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Matrix Product Density Operators:
Renormalization Fixed Points and Parent Lindbladians

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MPQ
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2501.10552
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Take-away message:

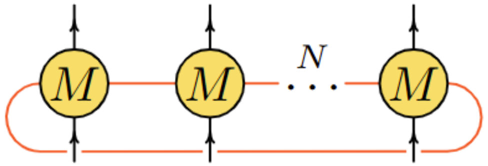
In our work, we use tensor network for mixed states.

1. How do MPDO renormalization fixed points look like?

→ *[YL*, Molnar, Sun, Verstraete, Kato, Lootens*, 2509.03600]*

2. Is MPDO a steady state of local Lindbladian?

→ *[YL, Ruiz-de-Alarcón, Styliaris, Sun, Pérez-García, Cirac, 2501.10552]*

$\rho^{(N)}(M) =$ 

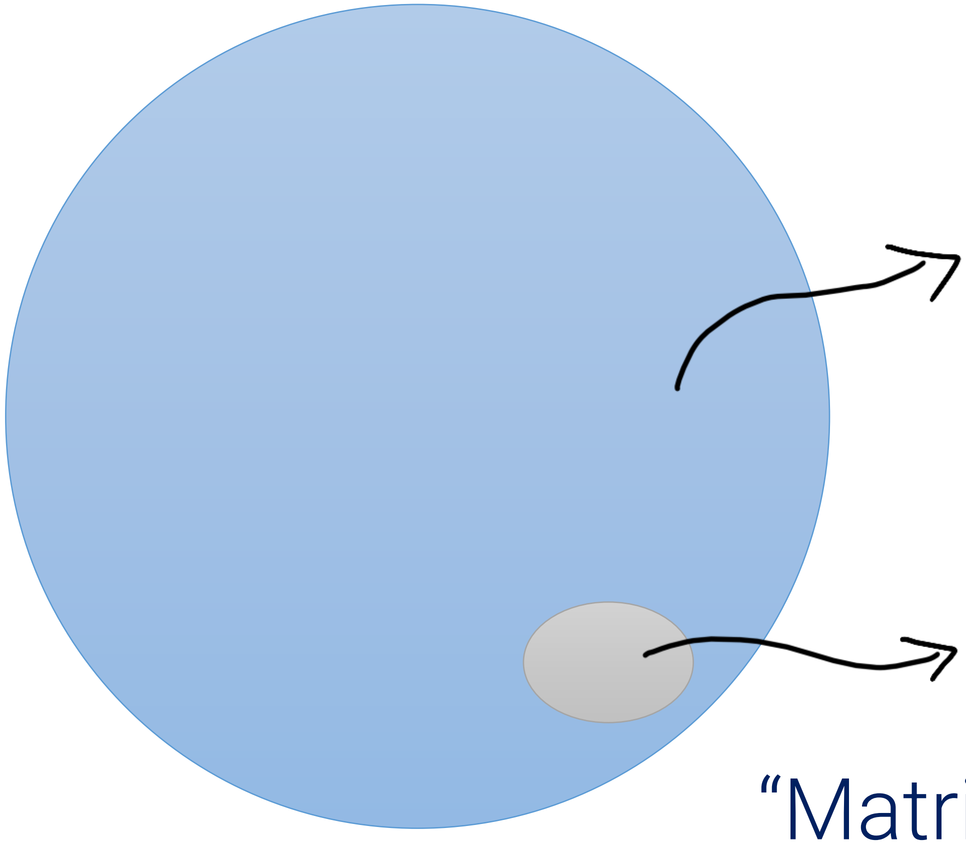
Yet another Take-away message:

Tensor network can be an **analytical** tool. 🤔

How to describe a quantum state?

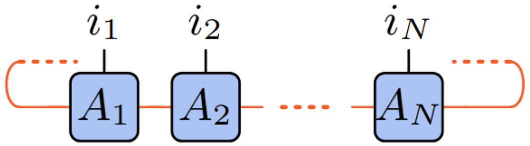
[Fannes, Nachtergaele, Werner, 1992]
[Perez-Garcia, Verstraete, Wolf, Cirac 2006]
[Hastings, 2006]
[Cirac, Perez-Garcia, Schuch, Verstraete, 2021]

- A general state of N spin: Hilbert space dimension d^N
- Ground state of a local Hamiltonian



$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_N} c_{i_1,i_2,\cdots,i_N} |i_1, i_2, \cdots, i_N\rangle$$

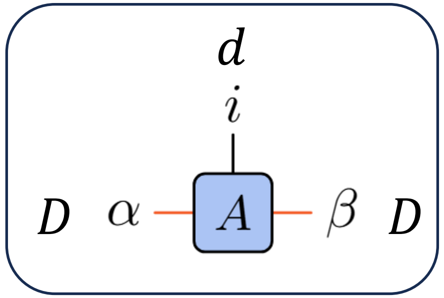
d^N parameters



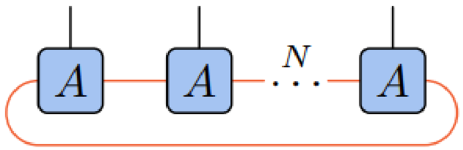
$$c_{i_1,i_2,\cdots,i_N} = \mathrm{Tr}\left(A_1^{i_1} A_2^{i_2} \cdots A_N^{i_N}\right)$$

$|\psi_{\mathrm{GS}}\rangle$ ~N parameters

“Matrix Product State”



MPS canonical form

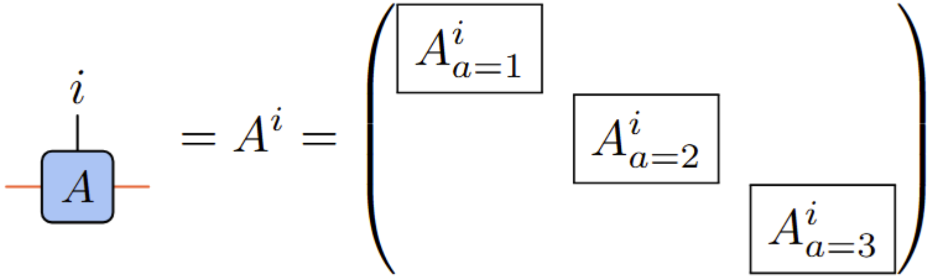
$|\psi^{(N)}(A)\rangle =$ 

- One can always bring an MPS tensor into the **canonical form** that represents **same state**:

$$A^i = \bigoplus_{a=1}^{g} (\mu_a \otimes A_a^i),$$

where $\mu_a = \text{diag}\{\mu_{a,1}, \mu_{a,2}, \cdots, \mu_{a,r_a}\}.$

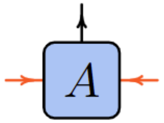
Basis of normal tensor (BNT)



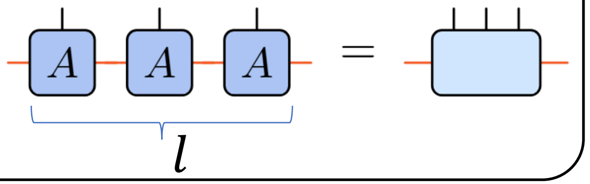
Example: GHZ state has 2 blocks

- Each block is a **normal tensor**

Definition (Injective tensor): *An MPS tensor A is injective if viewed as a linear map, the linear map is injective.*



Definition (Normal tensor): *An MPS tensor A is normal if after blocking, it is injective tensor.*



l is called injective length

[Cirac, Perez-Garcia, Schuch, Verstraete, 2021]

Examples of MPS

Example 1: cluster state

$$A^{0} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \qquad A^{1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}$$

$$|\psi\rangle = \left(\prod_i \mathrm{CZ}_{i,i+1}\right)|++\cdots+\rangle$$

Example 2: AKLT state

$$A^{1} = \sqrt{\frac{2}{3}}\sigma^{+}, \qquad A^{0} = -\sqrt{\frac{1}{3}}\sigma^{z}, \qquad A^{-1} = -\sqrt{\frac{2}{3}}\sigma^{-}$$

Example 3: GHZ state

$$A^{0} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad A^{1} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

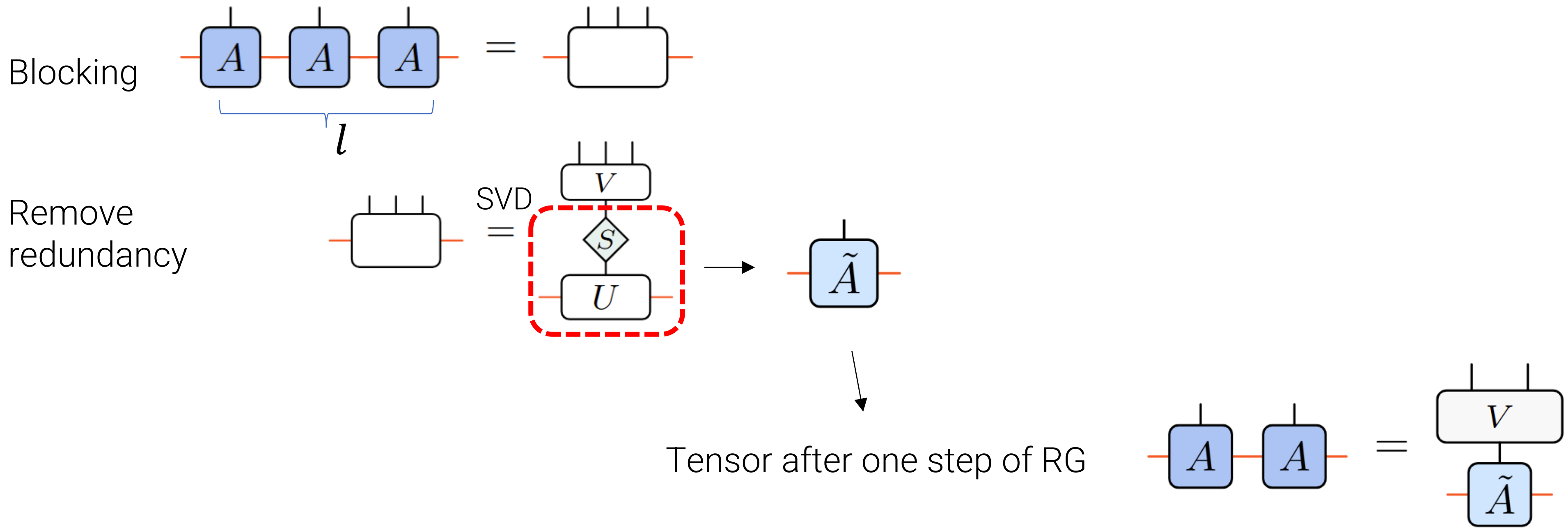
$$|\psi_{\mathrm{GHZ}}\rangle = |00\cdots0\rangle + |11\cdots1\rangle$$

Exercise

	cluster	AKLT	GHZ
Injective?	✗	✗	✗
Normal?	✓ $l = 2$	✓ $l = 2$	✗

MPS Renormalization

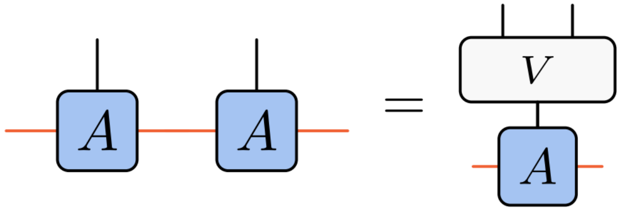
- **Coarse graining**: blocking, and remove redundancy by isometry



[Verstraete, Cirac, Latorre, Rico, and Wolf, 2005]

MPS Renormalization Fixed Points

Definition (Renormalization fixed point, RFP): *A tensor A appears as a limit in the above renormalization flow if and only if*



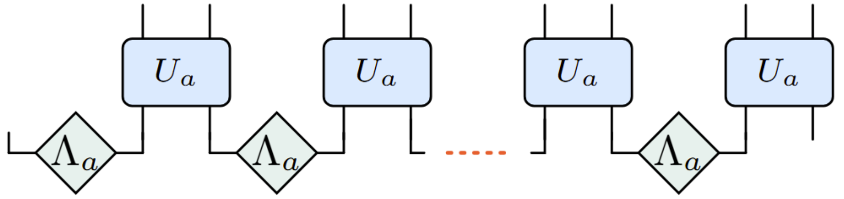
for an isometry V.

Theorem 1(structure theorem): *A tensor A in canonical form is an RFP iff*

$$A^i = \bigoplus_{a=1}^{g} \bigoplus_{q=1}^{r_a} \mu_{a,q} X_{a,q} \Lambda_a U_a^i X_{a,q}^{-1}$$

where Λ_a is diagonal, positive, and $\mathrm{tr}(\Lambda_a^2) = 1$.

Graphically, each block a generates:



Product of entangled pairs, up to local unitaries

[Cirac, Perez-Garcia, Schuch, and Verstraete, 2017]

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Why should I care about canonical form & fixed point? 🤔

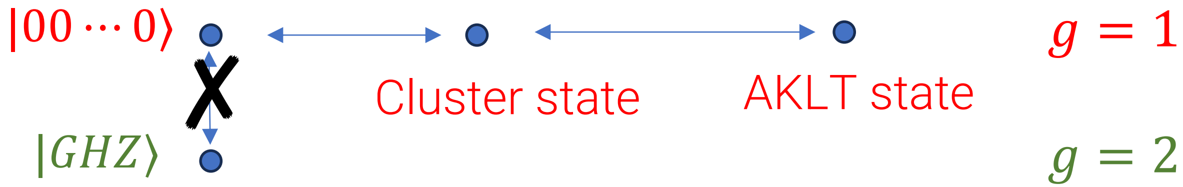
One application: phase classification.

Phase classification (pure)

- **Definition (phase equivalence)**: *Two systems are in the same* ***phase*** *if they can be connected by a continuous path of gapped local Hamiltonian.*

Claim: An MPS state $|\psi(A)\rangle$ is in the same phase as its corresponding fixed point $|\psi(A_{FP})\rangle$

Claim: Two fixed-point MPS state $|\psi(A_{FP})\rangle$ and $|\psi(B_{FP})\rangle$ are in the same phase if they have the same number of BNT elements, namely, $g_A = g_B$.



Claim: 1D phases of matter are fully classified by g

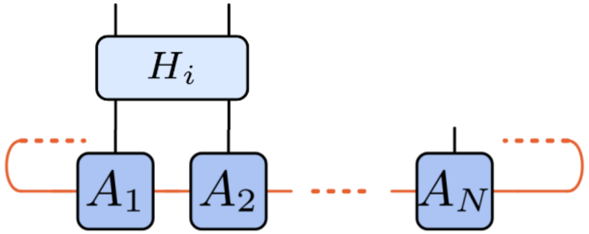
Proof: Use parent Hamiltonian.

[Chen, Gu, Wen, 2011]
[Schuch, Pérez-García, Cirac 2011]

Parent Hamiltonian

- Given a Matrix Product State $|\psi\rangle$ can we find a geometrically local Hamiltonian H such that $|\psi\rangle$ is its ground state?
- A "naïve" guess: $H = \mathbb{1} - |\psi\rangle\langle\psi|$
- However, this Hamiltonian is non-local

$$H^{(N)} = \sum_{i=1}^{N} H_i$$



- Yes! There exists a construction called **parent Hamiltonian**, which is geometrically local

[Perez-Garcia, Verstraete, Wolf, Cirac 2006]

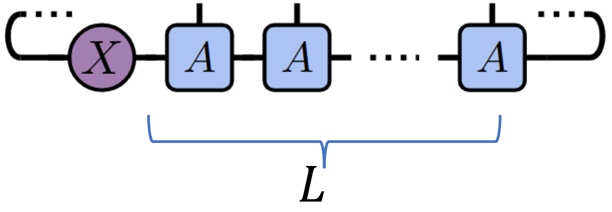
Parent Hamiltonian Construction

$$A^i = \begin{pmatrix} \boxed{A^i_{a=1}} & & \\ & \boxed{A^i_{a=2}} & \\ & & \boxed{A^i_{a=3}} \end{pmatrix}$$

L: injective length + 1

- Consider the space of L consecutive sites with arbitrary boundary condition X

$$G_L = \{|\psi^{(L)}(A)\rangle_X | \forall X \in \mathcal{M}_D\}$$



- Local term h in parent Hamiltonian satisfies:

$$h \geq 0$$

$$\ker(h) = G_L$$

- A simple choice of h: $\text{projector to } G_L^{\perp}$

See, Andras's talk

Properties:

1. **Frustration freeness**

$$H^{(N)}|\mathrm{GS}\rangle = 0 \quad \Leftrightarrow \quad h_i|\mathrm{GS}\rangle = 0$$

2. **Ground-state degeneracy** = # elements in BNT = g, which is **minimal**

3. A tensor A is a **renormalization fixed point** iff $|\psi^{(N)}(A)\rangle$ is a ground state of a **commuting** parent Hamiltonian

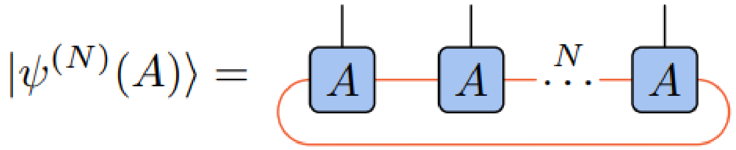
Now: How about mixed state? 🤔

How to describe a *mixed* quantum state?

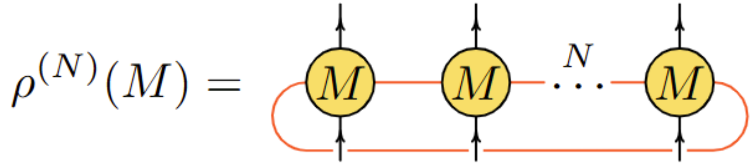
[Cirac, Perez-Garcia, Schuch, and Verstraete, 2017]

- A natural "guess" for "low entanglement" states:

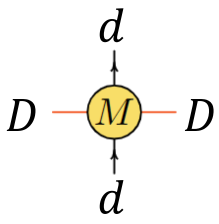
Subtlety: [Kliesch, Gross, Eisert 2014]



Matrix Product State



"Matrix Product Density Operator"



$D^2 d^2$ parameters

Fixed points are sum of maximally entangled pairs up to local unitary

MPS state is ground state of local Hamiltonian

How do fixed points look like?

Is MPDO steady state of local Lindbladian?

Examples of MPDO

- Many interesting states are matrix product density operators (MPDO)
- **Example 1**: 1D Gibbs state of quasi-local Hamiltonian

$$e^{-H}$$

[Molnar, Schuch, Verstraete, Cirac, 2015]
[Chen, Kato, Brandao, 2020]

- **Example 2**: boundary of 2D topological order
- For example, boundary of 2D toric code

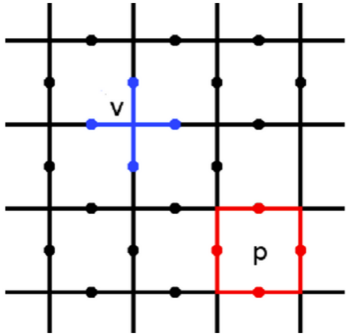
$$\rho^{(N)}(M) = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}).$$

And, more interestingly,

$$\rho^{(N)}(M) = \frac{1}{2^N}(\mathbb{1}_2^{\otimes N} + CZX)$$

We will come back to this later

Picture from Wikipedia



[Molnar, Ruiz-de-Alarcon, Garre-Rubio, Schuch, Cirac, Perez-Garcia, 2022]
[Ruiz-de-Alarcon, Garre-Rubio, Molnar, Perez-Garcia, 2024]

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In our work, we answer:

1. How do fixed points look like?

→ *[YL*, Molnar, Sun, Verstraete, Kato, Lootens*, 2509.03600]*

2. Is MPDO steady state of local Lindbladian?

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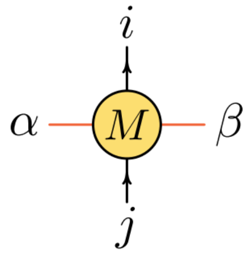
MPDO canonical form

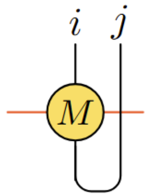
$(M^{ij})_{\alpha\beta} =$  $\in \mathbb{C}$

- For MPDO, canonical form can be defined horizontally, or vertically

- **Proposition (Horizontal canonical form):** *Any rank-4 tensor can always be brought into another rank-4 tensor M that generates the same MPDO state, and M is in the horizontal canonical form,*

$$M^{ij} = \bigoplus_{a_\mathrm{h}=1}^{g_\mathrm{h}} (\mu_{a_\mathrm{h}} \otimes M^{ij}_{a_\mathrm{h}}),$$

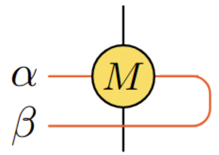
$$\text{where} \quad \mu_{a_\mathrm{h}} = \mathrm{diag}\{\mu_{a_\mathrm{h},1}, \mu_{a_\mathrm{h},2}, \cdots, \mu_{a_\mathrm{h},n_{a_\mathrm{h}}}\},$$

$M^{ij} =$ 

where the tensors $\{M_{a_h}\}$ form a BNT.

- **Proposition (Vertical canonical form):** *For any tensor M generating MPDO and in the horizontal canonical form, there exists an isometry U such that $UM_{(\alpha\beta)}U^\dagger$ is in the vertical canonical form*

$$UM_{(\alpha\beta)}U^\dagger = \bigoplus_{a=1}^{g} \mu_a \otimes M_{(\alpha\beta),a}$$

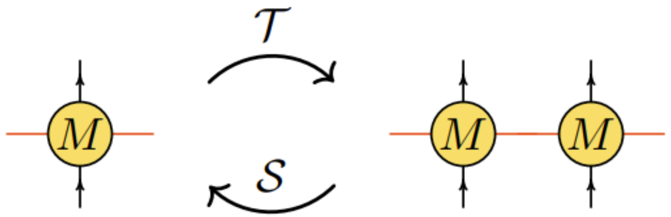
$M_{\alpha\beta} =$ 

where μ_a are diagonal and positive matrices, and the tensors $\{M_a\}$ form a BNT.

MPDO Renormalization Fixed Points

[Cirac, Perez-Garcia, Schuch, and Verstraete, 2017]

- **Definition (Renormalization fixed point, RFP):** *A tensor M generating MPDO is a renormalization fixed point (RFP) if there exists two quantum channels T and S, that fulfill:*

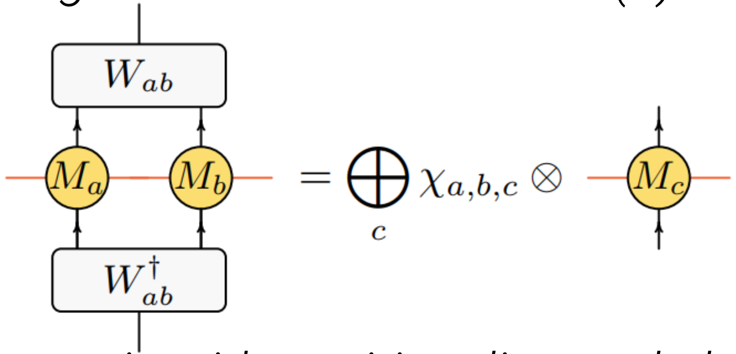


Consider the vertical canonical forms for tensor M,

$$U M_{(\alpha\beta)} U^\dagger = \bigoplus_a \mu_a \otimes M_{(\alpha\beta),a}$$

and define $m_a = \mathrm{tr}[\mu_a]$.

Theorem 2: *A tensor M generating an MPDO is an RFP iff (1) there exists isometries W_{ab} such that*



Where each $\chi_{a,b,c}$ is a diagonal matrix with positive diagonal elements, and (2) $m_c = \sum_{ab} \mathrm{Tr}[\chi_{a,b,c}] m_a m_b$

How do fixed points look like?

To answer this, we need a little bit of math

A quick intro to C*-pre-bialgebra

- **Algebra (over a field $\mathbb{C}$):** *An algebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ with bilinear operator called "product",*

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

which is associative, meaning $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in \mathcal{A}$;

and unital, meaning there exist $1 \in \mathcal{A}$ such that $1 \cdot a = a \cdot 1 = a$ for any $a \in \mathcal{A}$.

- **Pre-bialgebra**: *A pre-bialgebra A is a unital associative algebra together with a linear map called "coproduct",*

$$\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$$

which is associative $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$

and counital, meaning there exists $1_{\mathcal{A}^*} : \mathcal{A} \to \mathbb{C}$ such that

$$(1_{\mathcal{A}^*} \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes 1_{\mathcal{A}^*}) \circ \Delta = \mathrm{Id}$$

and coproduct is multiplicative, meaning $\Delta(xy) = \Delta(x)\Delta(y)$

For finite-dimensional $\mathcal{A}$

basis $\{e_I\}$ of $\mathcal{A}$,

product

$$e_I \cdot e_J = \sum_K \lambda_{IJ}^K e_K$$

Coproduct

$$\Delta(e_I) = \sum_{JK} \Lambda_I^{JK} e_J \otimes e_K$$

[Bohm and Szlachanyi,1996]

A quick intro to C*-pre-bialgebra

Example: $\mathbb{C}[\mathbb{Z}_2]$ group algebra

- Z_2 group $G = \{1, z\}$
- The algebra basis: $e_1 = 1, e_2 = z$ (so the vector space is $V = \{c_1 e_1 + c_2 e_2 | c_1, c_2 \in \mathbb{C}\}$)

- Product: inherited from group product $e_1 e_1 = e_2 e_2 = e_1, e_1 e_2 = e_2 e_1 = e_2$
- Coproduct Δ: we define $\Delta(e_i) = e_i \otimes e_i$

- You can verify that all axioms of pre-bialgebra are satisfied.

A quick intro to C*-pre-bialgebra

- **Claim**: Given a pre-bialgebra $\mathcal{A}$, its dual vectors space $\mathcal{A}^*$ inherits a pre-bialgebra structure.

$$\text{dual basis } \{e^I\} \text{ of } \mathcal{A}^* \text{ such that } e^J(e_I) = \delta_I^J$$

The product and coproduct are inherited from:

$$e_I e_J = \sum_K \lambda_{IJ}^K e_K \quad \Rightarrow \quad \Delta(e^I) = \sum_{JK} \lambda_{JK}^I e^J \otimes e^K$$

$$\Delta(e_I) = \sum_{JK} \Lambda_I^{JK} e_J \otimes e_K \quad \Rightarrow \quad e^I e^J = \sum_K \Lambda_K^{IJ} e^K$$

A direct result is:

$$\boxed{\sum_{IJ} e_I \otimes e_J \otimes e^I e^J = \sum_K \Delta(e_K) \otimes e^K}$$

Will be useful for MPDO!

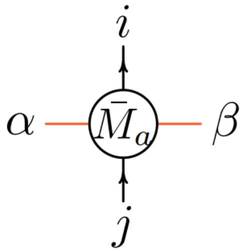
C*-pre-bialgebra: is a pre-bialgebra endowed with an anti-linear map $*: \mathcal{A} \to \mathcal{A}$ which is an involution $a^{**} = a$, anti-homomorphism $(ab)^* = b^* a^*$, and cohomomorphism

$$\Delta \circ * = (* \otimes *) \circ \Delta$$

How do fixed points look like?

- **Claim:** C*-pre-bialgebra, which may lack counit, generates MPDO fixed points.

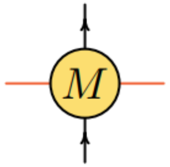
Given a C*-pre-bialgebra (which may lack counit), an irreducible *-representation ϕ_a of $\mathcal{A}$, and a representation ψ of $\mathcal{A}^*$, define a rank-4 tensor



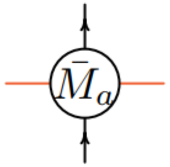
$$= \sum_I [\phi_a(e_I)]_{ij} [\psi(e^I)]_{\alpha\beta}$$

Note: I use a to label both vertical BNT elements and irrep of $\mathcal{A}$

Theorem 3: *Let $\mathcal{A}$ be an associative semisimple C*-pre-bialgebra, possibly lacking a counit and not necessarily cosemisimple. If (1) the fusion multiplicities N_{ab}^c are transitive, and (2) for any $a \in Irr(A)$ there exists $a^* \in Irr(A)$ such that $N_{a^*} = N_a^T$, then the MPO*



$$= \bigoplus_{a \in Irr(\mathcal{A})} \frac{d_a}{\mathcal{D}^2}$$



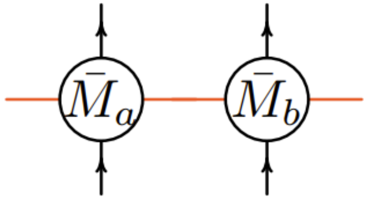
Where the direct sum is in the vertical direction, with d_a being the spectral radius of N_a, and $\mathcal{D}^2 = \sum_a d_a^2$, satisfies the renormalization fixed point condition.

N_{ab}^c will be defined on next page

[YL*, Molnar, Sun, Verstraete, Kato, Lootens*, 2509.03600]

Why this construction work?

This construction satisfies **Theorem 2** because

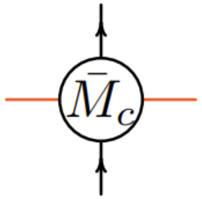


$$= \sum_{IJ} \phi_a(e_I) \otimes \phi_b(e_J) \otimes \psi(e^I e^J)$$

$$= \sum_{I} [(\phi_a \otimes \phi_b) \circ \Delta(e_I)] \otimes \psi(e^I)$$

$$\simeq \sum_{I} \left[\bigoplus_c \mathbb{1}_{N_{ab}^c} \otimes \phi_c(e_I) \right] \otimes \psi(e^I)$$

$$= \bigoplus_c \mathbb{1}_{N_{ab}^c} \otimes$$



,

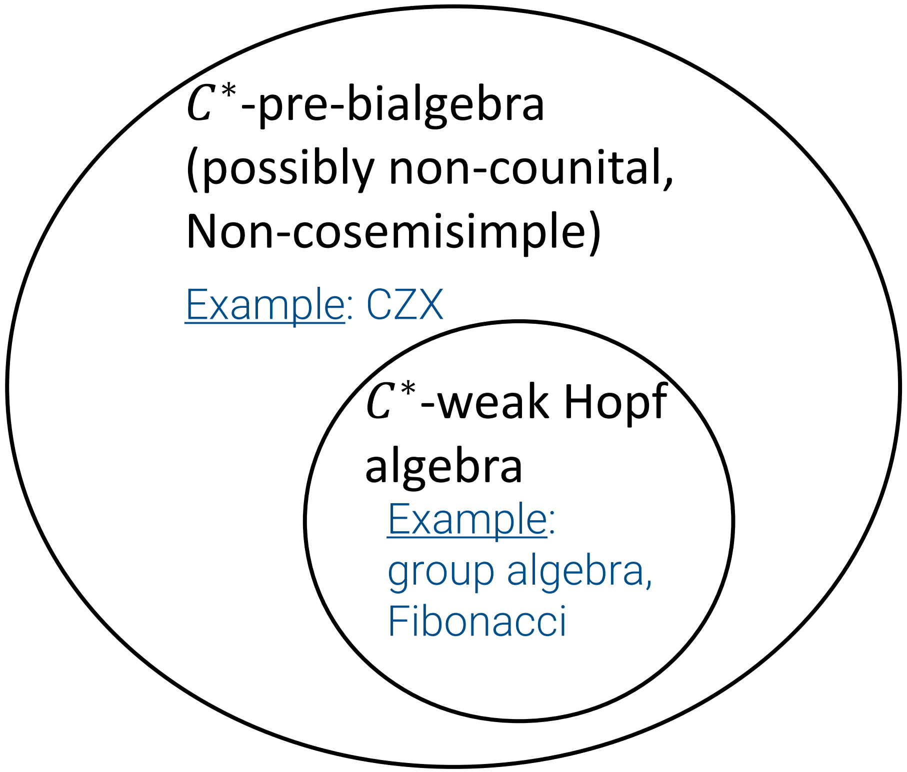
$$\sum_{IJ} e_I \otimes e_J \otimes e^I e^J = \sum_K \Delta(e_K) \otimes e^K$$

A is semisimple

The coefficients $d_a/\mathcal{D}^2$ is chosen to satisfy $m_c = \sum_{ab} \mathrm{Tr}[\chi_{a,b,c}] m_a m_b$

Why this construction work?

!! Note: our conditions are much relaxed compared to previous works based on C*-weak Hopf algebra or unitary fusion category



A C*-weak Hopf algebra is a C*-pre-bialgebra with more structures (more operators, more consistency relations)

Non-counital:
There is no identity object in the fusion category!

Our conditions include ALL known MPDO fixed points (up to 2025.08)

[Molnar, Ruiz-de-Alarcon, Garre-Rubio, Schuch, Cirac, Perez-Garcia, 2022]
[Ruiz-de-Alarcon, Garre-Rubio, Molnar, Perez-Garcia, 2024]
[YL, Molnar, Sun, Verstraete, Kato, Lootens*, 2509.03600]*

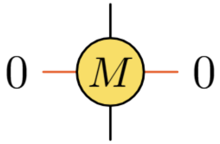
A first example: toric code boundary

Toric code boundary can be generated by $\mathbb{C}[\mathbb{Z}_2]$ group algebra. A is semisimple and co-semisimple (A^* is semisimple).

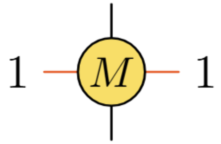
- Take ψ as the faithful representation,

$$\psi(e^1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi(e^2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- The tensor is



$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- It generates

$$\rho^{(N)}(M) = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}).$$

An interesting example: CZX

[Lessa, Cheng and Wang, 2025]
[*YL**, *Molnar, Sun, Verstraete, Kato, Lootens, 2509.03600*]

- Consider the density matrix

$$\rho_{CZX}^{(N)} = \frac{1}{2^N}(\mathbb{1}_2^{\otimes N} + U_{CZX}^{(N)})$$

Where U_{CZX} is a nontrivial representation of Z_2 group element associated with a 3-cocyle

$$U_{CZX}^{(N)} = \prod_{i=1}^{N} CZ_{i,i+1} \prod_{i=1}^{N} X_i$$

- It has an MPDO representation with $D = 3$

$$\tilde{M}^{11} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{M}^{22} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{M}^{12} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{M}^{21} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

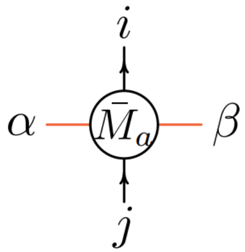
- The MPDO is generated by a C*-pre-bi-algebra which is **non-counital** and **non-cosemisimple**

!! Note: to obtain its full MPO algebra, non-semisimple indecomposable rep of A^* is required

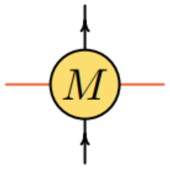
Summary I

1. How do MPDO renormalization fixed points look like?

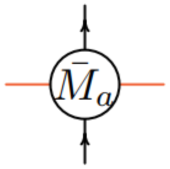
➡ They are generated by C*-pre-bialgebra



$$= \sum_I [\phi_a(e_I)]_{ij}[\psi(e^I)]_{\alpha\beta}$$



$$= \bigoplus_{a\in Irr(\mathcal{A})} \frac{d_a}{\mathcal{D}^2}$$



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In our work, we answer:

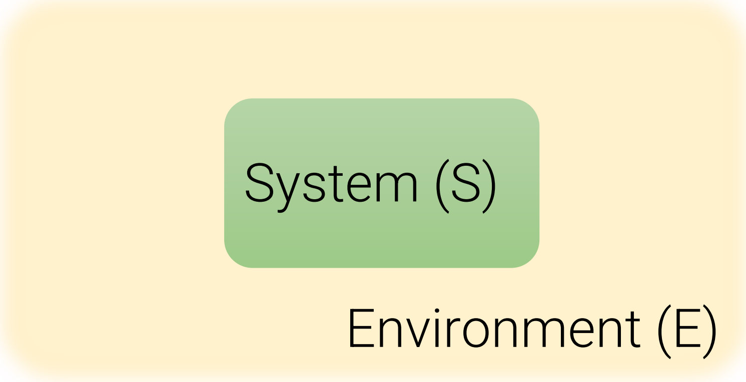
How do fixed points look like?

→ *[YL*, Molnar, Sun, Verstraete, Kato, Lootens*, 2508.*****]*

2. Is MPDO steady state of local Lindbladian?

→ *[**YL**, Ruiz-de-Alarcón, Styliaris, Sun, Pérez-García, Cirac, 2501.10552]*

In real-world systems, coupling to the environment is inevitable



After tracing out the environment,

S+E		S
Pure state		Mixed state
$\|\psi\rangle$	→	$\rho = \mathrm{Tr}_{\mathrm{E}}\|\psi\rangle\langle\psi\|$
Hamiltonian evolution		Lindbladian evolution
e^{-iHt}	→	$e^{\mathcal{L}t}$

$$|\dot{\psi}\rangle = -iH|\psi\rangle$$

$$\dot{\rho} := \mathcal{L}(\rho) = -i[H,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

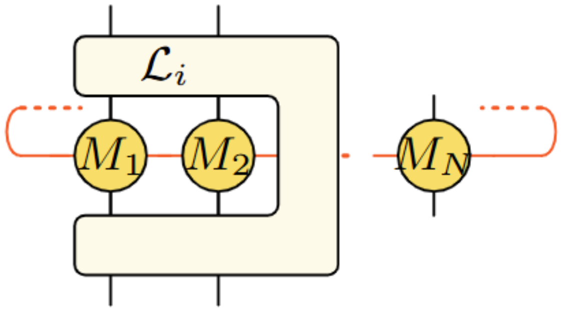
Parent Lindbladian

Now,

$$\dot{\rho} := \mathcal{L}(\rho) = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

- Given a Matrix Product Density Operator ρ, can we find a geometrically local Lindbladian $\mathcal{L}$ such that ρ is its steady state?

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i$$



Steady state

$$\mathcal{L}(\rho_\infty) = 0$$

$$\lim_{t\to\infty} e^{\mathcal{L}t}(\rho_{\text{init}}) = \rho_\infty$$

- Yes*! We construct the **parent Lindbladian**

* For Renormalization fixed points

Features of parent Lindbladian

- We construct parent Lindbladians for MPDO renormalization fixed-points, and they satisfy:

1. Geometrically local
2. Frustration-free
3. Reaches **minimal** steady-state degeneracy

	Parent Hamiltonian H	Parent Lindbladian $\mathcal{L}$ (RFP)
Geometrically local	$H = \sum_i h_i$	$\mathcal{L} = \sum_i \mathcal{L}_i$
Frustration-free	$h_i \lvert\psi_{\mathrm{MPS}}\rangle = 0\ \forall i \quad (H \geq 0)$	$\mathcal{L}_i(\rho_{\mathrm{MPDO}}) = 0\ \forall i$
Degeneracy	Minimal ground-state degeneracy	Minimal steady-state degeneracy
Degeneracy implies long-range correlation	✓	Not always
Commuting for RFP?	✓	Not possible for a class of RFP

Table I: Properties of parent Lindbladian for MPDO renormalization fixed-point, in comparison with parent Hamiltonian for MPS.

*[**YL**, Ruiz-de-Alarcón, Styliaris, Sun, Pérez-García, Cirac, 2501.10552]*

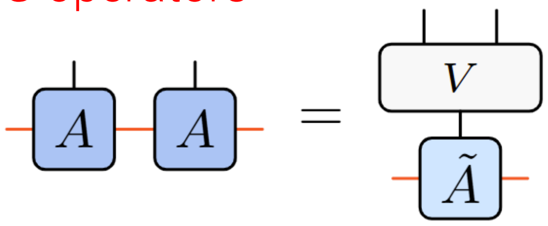
Here come the key insights…

Parent Lindbladian construction

- Is actually very intuitive, by revisiting **renormalization**

The parent Hamiltonian

$h =$ projector to $G_L^{\perp}$ can be **rewritten** using RG operators



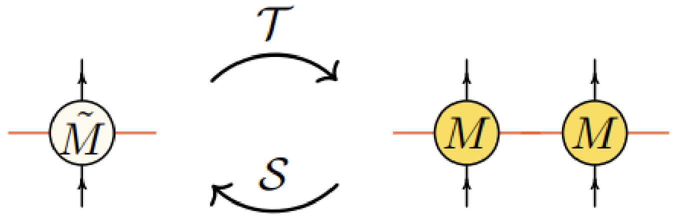
$V^{\dagger}V = \mathbb{1}$

$$h = \mathbb{1} - VV^{\dagger}$$

$$H^{(N)} = \sum_{i=1}^{N} h_i$$

where $h_i = \tau_i(h)$ and τ_i translates the sites by i.

By analogy,



$$\mathcal{E} = \mathcal{T} \circ \mathcal{S},$$

$$\mathcal{L}^{(N)} = \sum_{i=1}^{N} \mathcal{L}_i$$

$$\mathcal{L}_i = \mathcal{E}_i - \mathbb{1},$$

denote $\mathcal{E}_i = \tau_i(\mathcal{E})$ where τ_i translates the sites by amount i.

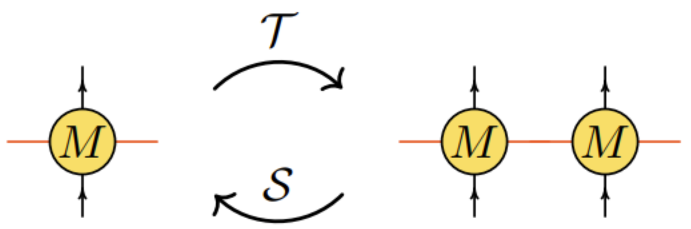
This Lindbladian only has jump operators

Yes, but...

Is this construction useful?

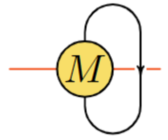
- How do you define RG and find RG channels?
- Is such construction frustration-free?
- Does steady-state degeneracy (SSD) grow with system size?
- ...

We show, all the above questions can be settled, for renormalization fixed-point (RFP)



✓ Explicit construction of RG channels
✓ Frustration-free
✓ Minimal SSD

RFPs have nice properties:
1. Zero correlation length $E^2 = E$
2. CMI $I(A\colon C|B) = 0$ if their complement is non-empty

$E =$ 

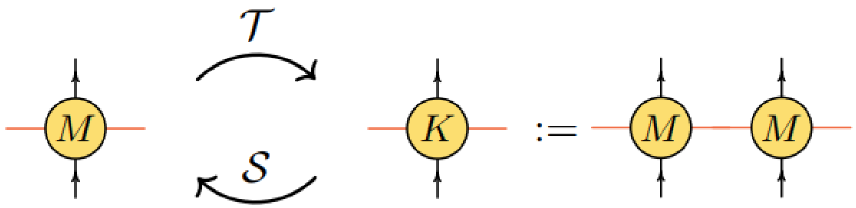
Parent Lindbladian construction

- To construct the parent Lindbladian, we need to construct the RG channels T and S.
- Here is one construction based on the **vertical canonical form**.

Consider the vertical canonical forms for tensor M,

$$U M_{(\alpha\beta)} U^\dagger = \bigoplus_a \mu_a \otimes \boxed{M_{(\alpha\beta),a}}$$

$$V K_{(\alpha\beta)} V^\dagger = \bigoplus_a \nu_a \otimes \boxed{K_{(\alpha\beta),a}}$$



and define $m_a = \mathrm{tr}[\mu_a], n_a = \mathrm{tr}[\nu_a]$.

<u>Theorem 4</u>: *A tensor M generating an MPDO is an RFP iff the elements in vertical BNTs $\{M_a\}$ and $\{K_a\}$ are related by a unitary acting on the physical index*

$$K_{(\alpha\beta),a} = U_a M_{(\alpha\beta),a} U_a^\dagger$$

for any a, and $m_a = n_a$

BNT elements are the same for M and K!!

Parent Lindbladian construction

- Therefore, the channels are (in the example of two blocks $a = 2$)

$$\mathcal{T}(X) =$$
$$V^\dagger \left[(\frac{\nu_1}{m_1} \otimes P_1 U X U^\dagger P_1^\dagger) \oplus (\frac{\nu_2}{m_2} \otimes P_2 U X U^\dagger P_2^\dagger) \right] V$$

$$\mathcal{S}(X) =$$
$$U^\dagger \left[\mathrm{Tr}_1(P_1' V X V^\dagger (P_1')^\dagger) \oplus \mathrm{Tr}_1(P_2' V X V^\dagger (P_2')^\dagger) \right] U$$

"project, partial trace, and replace"

Interesting result 1: Gibbs state

- **I will mention 4 interesting results**

- One interesting class of MPDO RFP is **Gibbs state** of nearest-neighboring commuting Hamiltonian with zero correlation length

$$\rho = e^{-H}, \quad H = \sum_i h_i, \quad [h_i, h_j] = 0$$

- They can be written using MPDO

$$\rho^{(N)}(M) = \bigoplus_{a_{\mathrm{h}}=1}^{g_{a_{\mathrm{h}}}} n_{a_{\mathrm{h}}} \bigoplus_{k_1,\cdots,k_N \in S_{a_{\mathrm{h}}}} \eta_{k_1,k_2} \otimes \eta_{k_2,k_3} \otimes \cdots \otimes \eta_{k_N,k_1}$$

Different from Davies! Better?

- When there is a single a_{h}: **Commuting local Lindbladian → rapid mixing; unique steady state**

*What is rapid-mixing?

Definition 6.1. *Rapid mixing is the assumption that the convergence of the density matrix ρ_0 to its steady states ρ_∞ is of the form*

$$\|e^{t\mathcal{L}}(\rho_0) - \rho_\infty\|_1 \leq \text{cpoly}(N)e^{-\mu t} \tag{6.38}$$

where c, μ are constants independent of the system size N.

Note that after a time $t \gtrsim \text{poly}\log N$, the upper bound will vanish in the thermodynamic limit.

Why this matter? ⇒ mixed state phase

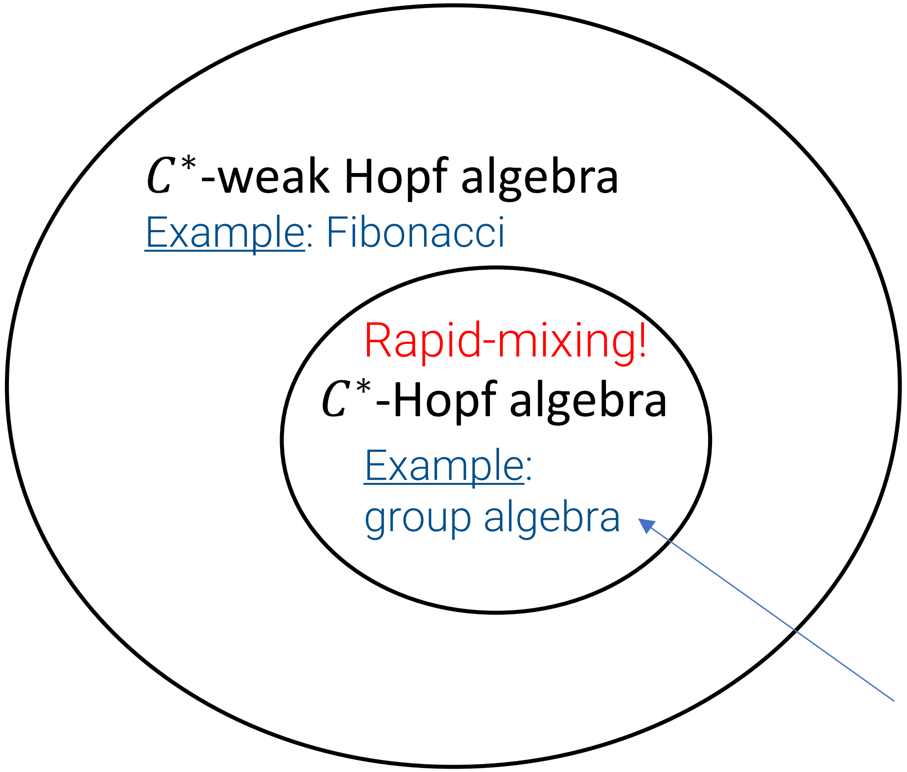
We say a state ρ_0 can be driven fast to another state ρ_1 and we write $\rho_0 \to \rho_1$, if it exists a dissipative evolution generated by a local and time-independent Lindbladian $\mathcal{L}$ such that for $t \gtrsim \text{poly}\log N$,

$$\|e^{t\mathcal{L}}(\rho_0) - \rho_1\|_1 \lesssim \text{poly}(N)e^{-\mu t}. \tag{6.39}$$

[Coser, Perez-Garcia, 2019]

Interesting result 2: Hopf vs weak Hopf

- Another interesting class of MPDO RFP is state generating from C^*-weak Hopf algebra.
- These are boundaries of 2D topological order



Theorem VI.3. *An MPDO RFP generated by C^*-Hopf algebras exhibits only short-range correlations and admits a commuting local rapid-mixing parent Lindbladian with minimal degenerate steady states under periodic boundary conditions. The minimal degeneracy equals g_h the number of BNT elements in the horizontal canonical form of the MPDO tensor, which is larger than 1.*

Example: toric code boundary is generated by group algebra $\mathbb{C}[Z_2]$

Interesting result 3: not always commuting for RFP

- Parent Hamiltonian is always commuting for RFP,
- While parent Lindbladian is *not* always commuting for RFP

No-go theorem for local commuting Lindbladian

Theorem V.4. *A non-injective simple MPDO generated by an RFP tensor M does not admit a local Lindbladian that has $\rho^{(N)}(M)$ as its steady state, is commuting, has minimal steady-state degeneracy, and has no purely imaginary eigenvalues.*

- Example:

$$\rho^{(N)}(M) = |0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N}.$$

Interesting result 4: SSD $\neq$ phase equivalence

- Pure state: phase of matter is classified by ground state degeneracy of parent Hamiltonian
- Mixed state: phase of matter is ***not*** classified by steady state degeneracy of parent Lindbladian
- Example:

$$\rho^{(N)}(M) = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}).$$

→ Short range correlation

$$\mathcal{E}(X) = \frac{1}{4}\left(\mathrm{Tr}(X)\mathbb{1}_2^{\otimes 2} + \mathrm{Tr}(\sigma_z^{\otimes 2}X)\sigma_z^{\otimes 2}\right)$$

Steady states $\rho_\infty^{(N)} = \frac{1}{2^N}\mathbb{1}_2^{\otimes N} + c\sigma_z^{\otimes N}$

(SSD)=2

$$\rho^{(N)}(M) = |0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N}.$$

→ Long range correlation

The two projectors for the two BNT elements are

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting $P^\perp = \mathbb{1} - P_0 \otimes P_0 - P_1 \otimes P_1$, the local channel takes the form

$$\begin{aligned}\mathcal{E}_i(X) = &\ \mathrm{Tr}\left((P_0 \otimes P_0)X\right) P_0 \otimes P_0 \\ &+ \mathrm{Tr}\left((P_1 \otimes P_1)X\right) P_1 \otimes P_1 + \mathrm{Tr}\left((P^\perp)X\right)\rho_0,\end{aligned} \tag{75}$$

Steady states: $\rho_\infty^{(N)} = c_1|0\rangle\langle 0|^{\otimes N} + c_2|1\rangle\langle 1|^{\otimes N}$

(SSD)=2

Summary II

2. Is MPDO steady state of local Lindbladian?

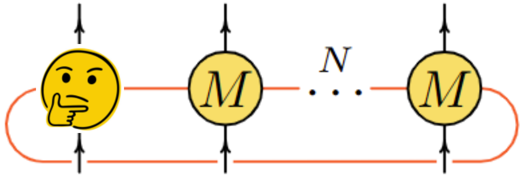
→ Yes, for renormalization fixed points. We construct parent Lindbladians which are local, frustration-free, and have minimal steady-state degeneracies.

$$\mathcal{E} = \mathcal{T} \circ \mathcal{S},$$

$$\mathcal{L}^{(N)} = \sum_{i=1}^{N} \mathcal{L}_i$$

$$\mathcal{L}_i = \mathcal{E}_i - \mathbb{1},$$

Future directions

$$\rho^{(N)}(M) =$$ 

- Can our **Theorem 3** be "if and only if"?

  "Our conditions include *ALL known* MPDO fixed points (up to 2025.08)"

  ➡ Are all the MPDO fixed points constructed as in theorem 3?

  Let me know if you find an exception ☺

- Parent Lindbladian Beyond RFPs? *[Kato 2024]*
- Parent Lindbladian for higher dimension (does temperature β play a role?) *Ongoing work*

- Establish full classification for the 1D MPDO phase using algebraic approach?
- MPDO phases in the presence of symmetry

Thank you!