Parent Hamiltonians

ground space degeneracies in 1D

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quantA

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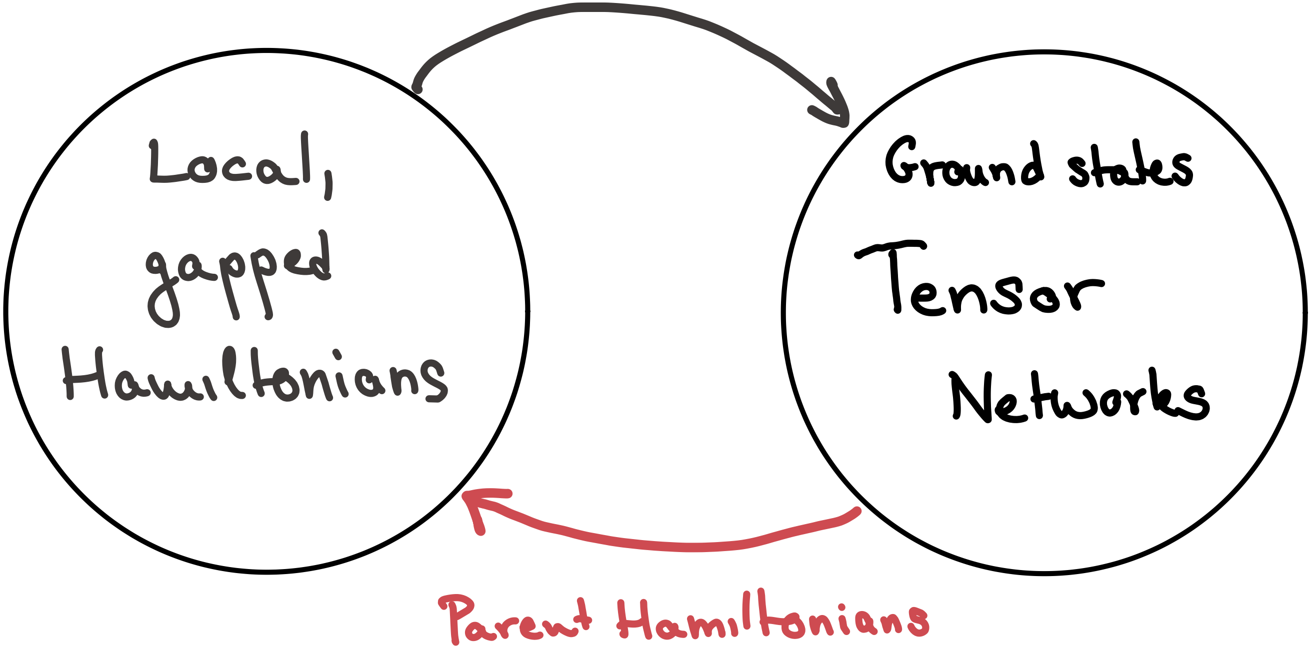
Local Hamiltonians

$$H = \sum_{i=0}^{n-2} 1^{\otimes i} \otimes h \otimes 1^{\otimes (n-i-2)} \in B(\mathcal{H}^{\otimes n})$$

Questions:

- Ground states?
- Spectral gap $\Delta = E_1 - E_0$?
- Excitations?



AKLT '88, Fannes '92

Motivation for parent Hamiltonians

- TNS $\supseteq$ Ground states
  (gapped, non-chiral, ...)
- parent Hamiltonian TNS $\subseteq$ GS
- $\Rightarrow$ Physics $=$ TNS

Matrix Product States

- MPS tensor

$$A = \sum_i A_i \otimes |i\rangle \in \mathcal{M}_D \otimes \mathcal{H}$$

- State $\psi_A^{(n)} \in \mathcal{H}^{\otimes n} \quad \forall n \in \mathbb{N}$

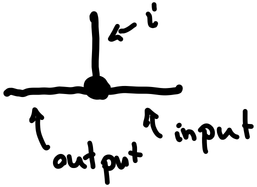
$$\psi = \sum_i \operatorname{tr}\{A_{i_1} A_{i_2} \ \ A_{i_n}\} |i_1 . \ i_n\rangle$$

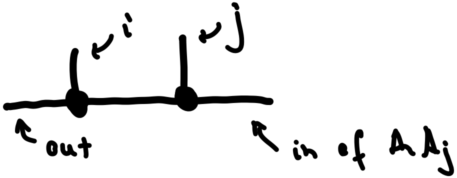
- States w/ boundary $\psi(X) \in \mathcal{H}^{\otimes n}$

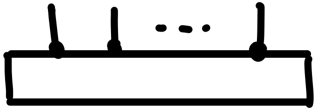
$$\psi(X) = \sum_i \operatorname{tr}\{A_{i_1} A_{i_2} \cdots A_{i_n}\} |i_1 \ldots i_n\rangle$$

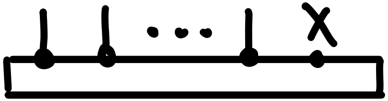
$$S_n := \{\psi(X) \mid X \in \mathcal{M}_D\}$$

Graphical notation of TNS

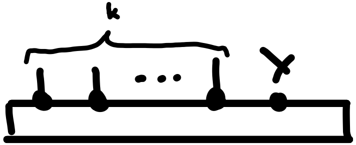
- $A = \sum_i A_i \otimes |i\rangle =$ 

- $\sum_{ij} A_i A_j \otimes |ij\rangle =$ 

- $\psi = \sum_i \mathrm{tr}\{A_{i_1} \cdot A_{i_n}\} |i_1 \ldots i_n\rangle =$ 

- $\psi(X) = \sum_i \mathrm{tr}\{X A_{i_1} \ldots A_{i_n}\} |i_1 \ldots i_n\rangle =$ 

Parent Hamiltonians

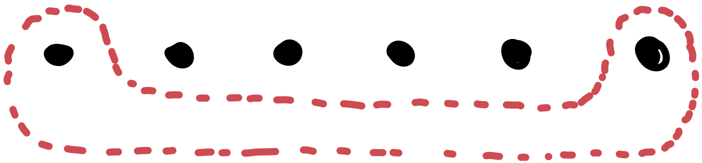
- $S_k = \Big\{$ 

$\Big| \, x \in M_D \Big\}, \quad k \geq 2$

- h = orthogonal projector to $S^\perp$

- $$H_{OBC} = \sum_{i=0}^{n-k} \mathbb{1}^{\otimes i} \otimes h \otimes \mathbb{1}^{\otimes (n-k-i)} \geq 0$$

- $$H_{PBC} = \sum_{i=0}^{n} \mathbb{1}^{\otimes i} \otimes h \otimes \mathbb{1}^{\otimes (n-k-i)} \geq 0$$



Boundary

$i = n$

Main message

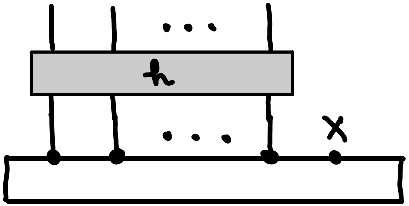
1. The MPS is ground state
2. There are no other ground states
3. The Hamiltonian is gapped

Main message

1. The MPS is ground state
2. There are no other ground states
3. The Hamiltonian is gapped

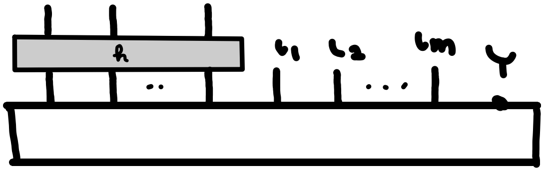
⚠ If...

Mechanism



$= 0 \quad \forall X$

Choose $X = A_{i_1} \cdots A_{i_m} Y$



$= 0 \quad \forall i$

So $h\psi(Y) = 0 \Rightarrow H\psi(Y) = 0$

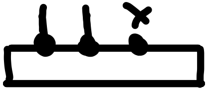
$\psi(x)$ is GS of OBC p H if $\psi(x) \neq 0$

First subtlety: if MPS = 0

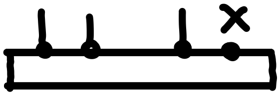
Example

$$A_0 = |0\rangle\langle 1| , \quad A_1 = |1\rangle\langle 2|$$

So that

$S_2 = \{$  $| \, X \in \mathcal{M}_2 \} = \mathbb{C}\,|01\rangle \neq 0$

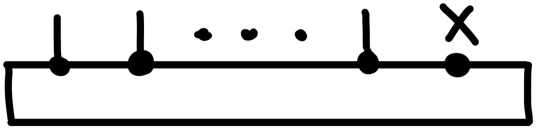
But for $n > 2$:

$S_n = \{$  $| \, X \in \mathcal{M}_D \} = 0$

$H = \sum_i h_i$, $h = \mathbb{1} - |01\rangle\langle 01|$ is frustrated!

"MPS is groud state..."

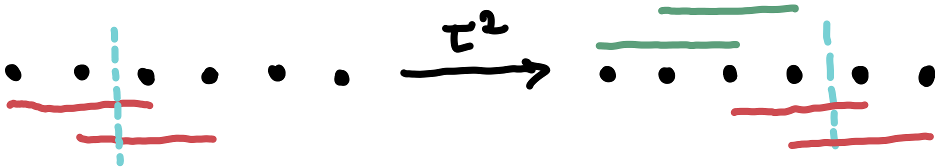
- Open boundary ground states

$S_n = \Big\{ \psi(X) =$  $\Big| \; X \in M_D \Big\}$

- Periodic boundary ground states

$$S_n \cap \tau^{k-1}(S_n) \quad \text{if} \quad n \geq 2k-1$$

↑ translation



Rigorous results in literature

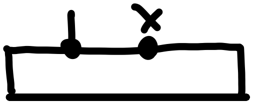
Behavior understood for 2 classes of MPS

1. Normal MPS

2. MPS in canonical form

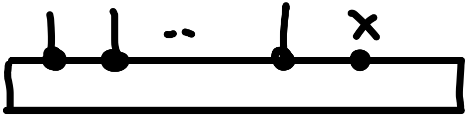
Normal MPS

1. Injective MPS

$X \longmapsto \Psi_1(X) =$ 

injective

2. Normal MPS. $\exists\, n \in \mathbb{N}$ s.t

$X \mapsto \Psi_n(X) =$ 

injective

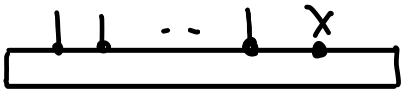
Normal MPS remarks

1. $X \mapsto \Psi_n(X)$ injective $\Rightarrow$
   $X \mapsto \Psi_m(X)$ as well $\forall m \geq n$
   min such n injectivity length
2. Normal $\Leftrightarrow$
   (a) Transfer mx has uniqe largest eigenvalue, non-degenerate
   (b) The corr eigen vector is positive definite (invertible!)

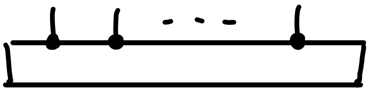
Normal MPS results

For normal MPS w/ inj length k

- GS of $(k+1)$-local OBC Ham is OBC MPS.

$$S_n = \Big\{ \quad$$  $$\Big| \; X \in \mathcal{M}_D \Big\}$$

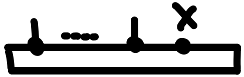
- GS of $(k+1)$-local PBC Ham. is PBC MPS.

$$\Psi_n = \mathbb{C} \quad$$ 

- The $(k+1)$-local Ham is gapped.

Fannes '92

Proof for injective MPS

We have seen: $GS \supseteq S_n = \{$  $\mid x \in M_D \} \neq 0$

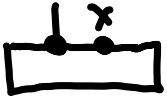
Need $GS \subseteq S_n$

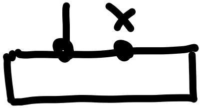
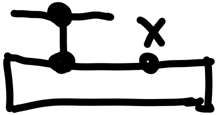
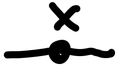
$$GS = \ker H = \bigcap_{i=1}^{n-1} \ker\left(\mathbb{1}^{\otimes i} \otimes h \otimes \mathbb{1}^{\otimes(n-i-2)}\right) =$$

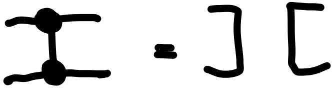
$$= \bigcap_{i=1}^{n-1} \mathcal{H}^{\otimes i} \otimes S_2 \otimes \mathbb{1}^{\otimes(n-i-2)}$$

I will show $(S_2 \otimes \mathcal{H}) \cap (\mathcal{H} \otimes S_2) \subseteq S_3$

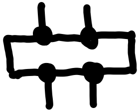
rest is similar

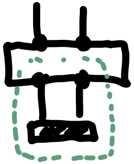
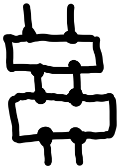
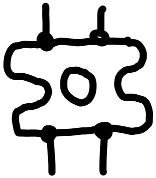
$X \mapsto$  injective $\Rightarrow \exists$ inverse.

$X \mapsto$  $\longmapsto$  $=$ 

That is 

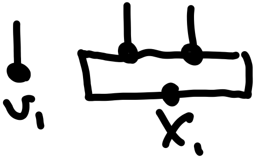
A non-Hermitian projector onto S_2.

$P = \frac{1}{D}$ 

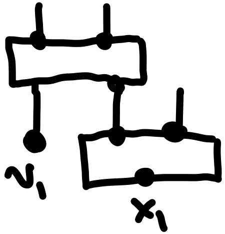
$P\sigma =$  $\in S_2$, $P^2 = \frac{1}{D^2}$  $= \frac{1}{D^2}$  $= P$

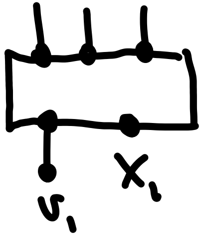
Consider $\varphi \in \mathcal{H} \otimes S_2 \cap S_2 \otimes \mathcal{H}$

Then

$\varphi = \sum_i$ 

And

$\varphi = (P \otimes \mathrm{Id})\varphi = \sum_i$ 

$=$ 

$\in S_3.$

$\square$

Properties of parent Hamiltonian

1. The MPS is ground state
2. There are no other ground states
3. The Hamiltonian is gapped

Literature:

- normal MPS
- MPS in canonical form

Questions

1. Can parent Hamiltonian "work" outside of normal MPS/canonical form?

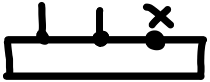
2 For normal MPS do we need inj. length +1?

3 Non-trivial examples where parent Hamiltonian doesn't work?

Non-normal MPS: Example

W-state: $\Psi = |100..0\rangle + |010.0\rangle + .. + |0..01\rangle$

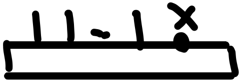
MPS tensor $A_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $A_1 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

$S_2 = \{$  $| \; X \in M_2 \} = \mathrm{Span}\{ |00\rangle, \underbrace{|01\rangle + |10\rangle}_{W_2} \}$

$h = \mathbb{1} - |00\rangle\langle 00| - |W_2\rangle\langle W_2|$, $H = \sum_{i=1}^{n/n-1} h_i$

<u>By visual inspection:</u>

$\mathrm{Ker}\, H = \mathrm{Span}\{ |00\rangle, \underbrace{|10..0\rangle + . + |0.01\rangle}_{W_n} \}$

$= \{$  $| \; x \in M_2 \} \Rightarrow$ Works!

Sufficient condition for good p.H

Let $A = \sum_{i=1}^{d} A_i \otimes |i\rangle$ be an MPS tensor,

$$V_n := \mathrm{Span}\{ A_{i_1} A_{i_2} \cdots A_{i_n} \mid i_1, \dots, i_n = 1, \dots, d \}$$

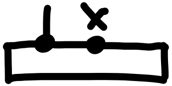
<u>Thm</u>: If $\exists\, Y_1, \dots, Y_d \in M_D$ and $k \in \mathbb{N}$ s.t.

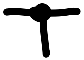
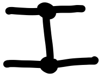
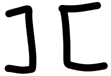
- $Y_i V_{k+1} \subseteq V_k \quad \forall i = 1, \dots, d$
- $Z = \sum_j A_j Y_j$ is s.t. $Z A_{i_1} \cdots A_{i_k} = A_{i_1} \cdots A_{i_k}$

Then the p.H has only the MPS as 0-energy state

Garre-Rubio 2501.17109

Application: Injective MPS

Injective MPS $\quad X \mapsto$  injective

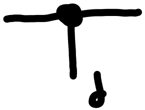
$\Longleftrightarrow \exists$  s.t.  $=$ 

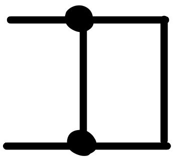
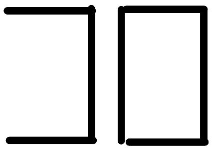
$\Longleftrightarrow V_1 = \mathrm{Span}\{A_i \mid i=1 \ldots d\} = M_D$

$\Longrightarrow V_n = \mathrm{Span}\{A_{i_1} \cdots A_{i_n}\} = M_D \quad \forall n \in \mathbb{N}.$

Conditions hold for $k = 1$.

- $Y_i V_2 \subseteq V_1 = \mathcal{M}_D$ trivial
- $Z = \sum_j A_j Y_j$ is s.t. $Z A_i = A_i$

Choose $Y_j =$ 

$\sum_j A_j Y_j$ =  =  $= D\,\mathbb{1}$

So Thm $\Rightarrow$ 2-local pH works for inj MPS

Application: W-state

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$Y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\Bigg\} \quad \sum_i A_i Y_i = \mathbb{1}$$

$$V_1 = V_2 = \ldots = V_n = \mathrm{Span}\{A_0, A_1\}$$

$$Y_i V_2 = A_i V_2 \subseteq V_3 = V_1 .$$

Condition of Thm holds for W.

Also for domain wall, etc

Questions

1. Sufficient condition on A for p.H. to work

2 For normal MPS do we need inj. length +1?

3 Non-trivial examples where parent Hamiltonian doesn't work?

Range of parent Hamiltonian

<u>Thm</u>: inj length $= k \Rightarrow k+1$-local p.H OK.

Can we do better? Example: AKLT

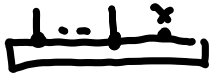
- $A_0 = X,\ A_1 = Y,\ A_2 = Z$
- Inj length $= 2 \rightarrow$ original thm applies for 3-local p.H only
- 2-local p H already OK
- Previous thm applies only for 3-local p.H

Range of parent Hamiltonian

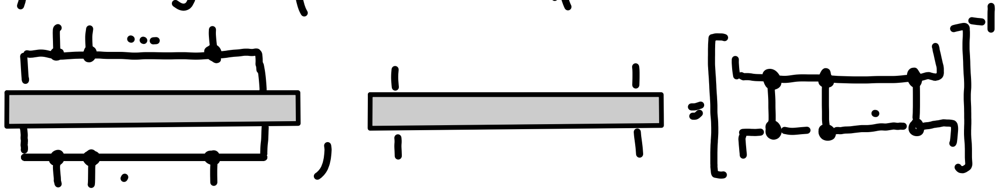
Let $A \in M_D \otimes \mathbb{C}^d$, $k = \min\{n \mid d^n > D^2\}$.

↑ 2 for AKLT

<u>Thm:</u> For generic A, the k-local p.H works either always or never

<u>Proof sketch</u> The orth proj onto $\{$  $\mid x\}$ is (real) analytic function of A.

$P_n =$



Schuch 2503 10767

Range of parent Hamiltonian

The n-body GS is too large if $\exists \Psi \neq 0$ s.t.

$$\Big[\underbrace{\sum_{i=1}^{n} (1 - P_2^i)}_{H} + \underbrace{P_L}_{\text{not an MPS}} \Big] \Psi = 0 .$$

$$\Rightarrow \det \Big(\sum_i (1 - P_2^i) + P_L \Big) = 0$$

But this is a real analytic function of A (if A injective)

$\Rightarrow$ Either 0 for all A or only on meas 0 set.

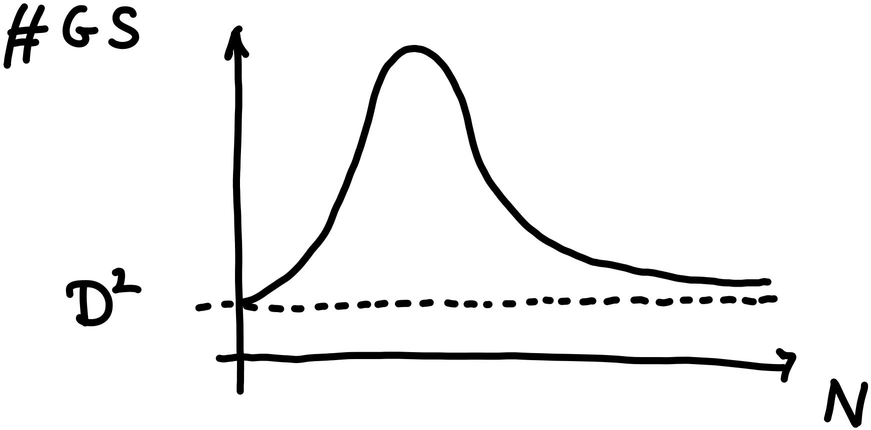
$\square$

Short parent Hamiltonians

Fix D, $\quad D^2 > d > D+1$, $A \in M_D \otimes \mathbb{C}^d$

A is not injective

Can define 2-local Ham.



Questions

1. Sufficient condition on A for p.H. to work

2 Generically short p H works

3 Non-trivial examples where parent Hamiltonian doesn't work?

4 Improved gap estimates?

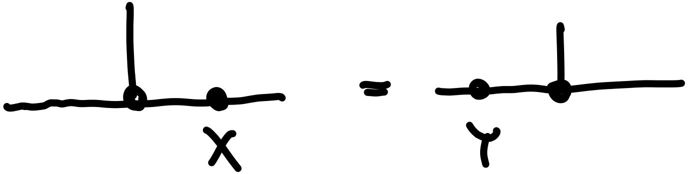
Failing parent Hamiltonians

Is there a (normal) MPS s.t. the p.H has <u>more</u> ground states than the MPS?

- Only short p H can fail if range > inj length + 1 $\Rightarrow$ works
- Non-generic
- Possible gs degeneracies?

Failing parent Hamiltonian I

Let A be normal MPS s.t. $\exists X, Y \neq \lambda \mathbb{1}$ s.t.



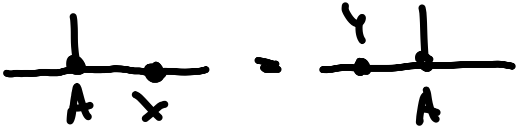
Remark 1. $X = Y \Rightarrow X = Y = \lambda I$.

$$A_{i_1} \cdots A_{i_n} X = X A_{i_1} \cdots A_{i_n}$$

$$M X = X M \quad \forall M \in \mathcal{M}_D$$

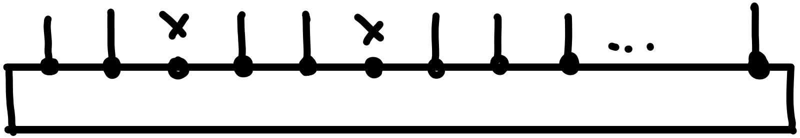
Remark 2. There are examples.

Failing parent Hamiltonian I

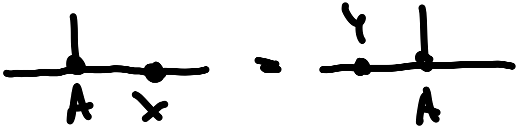


$\Rightarrow$ 2-local pH fails

A ground state that is not the MPS.



Failing parent Hamiltonian I



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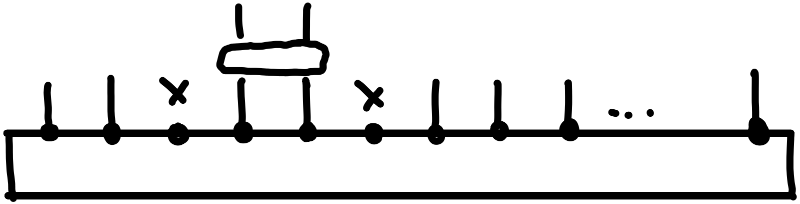

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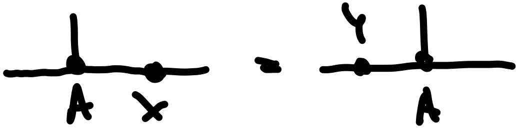

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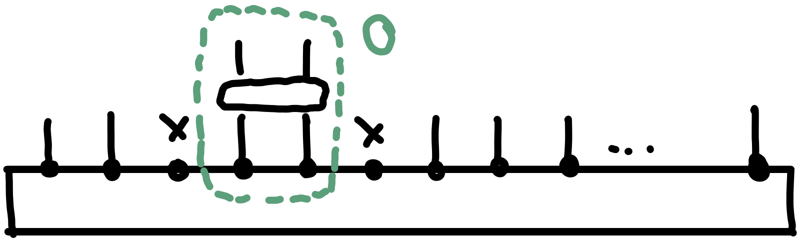
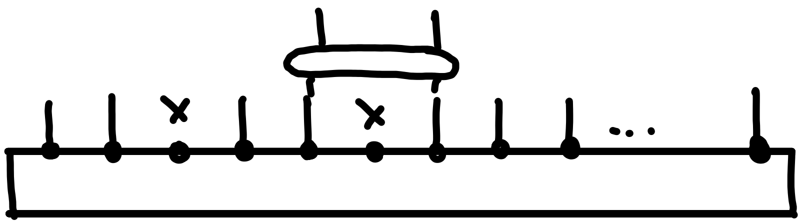


≈ 0

Failing parent Hamiltonian I

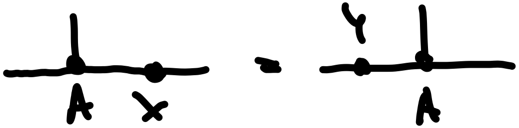
$\Rightarrow$ 2-local pH fails

A ground state that is not the MPS.



$=$

Failing parent Hamiltonian I



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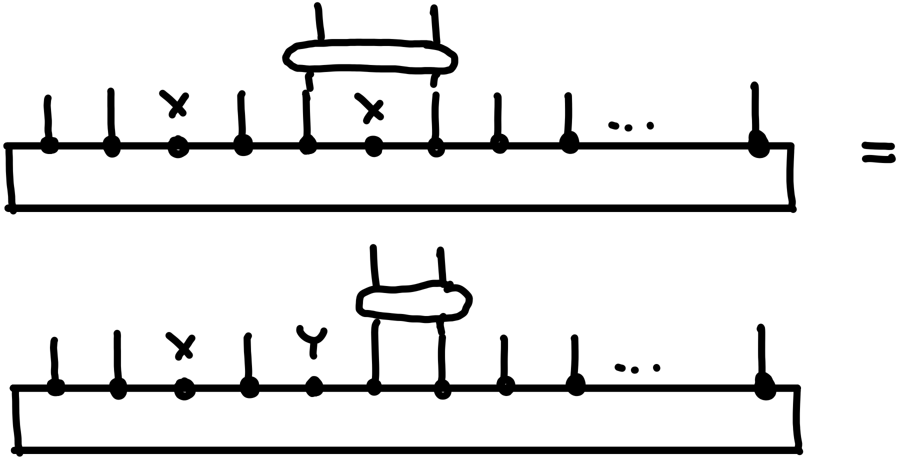
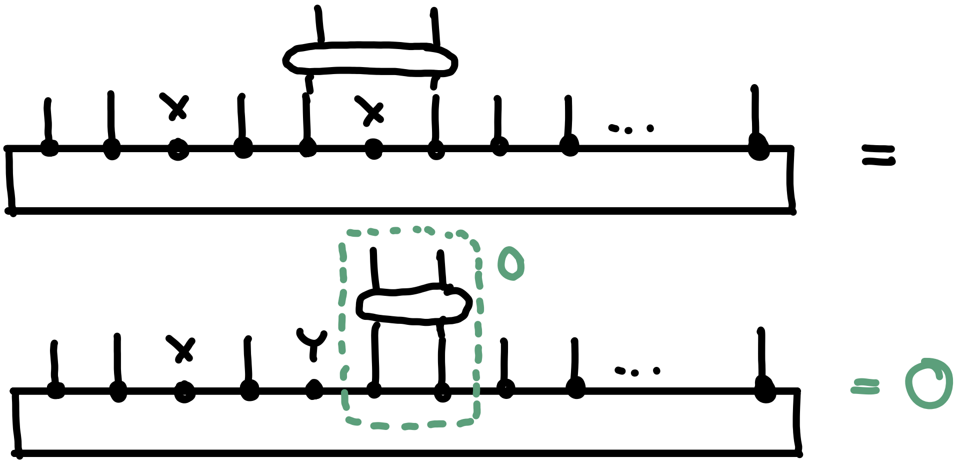
A ground state that is not the MPS.



Failing parent Hamiltonian I

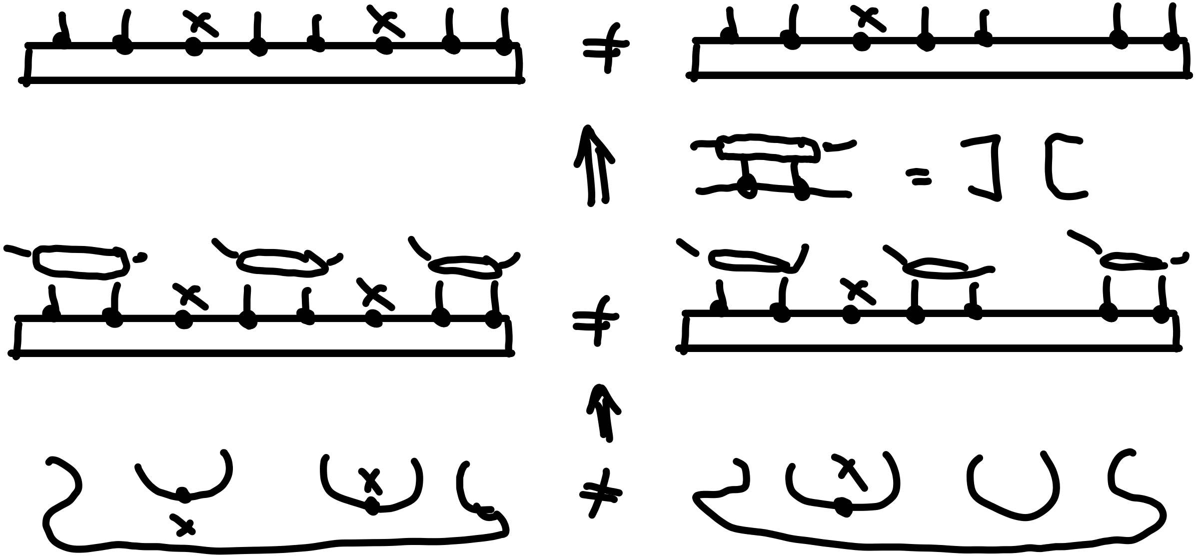
$\Rightarrow$ 2-local pH fails

A ground state that is not the MPS.



Failing parent Hamiltonian I

Exp many GS:



$\Rightarrow$ X/I on every 2^{nd} edge $\Rightarrow$ $\#GS \geq 2^{n/2}$

Failing parent Hamiltonians 2

- Generically parent Ham works
- When fails exp GS degeneracy

Can we find weirder examples?

→ poly(N) GS degeneracy

Poly(N) GS degeneracy

Similar to AKLT. SO(3)-invariant MPS

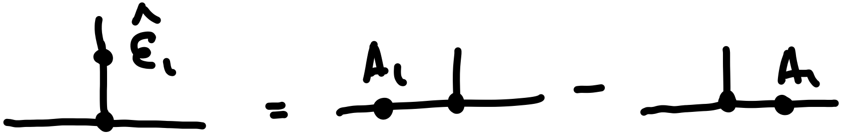
$$A \in \underbrace{\mathcal{M}_3}_{\text{3D } SO(3) \text{ irrep}} \otimes \underbrace{\mathbb{C}^4}_{\mathrm{Span}\{|e\rangle, |x\rangle, |y\rangle, |z\rangle\}}$$

$$A_e = \mathbb{1}, \underbrace{A_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, A_y = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ 2i & 0 & -2i \\ 0 & i & 0 \end{pmatrix}, A_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{Spin-1 } so(3) \text{ generators}}$$

$$[A_i, A_j] = \varepsilon_{ijk} A_k \qquad i,j,k \in \{x, y, z\}$$

SO(3) invariance.

$$\sum_k \varepsilon_{ijk} A_k = [A_i, A_j]$$



exp

ϕ 3D irrep

$\Rightarrow$

$\forall g \in SO(3)$

Also $[H, g^{\otimes n}] = 0 \Rightarrow$ GS also SO(3)-inv

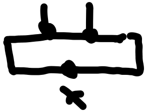
Thm. Let H be the OBC p.H of the MPS A. Then the GS decomposes.

$$GS = 0 \oplus 1 \oplus \ldots \oplus n$$

$$\Rightarrow \dim(GS) = 1 + 3 + \ldots + 2n+1 = (n+1)^2$$

Proof (sketch, dimension)

$$GS = \bigcap_{i=0}^{n-2} \mathcal{H}^{\otimes i} \otimes S_2 \otimes \mathcal{H}^{n-i-2},$$

$S_2 = \Big\{$  $\Big| \, x \in \mathcal{H}_3 \Big\}$ 9 dimensional

$S_2^{\perp}$ spanned by 7 vectors

- $|ie\rangle - |ei\rangle \quad i \in \{x, y, z\}$
- $|ij\rangle - |ji\rangle - |ke\rangle \quad (i,j,k) = \begin{cases} (1\,2\,3) \\ (2\,3\,1) \\ (3\,1\,2) \end{cases}$
- $|xx\rangle + |yy\rangle + |zz\rangle - 2|ee\rangle$

$$\dim GS = \dim\left(\mathcal{H}^{\otimes n} / GS^{\perp}\right)$$

$$GS^{\perp} = \sum_{i} \mathcal{H}^{\otimes i} \otimes S_2^{\perp} \otimes \mathcal{H}^{\otimes n-i-2}$$

We can construct a basis of $\mathcal{H}^{\otimes n}/GS^{\perp}$ in 3 steps

(1) $\mathcal{H}^{\otimes n} / \{ |ie\rangle - |ie\rangle \mid i \in x,y,z \} = V_1$

Representatives:

$$|xyyxzx\,.\,y\underbrace{ee..e}_{\text{at the end}}\rangle$$

(2) $V_1 / \mathrm{Span}\{ |xy\rangle - |yx\rangle - |ze\rangle, \dots \} \simeq V_2$

Representatives:

$$|x\boxed{yx}xzy\dots yee.\,e\rangle$$

$$\equiv |x\boxed{xy}xzy\dots yee\dots e\rangle +$$

$$\underbrace{|x\boxed{ze}xzy\dots yee\dots e\rangle}_{\text{one more } e}$$

$\Rightarrow$ Can order lexicographycally:

$$|xx\dots xyy\dots yzz\dots ze\dots e\rangle \to \binom{N+3}{3}$$

They are lin indep Poincaré-Birkhoff-de Witt

(Filtration of univ env alg)

(3) $V_2 / \mathrm{Span}\{|xx\rangle + |yy\rangle + |zz\rangle - 2|ee\rangle\}$

Eg change zz to others

$|xx\ldots xy\ldots y\,z\boxed{zz}e\ldots e\rangle$

$-xx-yy+ee$

Basis

$$\left.\begin{array}{l} |x\ldots xy\ldots ye\ldots e\rangle \to \binom{N+2}{2} \\ |x\ldots xy\ldots yze\ldots e\rangle \to \binom{N+1}{2} \end{array}\right\} (N+1)^2$$

$\square$

Poly(n) degeneracy for AKLT-like MPS:

- $A \in M_3 \otimes \mathbb{C}^4$, given by:

$$A_0 = 1, \quad A_x = X, \quad A_y = Y, \quad A_z = Z$$

- The MPS is SO(3)-symm
- Thus PH, and its GS as well
- $GS = 1 \oplus 3 \oplus \dots \oplus n \Rightarrow \dim GS = (n+1)^2$

Questions

1. Sufficient condition on A for p.H. to work

2 Generically short p H works

3 Short parent Ham of normal MPS can fail.

- exp GS degeneracy
- poly GS degeneracy

Conclusion

Variation:

Hamiltonian $\longrightarrow$ TN

Model generation

TN $\xrightarrow{\text{parent}}$ Hamiltonian

<u>Usually</u>: pH's GS = TN

↳ Formal statements, proofs

↳ Edge cases interesting models