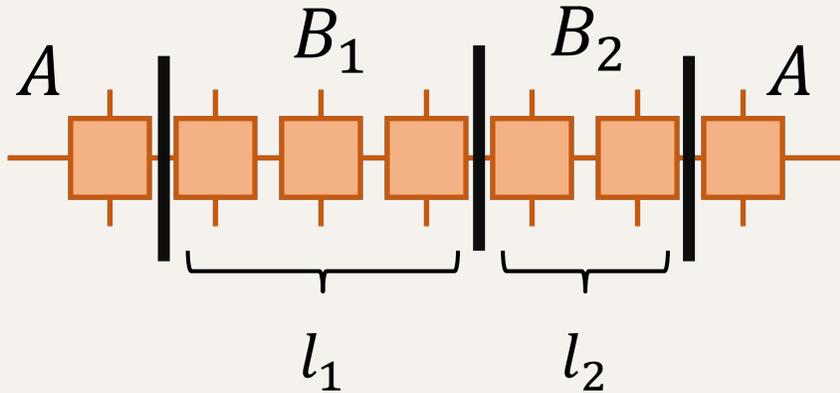
\*the number of block also depends on  $l$  in general

$\mathcal{A}_l := C^*$ -alg. generated by  $\{W_a^{(l)}\} \cong \bigoplus_a \mathcal{B}(\mathbb{C}^{d_a^{(l)}})$  ↑  $l$ -independent

**Condition 1:** there is a finite-dimensional  $C^*$ -algebra  $\mathcal{A} = \bigoplus_a \mathcal{B}(\mathbb{C}^{d_a})$  and **injective representations**  $\{\pi_l\}$  s.t.

$$\mathcal{A}_l = \pi_l(\mathcal{A}) \cong \bigoplus_a \mathcal{B}(\mathbb{C}^{d_a}) \otimes I_{\tilde{d}_a^{(l)}}$$

## Conditions for well-defined exact RG-flow



**Lemma:**  $\mathcal{A}_{l_1+l_2} \subset \mathcal{A}_{l_1} \otimes \mathcal{A}_{l_2}$

The inclusion  $\iota_{l_1+l_2}: \mathcal{A}_{l_1+l_2} \hookrightarrow \mathcal{A}_{l_1} \otimes \mathcal{A}_{l_2}$  induces  $\Delta_{l_1+l_2} := (\pi_{l_1}^{-1} \otimes \pi_{l_2}^{-1}) \circ \iota_{l_1+l_2} \circ \pi_{l_1+l_2}$

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\Delta_{l_1+l_2}} & \mathcal{A} \otimes \mathcal{A} \\
 \pi_{l_1+l_2} \downarrow & & \uparrow \pi_{l_1}^{-1} \otimes \pi_{l_2}^{-1} \\
 \mathcal{A}_{l_1+l_2} & \xrightarrow{\iota_{l_1+l_2}} & \mathcal{A}_{l_1} \otimes \mathcal{A}_{l_2}
 \end{array}$$

## MPDO with an exact RG-flow (definition)

➤ We say a MPDO has a  $(\pi_l, \mathcal{A})$  RG-flow if it satisfies the two conditions.

**Condition 1:**  $\mathcal{A}_l = \pi_l(\mathcal{A}), \quad \forall B, |B| = l.$

**Condition 2:**  $\exists \Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \quad \text{s. t.} \quad \Delta_{l_1+l_2} = \Delta, \quad \forall l_1, l_2 \in \mathbb{N}.$

$$\Delta_{l_1+l_2} := (\pi_{l_1}^{-1} \otimes \pi_{l_2}^{-1}) \circ \iota_{l_1+l_2} \circ \pi_{l_1+l_2}$$

### Proposition

The linear map  $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  becomes a **comultiplication**, i.e., it satisfies

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta =: \Delta^2.$$

$\Delta(xy) = \Delta(x)\Delta(y), \forall x, y \in \mathcal{A}$  holds by assumption.

# Coalgebra, bialgebra and Hopf algebra

**Coalgebra**  $(\mathcal{A}, \Delta, \epsilon)$   $\left\{ \begin{array}{l} (\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta =: \Delta^2. \\ \epsilon: \mathcal{A} \rightarrow \mathbb{C}, \text{ s.t. } (\text{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}) \circ \Delta = \text{id}. \end{array} \right.$

counit

Unital algebra + counital coalgebra +  $\Delta(xy) = \Delta(x)\Delta(y) =$  **pre-bialgebra**

**Theorem [Molnar, et al., '24]**

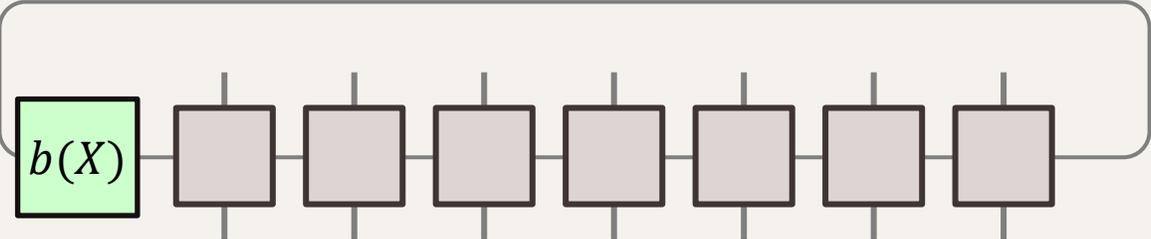
Any  $C^*$ -pre-bialgebra  $\Rightarrow$  MPO algebra as a rep.



## MPO representation [Molnar, et al., '24]

A pre-bialgebra  $\mathcal{A}$ , an injective rep  $\pi_1: \mathcal{A} \rightarrow \mathcal{B}(\mathbb{C}^d)$

$\forall X \in \mathcal{A}, \exists b(X)$  s.t.  $\pi_1^{\otimes N} \circ \Delta^{N-1}(X) \in \mathcal{B}((\mathbb{C}^d)^{\otimes N})$  is realized as an MPO algebra:

$$\pi_1^{\otimes L} \circ \Delta^{L-1}(X) =$$


$$\begin{array}{c} | \\ \square \\ | \end{array} := \sum_{e \in \text{basis}(\mathcal{A})} \pi_1(e) \otimes \psi(f^e)$$

$\{f^e\}$ : the dual basis of  $\mathcal{A}^*$   
 $\psi$ : an injective representation of  $\mathcal{A}^*$

# Structure theorem

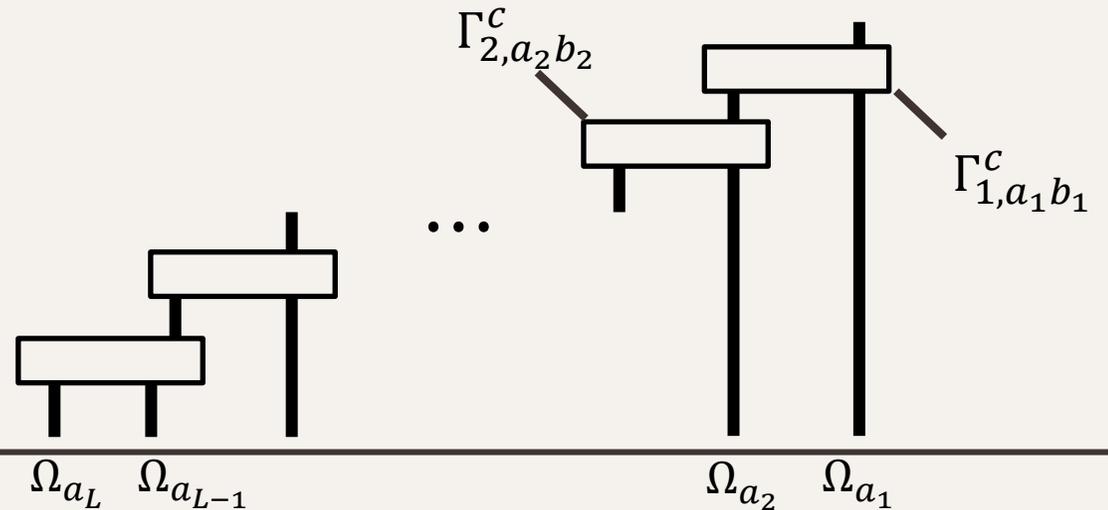
## Theorem

Any MPDO  $\rho \in \mathcal{B} \left( (\mathbb{C}^d)^{\otimes N} \right)$  with a  $(\pi_l, \mathcal{A})$  RG-flow can be written as

$$\rho \cong \pi_1^{\otimes N} \circ \Delta^{N-1} \left( w^{(N)} \right) \Omega^{(N)}, \quad \exists w^{(N)} \in \mathcal{A},$$

where  $[\pi^{\otimes N} \circ \Delta^{N-1} (a), \Omega^{(N)}] = 0, \forall a \in \mathcal{A}.$

$$\Omega^{(L)} = \bigoplus_{c, \mathbf{a}, \mathbf{b}} \Gamma_{1, a_1, b_1}^c \otimes \Gamma_{2, a_2, b_2}^{b_1} \dots \otimes \Gamma_{L-1, a_{L-1}, a_L}^{b_{L-2}} \otimes_k^L \Omega_{a_k} \in$$



# Example 1 : boundary state of toric code with a noise

$$\mathcal{A} = \mathbb{C}^2$$

$$\rho_{\text{MPDO}} = \frac{1}{2^L} (I^{\otimes L} + p^L X^{\otimes L}), \quad 0 \leq p \leq 1.$$

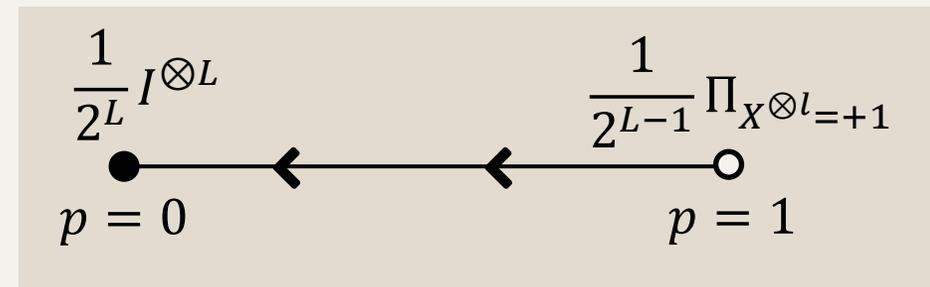
$$\Delta: c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1| \in \mathcal{A}$$

$$\epsilon(\cdot) = \langle 0| \cdot |0\rangle.$$

$$\mapsto c_0(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) + c_1(|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|)$$

$$\pi_l: (c_0, c_1) \in \mathcal{A} \mapsto c_0 \Pi_{X^{\otimes l}=+1} \oplus c_1 \Pi_{X^{\otimes l}=-1} \quad \Pi_{X^{\otimes l}=\pm 1}: \text{The projection onto } \pm 1 \text{ eigenspace of } X^{\otimes l}$$

$$w^{(L)} = \frac{1+p^L}{\sqrt{2^L}} |0\rangle\langle 0| + \frac{1-p^L}{\sqrt{2^L}} |1\rangle\langle 1|, \quad \Omega^{(L)} = \frac{1}{\sqrt{2^L}} I^{\otimes L}.$$



$$\rho_{\text{MPDO}} = \pi^{\otimes L} \circ \Delta^{L-1}(w^{(L)}) \Omega^{(L)} = \frac{1+p^L}{2^L} \Pi_{X^{\otimes L}=+1} + \frac{1-p^L}{2^L} \Pi_{X^{\otimes L}=-1}$$

## Example 2 : mixed dimer state (non-counital)

- In our framework,  $\mathcal{A}$  does not always have a counit.

Consider a bipartite state  $\omega \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$  and its operator-Schmidt decomposition:

$$\omega = \sum_{\alpha} F^{\alpha} \otimes G^{\alpha} .$$

Suppose  $\{F^{\alpha}\}, \{G^{\alpha}\}$  generate  $\mathcal{B}(\mathbb{C}^d)$ .

$$\mathcal{A} = \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) \text{ with comultiplication } \Delta: x \otimes y \mapsto (x \otimes I) \otimes (I \otimes y)$$

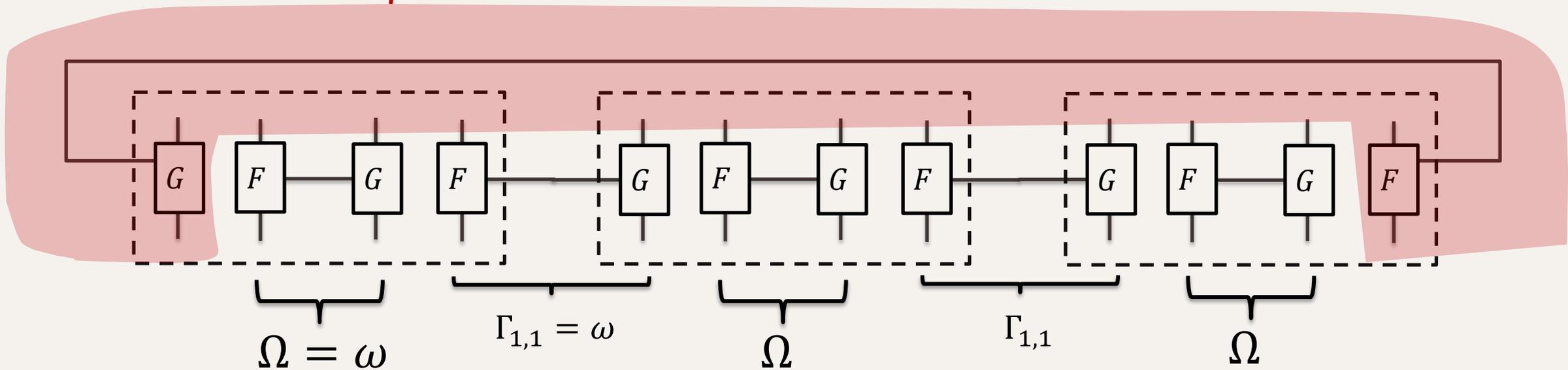
$$\epsilon: \mathcal{A} \rightarrow \mathbb{C}, \text{ s.t. } (\text{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}) \circ \Delta = \text{id}.$$

→  $\mathcal{A}$  cannot have a counit!

## Example 2 : mixed dimer state (non-counital)

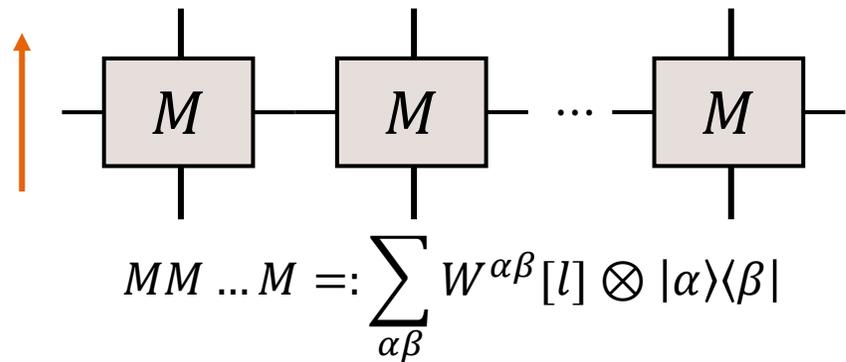
The structure theorem still works:

$$\rho = \pi^L \circ \Delta^{L-1} \left( \sum_{\alpha} G_{\alpha} \otimes F_{\alpha} \right) \Omega^{(L)} = \omega_{N,1} \otimes \left( \otimes_i \Gamma_{i,i+1} \right) \otimes \Omega^{\otimes L} = \omega^{\otimes 2L}$$



# Proof: structure theorem (Technical)

**Condition 1:**  $\mathcal{A}_l = \pi_l(\mathcal{A}), \quad \forall B, |B| = l.$



$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_l \left( \widehat{w}_a^{\alpha\beta}[l] \right) \otimes \Omega_a^{(l)}.$$

By **Condition 2**,

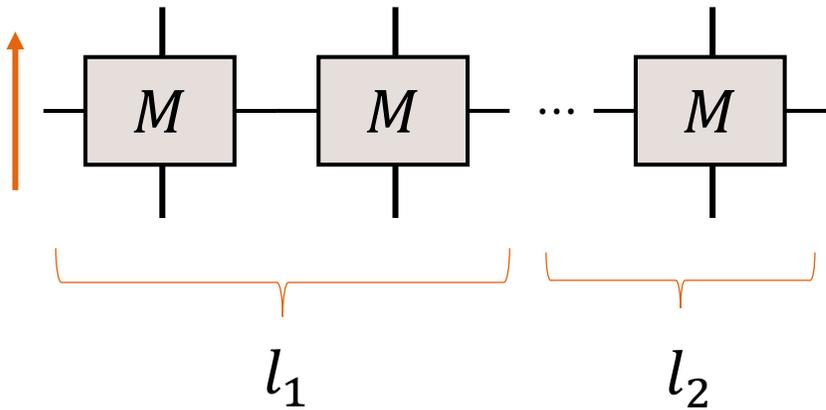
$$l_{l_1+l_2} \circ \pi_{l_1+l_2} = \left( \pi_{l_1} \otimes \pi_{l_2} \right) \circ \Delta \quad \Rightarrow \quad \pi_l = \pi_1^{\otimes l} \circ \Delta^{l-1}.$$

\*omit the inclusion maps.

$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_1^{\otimes l} \circ \Delta^{l-1} \left( \widehat{w}_a^{\alpha\beta}[l] \right) \otimes \Omega_a^{(l)}.$$

# Proof: structure theorem (Technical)

$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_1^{\otimes l} \circ \Delta^{l-1} \left( \widehat{w}_a^{\alpha\beta}[l] \right) \otimes \Omega_a^{(l)}. \quad (*)$$



$$W^{\alpha\beta}[l_1 + l_2] = \sum_{\gamma} W^{\alpha\gamma}[l_1] \otimes W^{\gamma\beta}[l_2]$$

Consistency between decomposition (\*) for LHS and RHS



**Lemma:**  $\exists V_{ab}, \sum_{\gamma} V_{ab} \left( \pi_{l_1} \left( \widehat{w}_a^{\alpha\gamma}[l_1] \right) \otimes \pi_{l_2} \left( \widehat{w}_b^{\gamma\beta}[l_2] \right) \right) V_{ab}^{\dagger} = \bigoplus_c \Gamma_{ab}^c[l_1, l_2] \otimes \Omega_l^c$

This lemma factorizes  $\Omega_a^{(l)}$  into small pieces.

$$\Omega^{(L)} = \bigoplus_{c,a,b} \Gamma_{1,a_1,b_1}^c \otimes \Gamma_{2,a_2,b_2}^{b_1} \cdots \otimes \Gamma_{L-1,a_{L-1},a_L}^{b_{L-2}} \bigotimes_k \Omega_{a_k} \in \mathcal{B}(\mathbb{C}^{d^L})$$

# Structure theorem

## Theorem

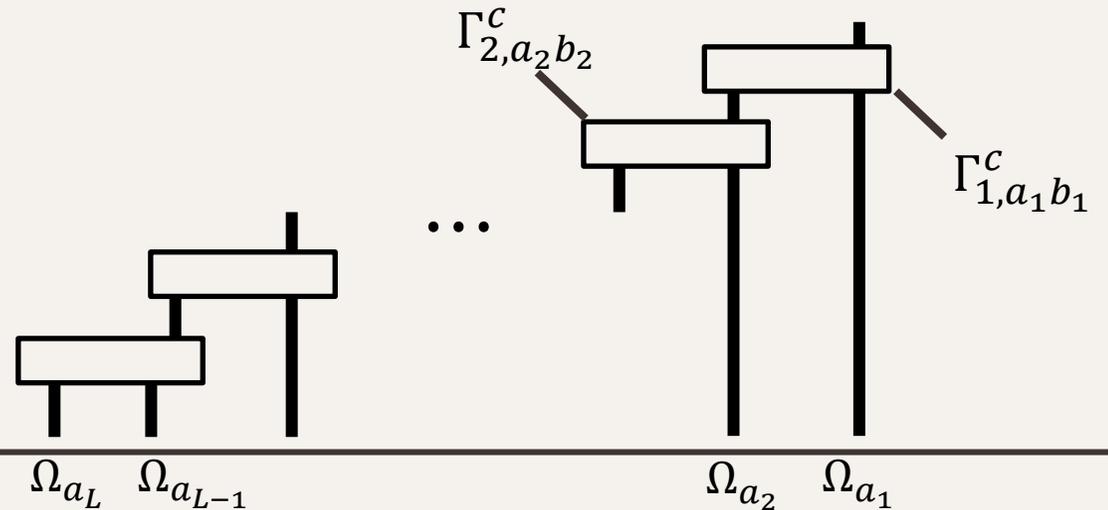
Any MPDO  $\rho \in \mathcal{B} \left( (\mathbb{C}^d)^{\otimes N} \right)$  with a  $(\pi_l, \mathcal{A})$  RG-flow can be written as  
 “thermal” part? (possibly anyonic thermal state)

$$\rho \cong \pi_1^{\otimes N} \circ \Delta^{N-1} \left( w^{(N)} \right) \Omega^{(N)}, \quad \exists w^{(N)} \in \mathcal{A},$$

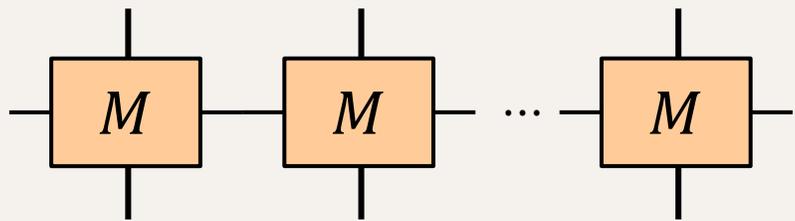
MPO-algebra + boundary condition?

where  $[\pi_1^{\otimes N} \circ \Delta^{N-1} (a), \Omega^{(N)}] = 0, \forall a \in \mathcal{A}.$

$$\Omega^{(L)} = \bigoplus_{c,a,b} \Gamma_{1,a_1,b_1}^c \otimes \Gamma_{2,a_2,b_2}^{b_1} \dots \otimes \Gamma_{L-1,a_{L-1},a_L}^{b_{L-2}} \otimes_k^L \Omega_{a_k} \in \mathcal{B} \left( \mathbb{C}^{d^L} \right)$$



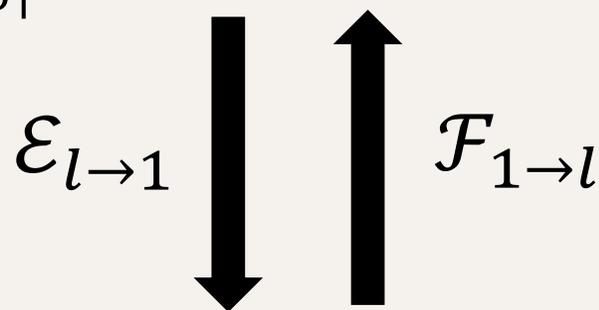
# Construction of exact RG-transformation



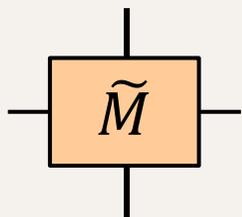
$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_1^{\otimes l} \circ \Delta^{l-1} \left( \widehat{w}_a^{\alpha\beta}[l] \right) \otimes \Omega_a^{(l)}.$$

$l$  - independent constant dimension

$$MM \dots M =: \sum_{\alpha\beta} W^{\alpha\beta}[l] \otimes |\alpha\rangle\langle\beta|$$



$\mathcal{E}_{l \rightarrow 1}, \mathcal{F}_{1 \rightarrow l}: \text{CPTP-maps}$



$$\widetilde{W}^{\alpha\beta} \cong \bigoplus_a \frac{\text{tr} \Omega_a^{(l)}}{\text{tr} \Omega_a^{(1)}} \pi_1^{\text{new}} \left( \widetilde{w}_a^{\alpha\beta} \right) \otimes \Omega_a^{(1)}.$$

$$\widetilde{M} =: \sum_{\alpha\beta} \widetilde{W}^{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$

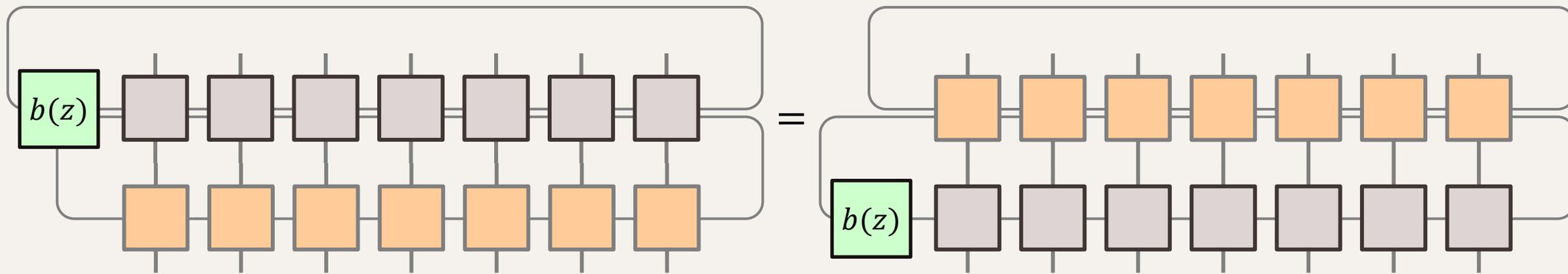
$$\pi_1^{\text{new}} \left( \widetilde{w}_a^{\alpha\beta} \right) := \pi_1^{\otimes l} \circ \Delta^{l-1} \left( \widehat{w}_a^{\alpha\beta}[l] \right)$$

## Non-invertible MPO symmetry of MPDOs (counital case)

Let  $\mathcal{Z}(\mathcal{A})$  be the center of  $\mathcal{A}$ . For any  $z \in \mathcal{Z}(\mathcal{A})$ , the MPDO  $\rho$  satisfies

$$\left[ \pi^{\otimes L} \circ \Delta^{L-1}(z), \rho \right] = 0.$$

\*Recall that  $\left[ \pi^{\otimes L} \circ \Delta^{L-1}(a), \Omega^{(L)} \right] = 0, \forall a \in \mathcal{A}$ .



# Summary & Open questions

## Summary

- We introduced an **exact RG-flow** for 1D mixed states based on MPDOs.
- A subclass of MPDOs admits **reversible, size-independent coarse-graining**.
- These are fully characterized by algebraic data  $(\mathcal{A}, \pi_1, \Delta)$ .
- The structure theorem generalizes RFP MPDOs and applies even in **non-counital** settings.

## Open questions

1. How to formulate approximate RG or stable renormalization flows for general MPDOs?
2. How can we characterize **MPO-algebra beyond weak Hopf algebra**? (even non-counital).
3. Connection to 2D-1D correspondence? **Is there an MPDO which cannot be realized by the boundary of 2D gapped ground states?**