Rigidity and Flow Near Jamming

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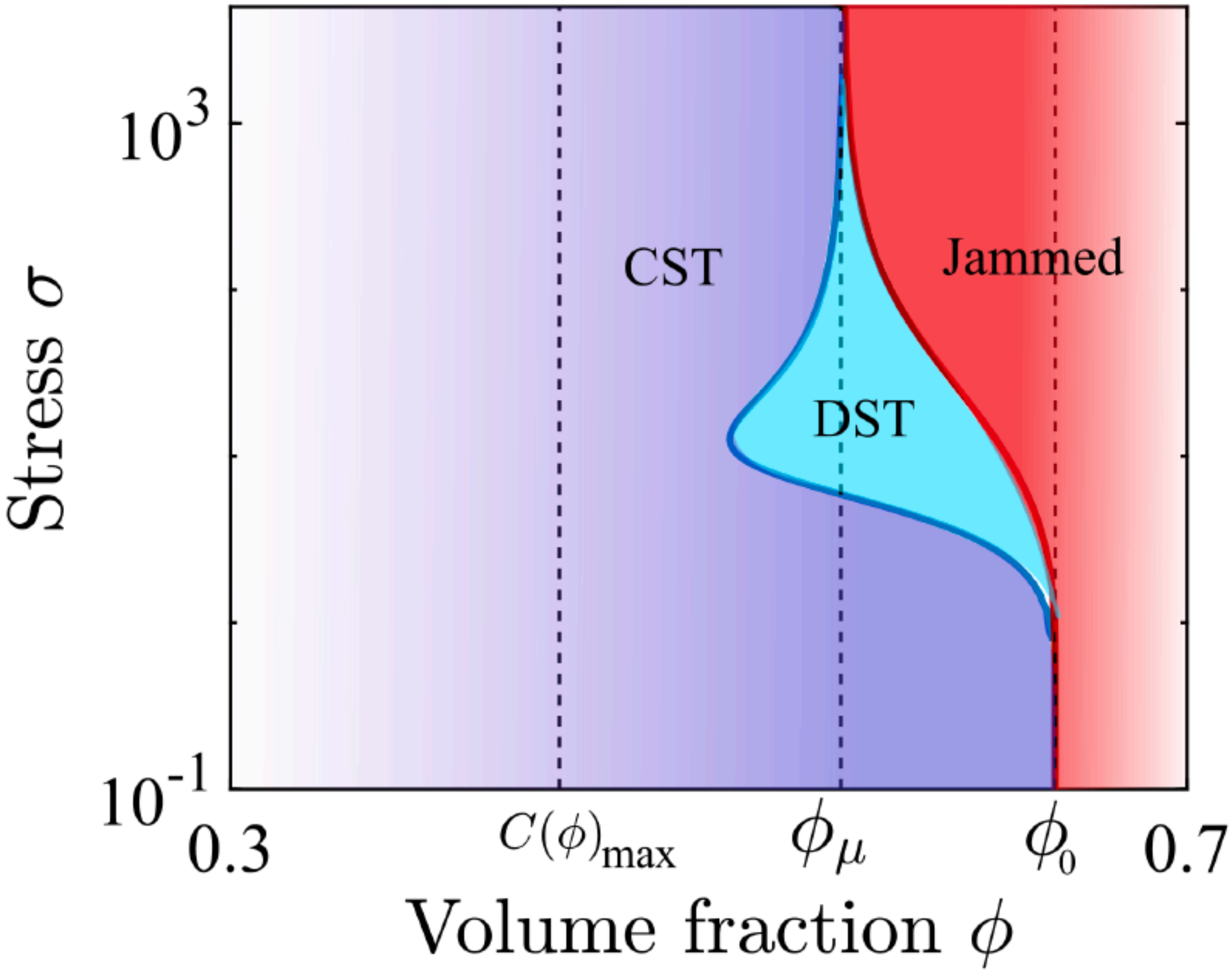
With eternal gratitude to Bob Behringer

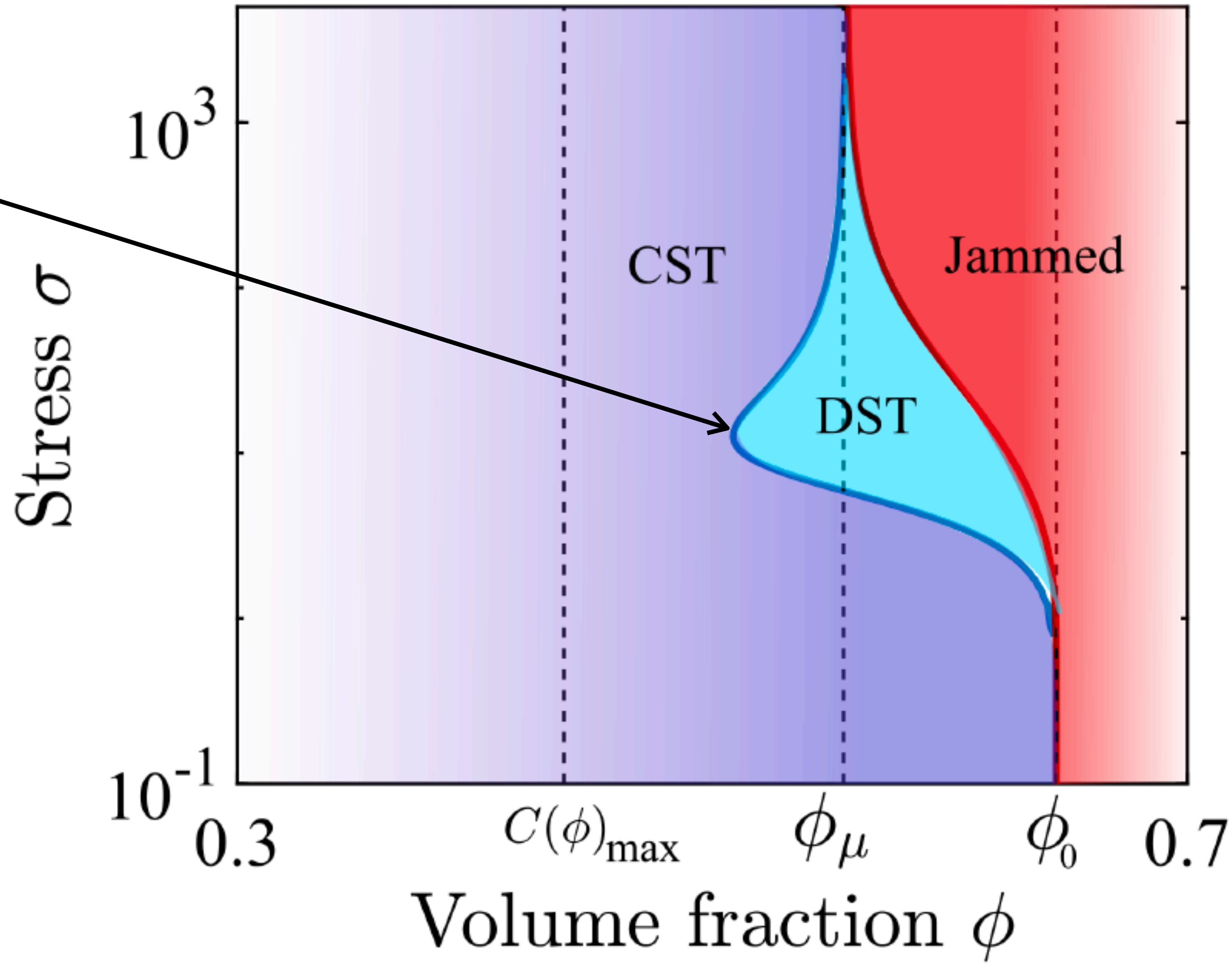
Shear-thickening Suspensions (Steady States)



Shear-thickening Suspensions (Steady States)

Locus of points :

$$\frac{d\dot{\gamma}}{d\sigma} = 0$$



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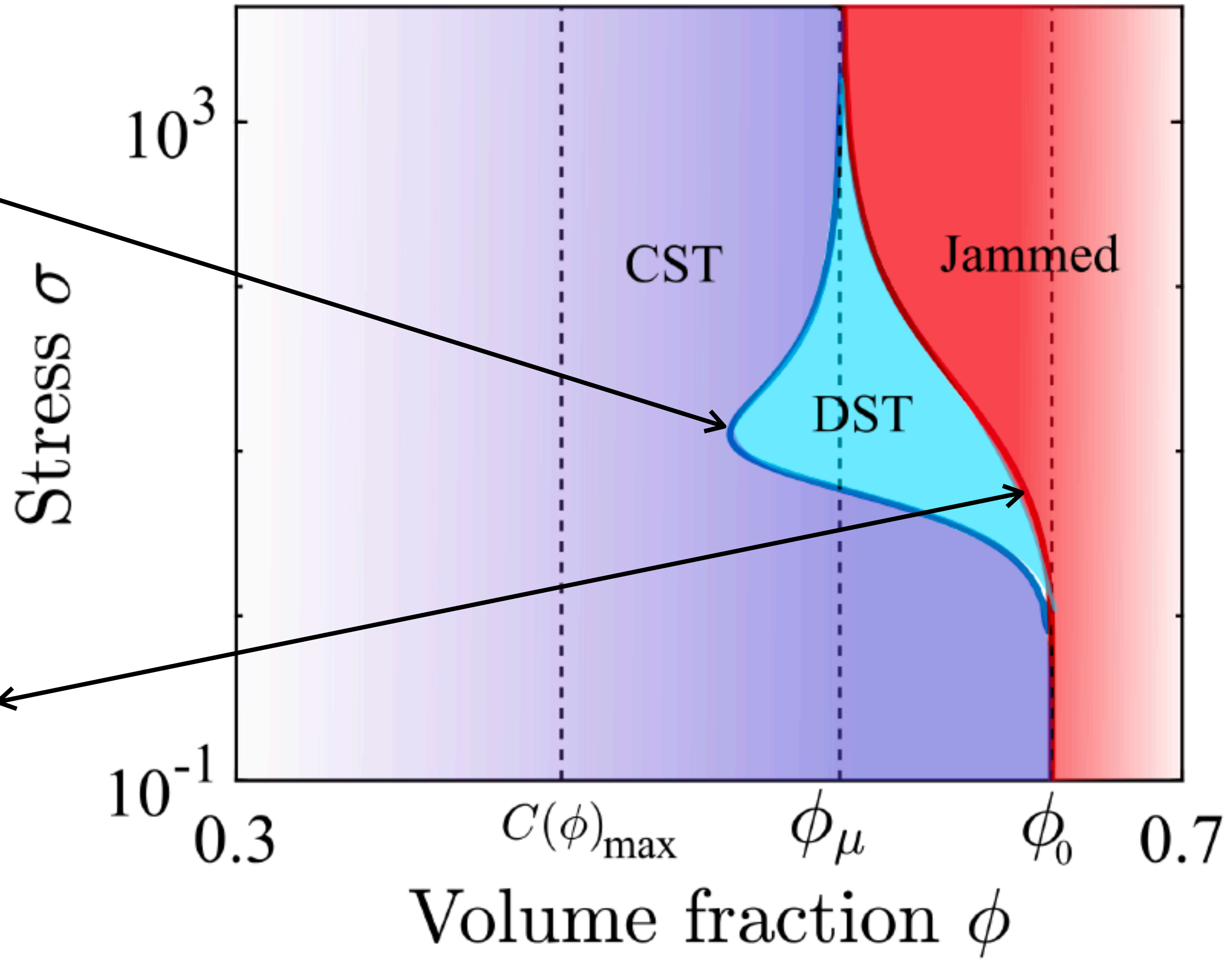

Line of critical points: viscosity divergence is controlled by interactions at particle contacts: friction, adhesion, …
Networks that emerge exhibit rigidity, a shear modulus

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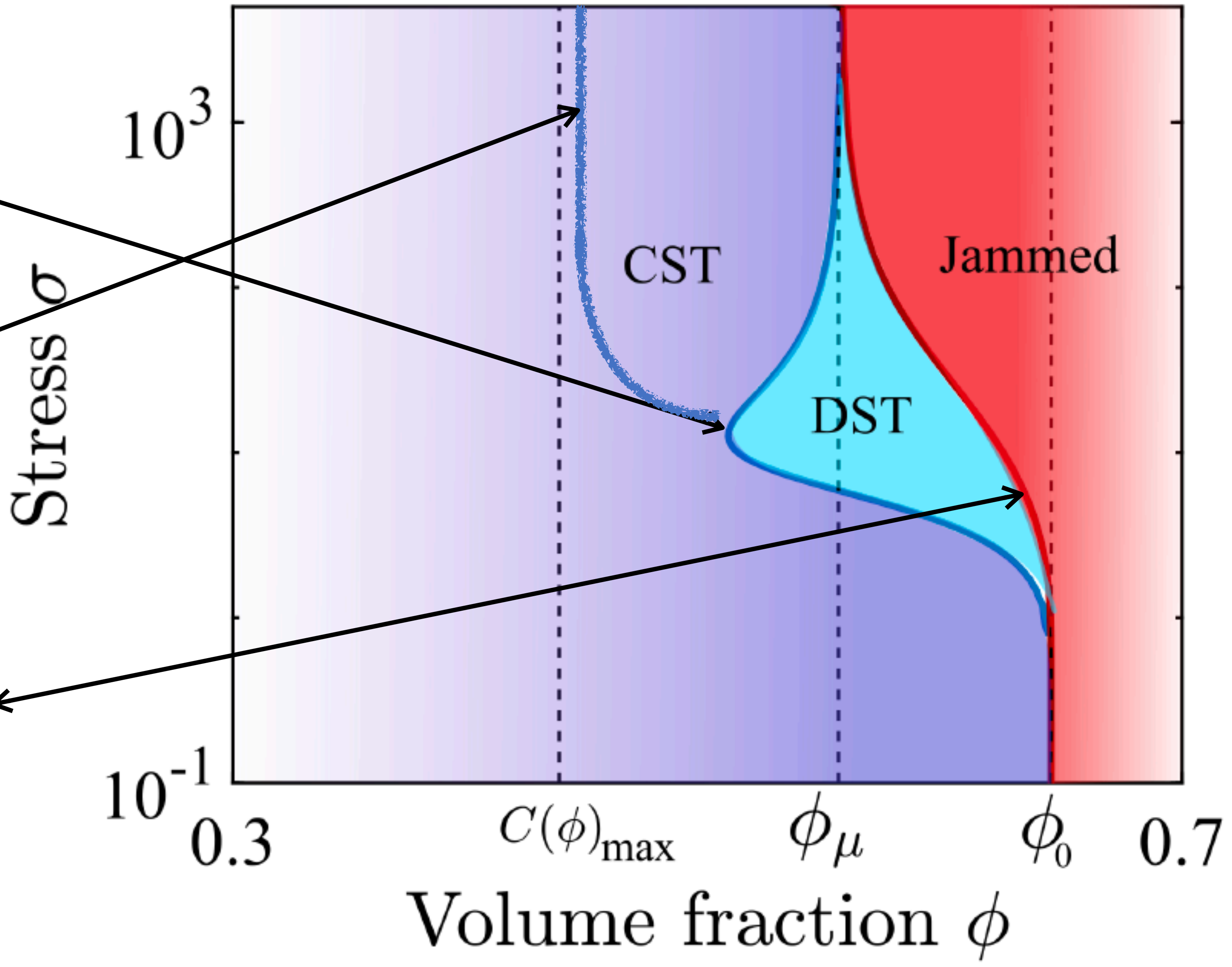
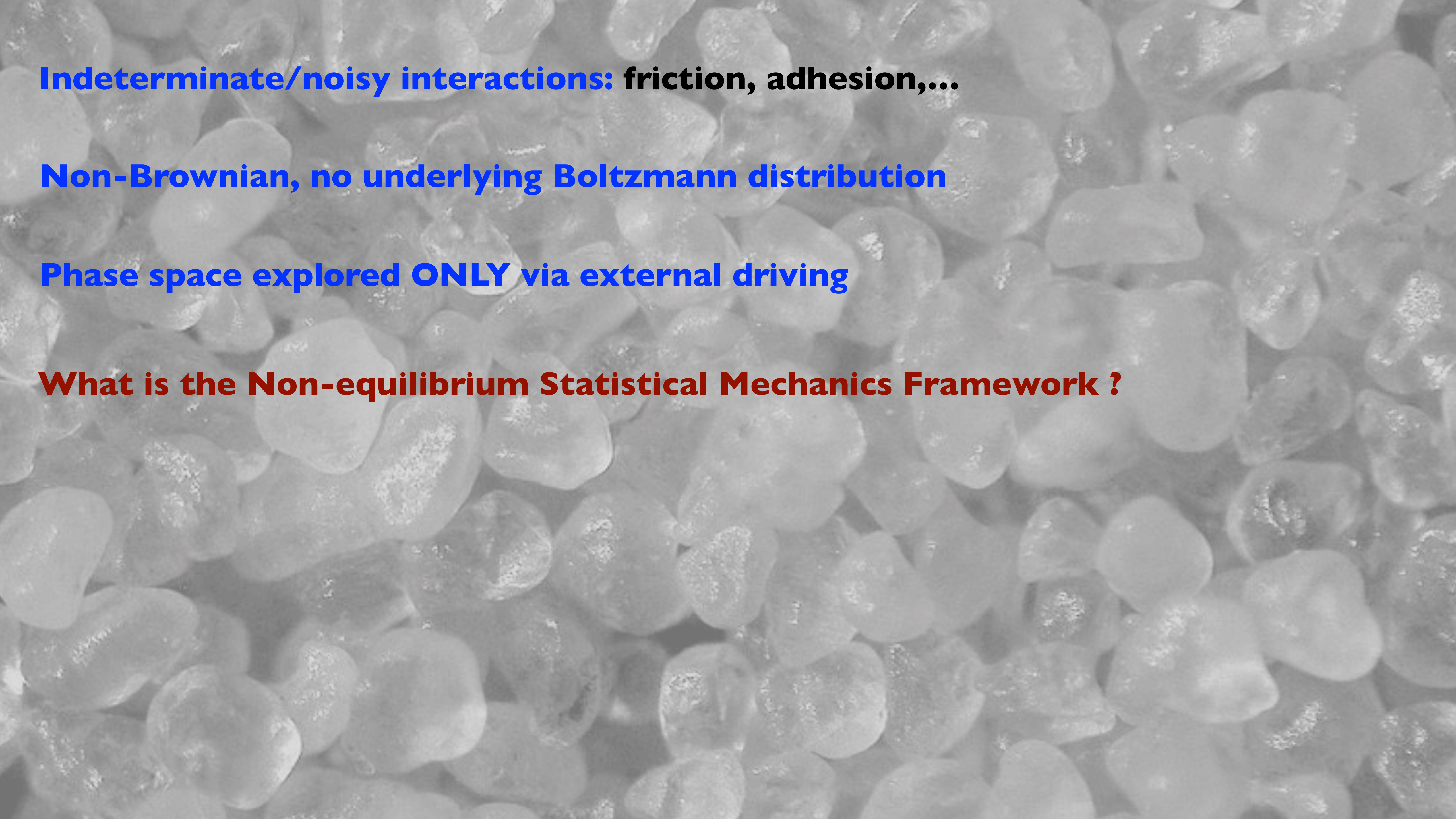


Indeterminate/noisy interactions: friction, adhesion,...

Non-Brownian, no underlying Boltzmann distribution

Phase space explored ONLY via external driving



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What is the Non-equilibrium Statistical Mechanics Framework ?

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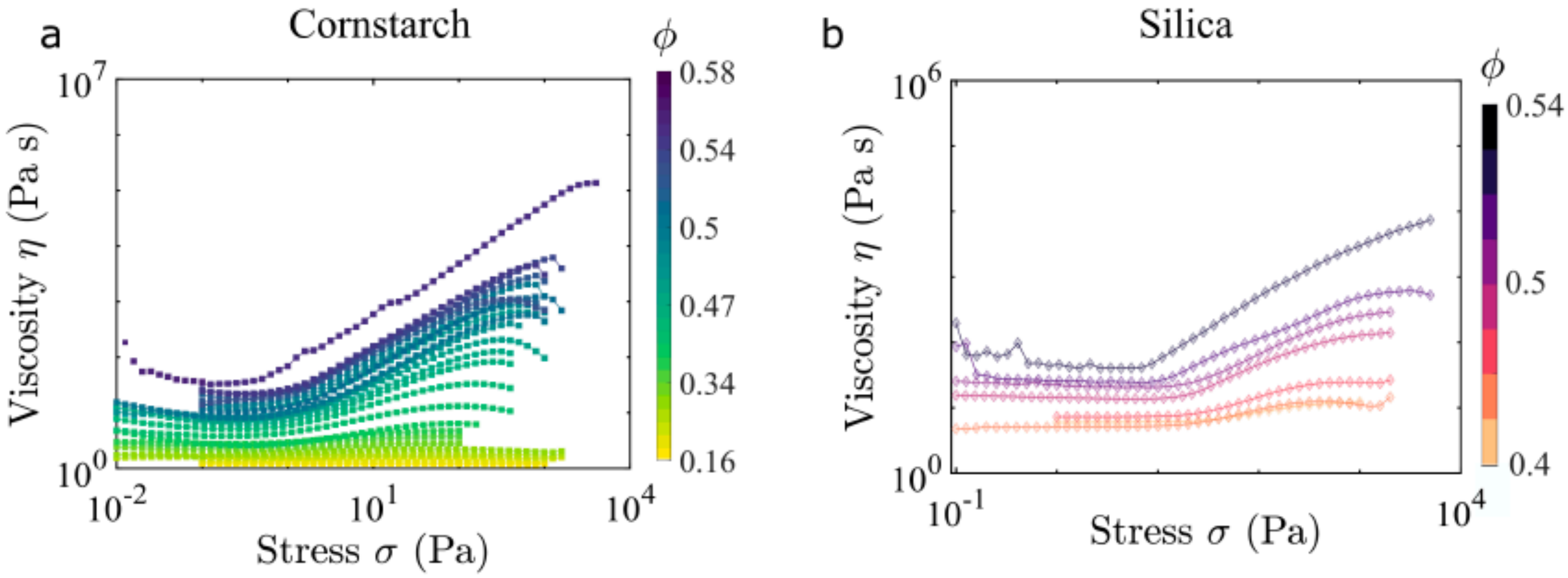
Strongly Rate-dependent Viscosity

Experimental results from Meera Ramaswamy (Cornell Collaboration)



Crossover Behavior Controlled by Multiple, Different Critical Points

Wyart & Cates *PRL* 2014
Key Idea: stress controls fraction of frictional contacts

Viscosity of stress independent states

$$\eta_r(\phi) = \alpha(\phi_J - \phi)^{-2}$$

Controlled by geometrical constraints: depends only on density

Key idea: interpolate jamming fraction

$$\phi_J(\sigma) = f(\sigma)\phi_\mu + (1 - f)\phi_0$$

Friction introduces a different type of constraint and shifts the critical point to a different packing fraction

Discontinuous shear thickening arises from a “crossover” between these two different critical points

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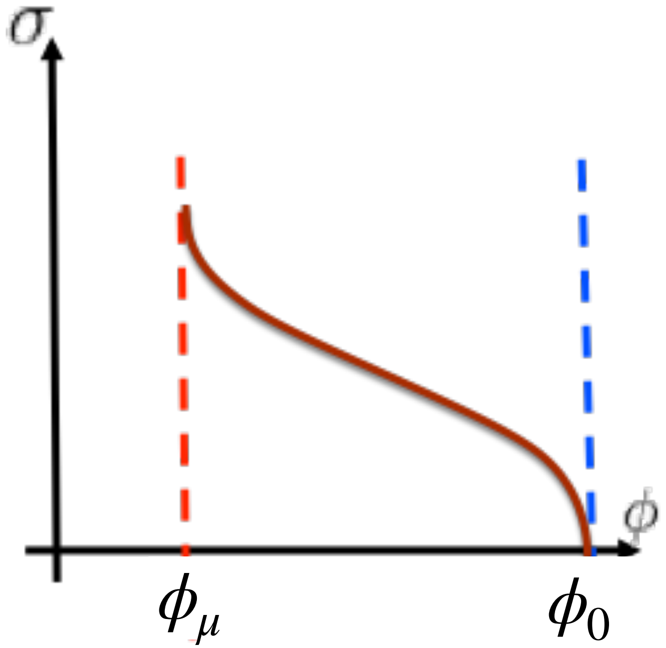

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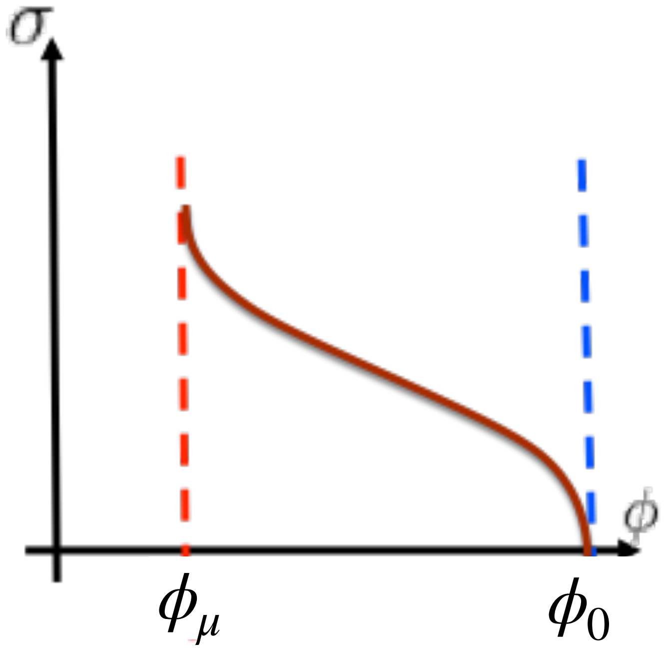

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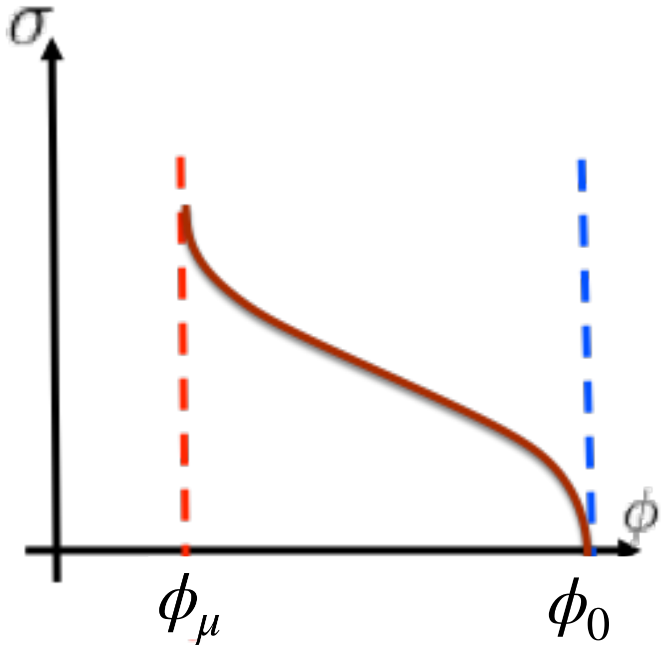

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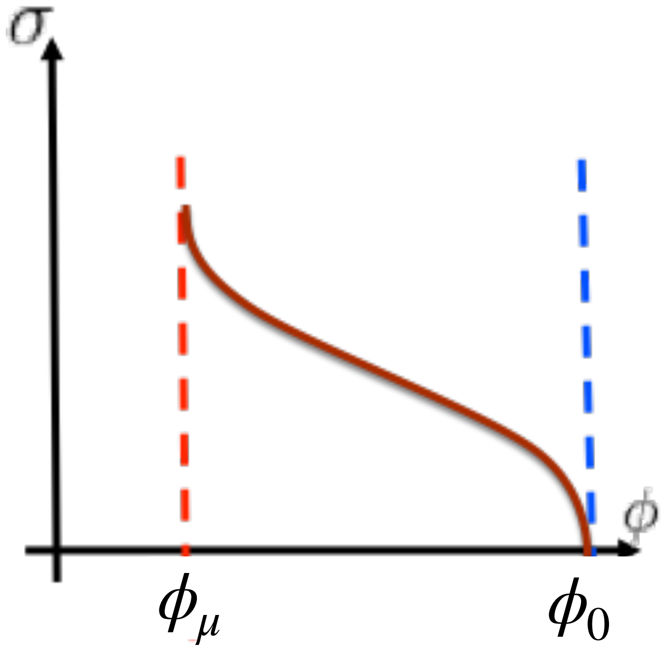

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Rate-dependent Viscosity is a signature of Crossover Scaling

Complex Phase Behavior from Crossover Scaling

Multiple, different critical points lead to complex phase behavior in equilibrium systems

Example: Magnetic Systems

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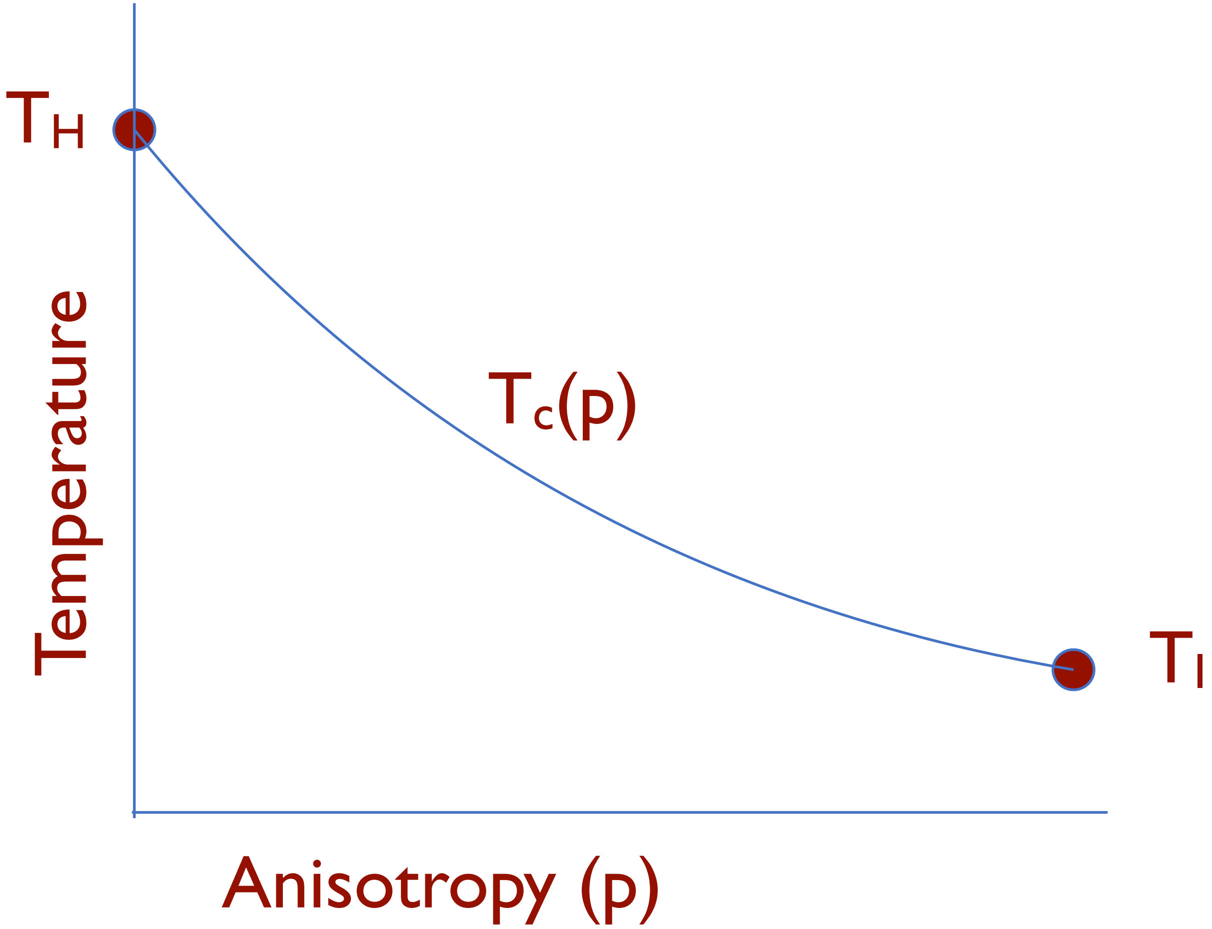

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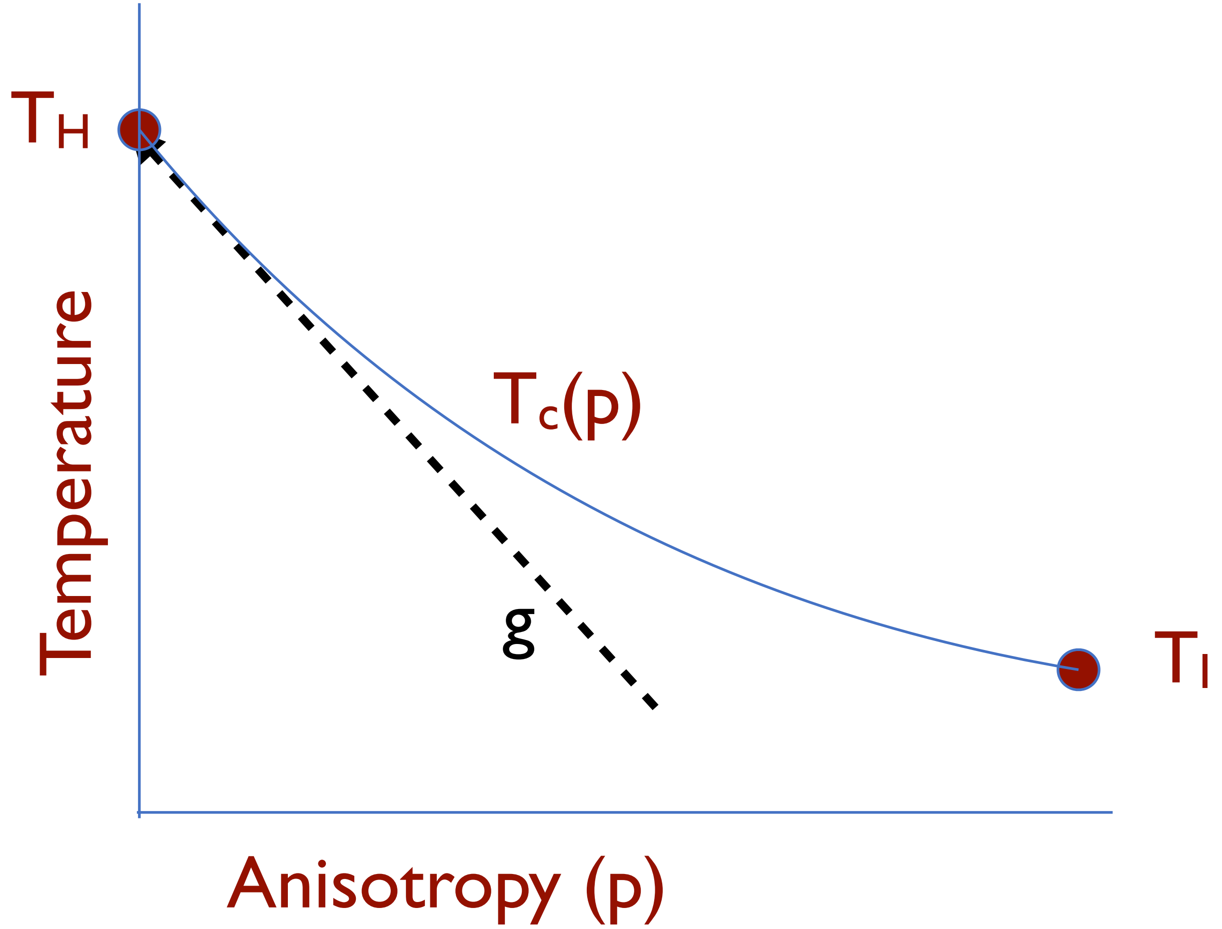

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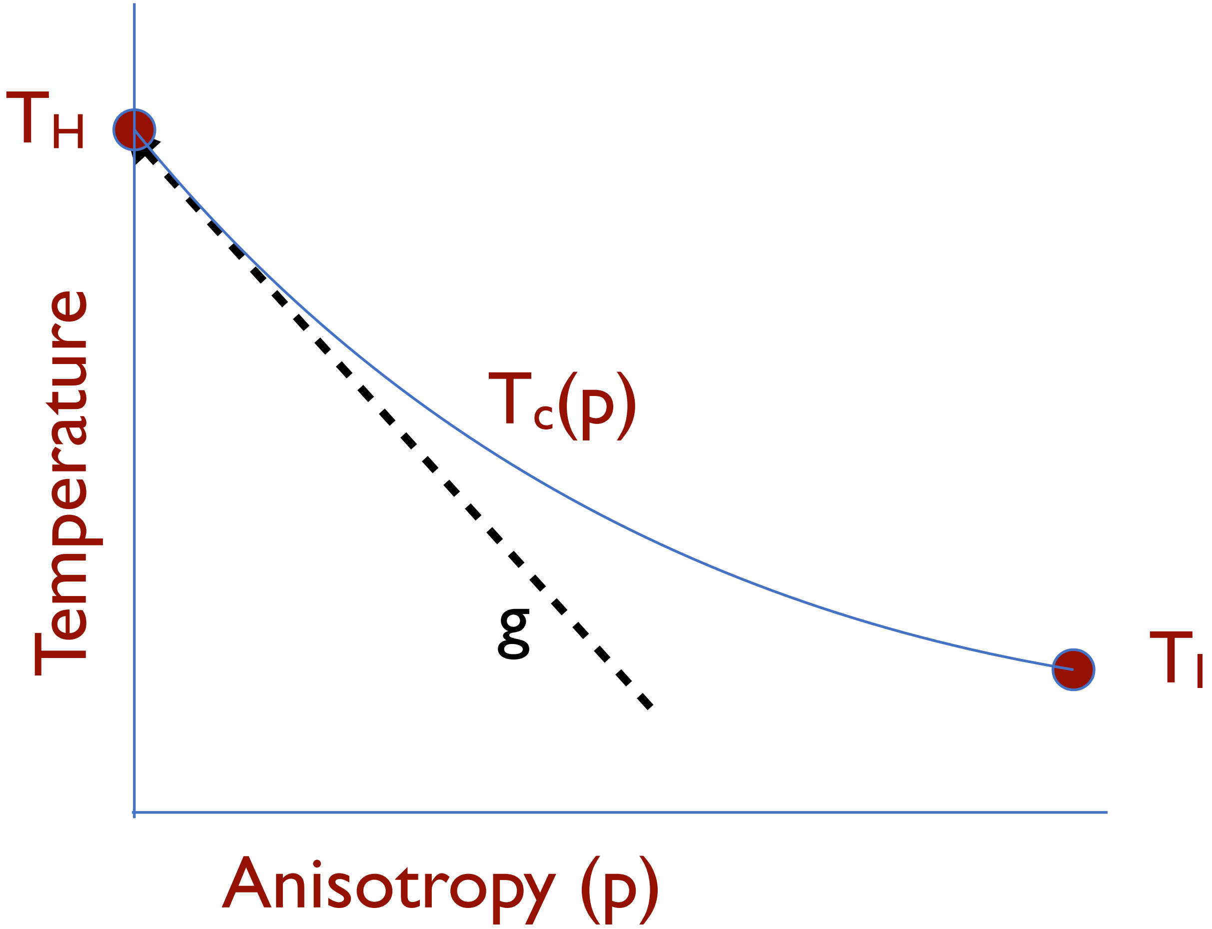

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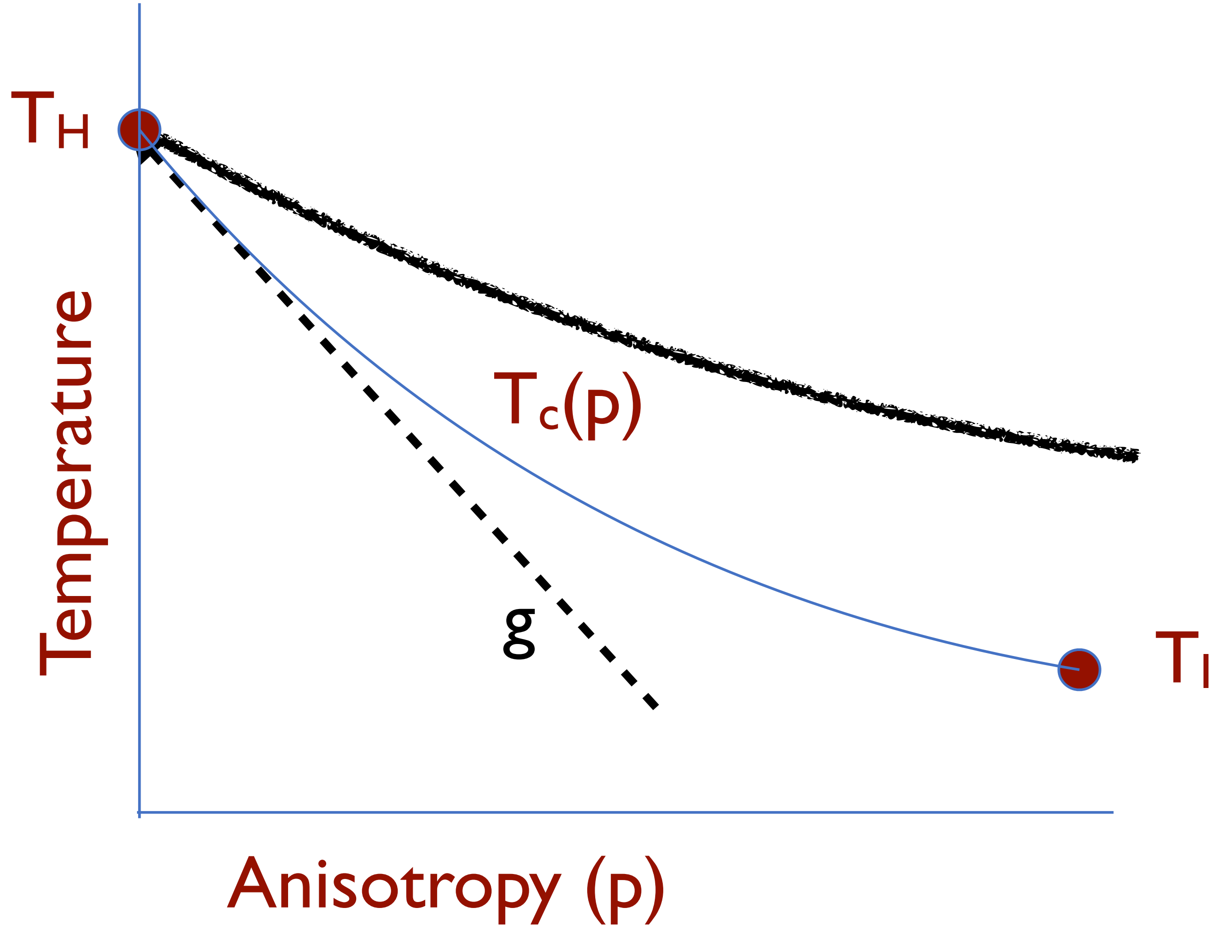

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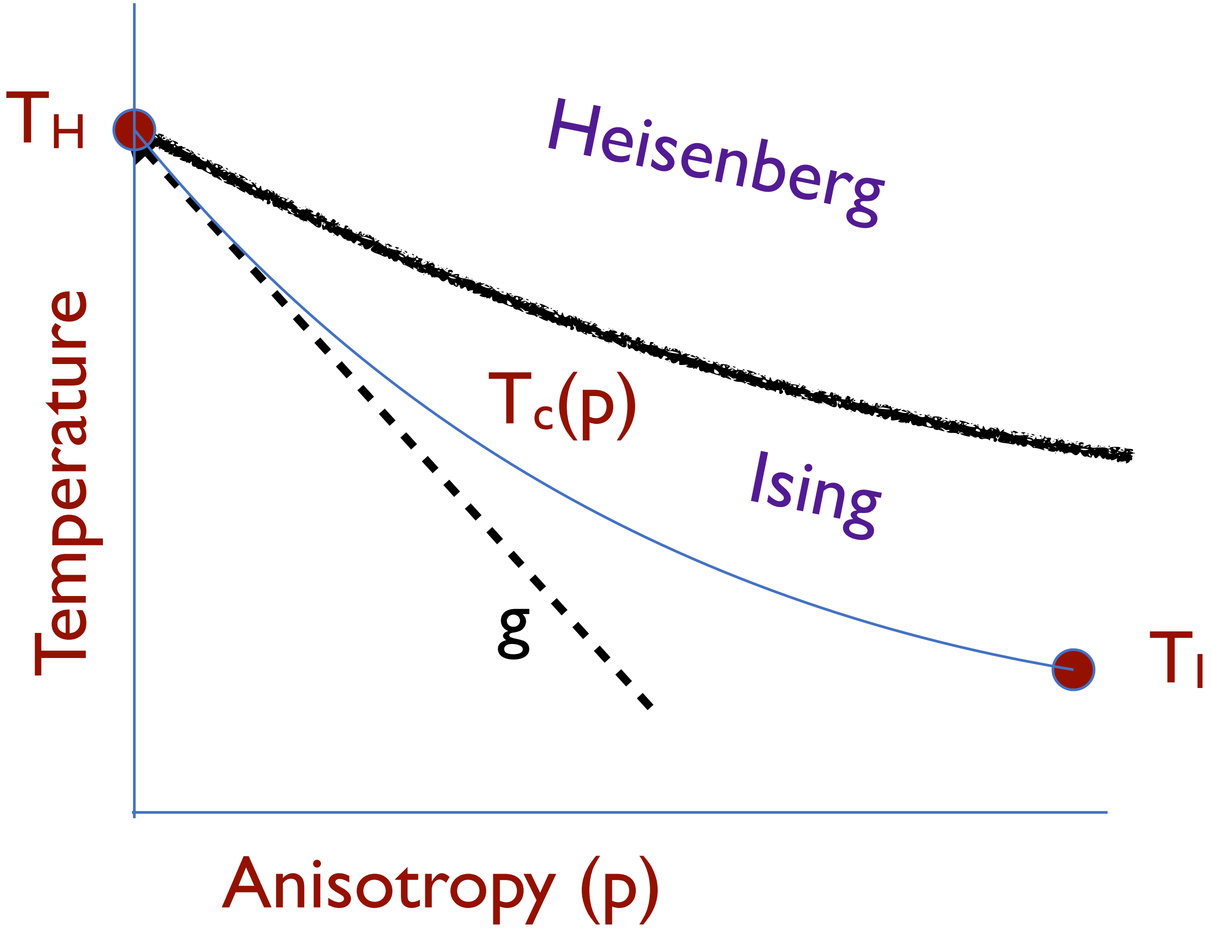

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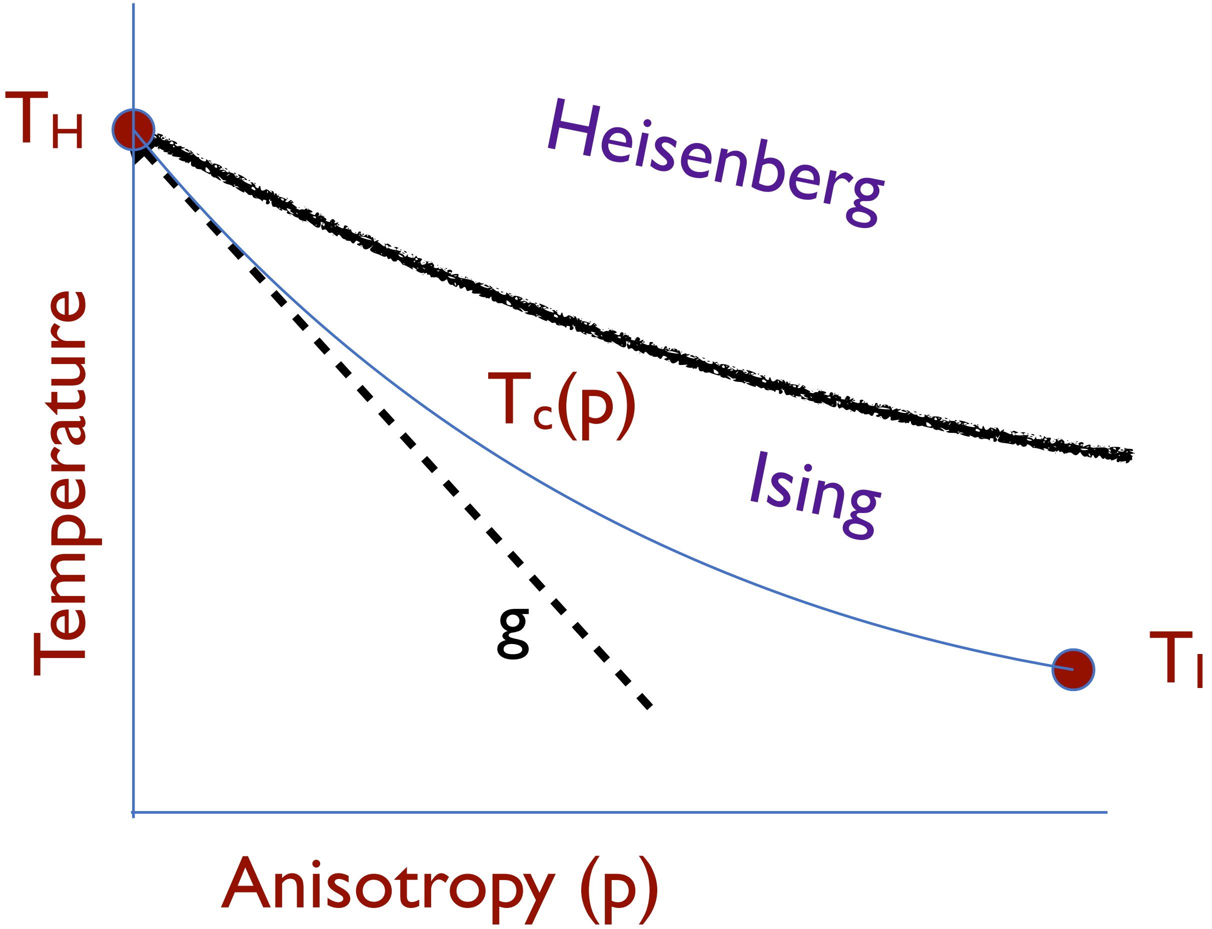

Note: Controlling p is not the same as controlling g or the distance x.

Crossover Scaling in Shear Thickening

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Does it work ?

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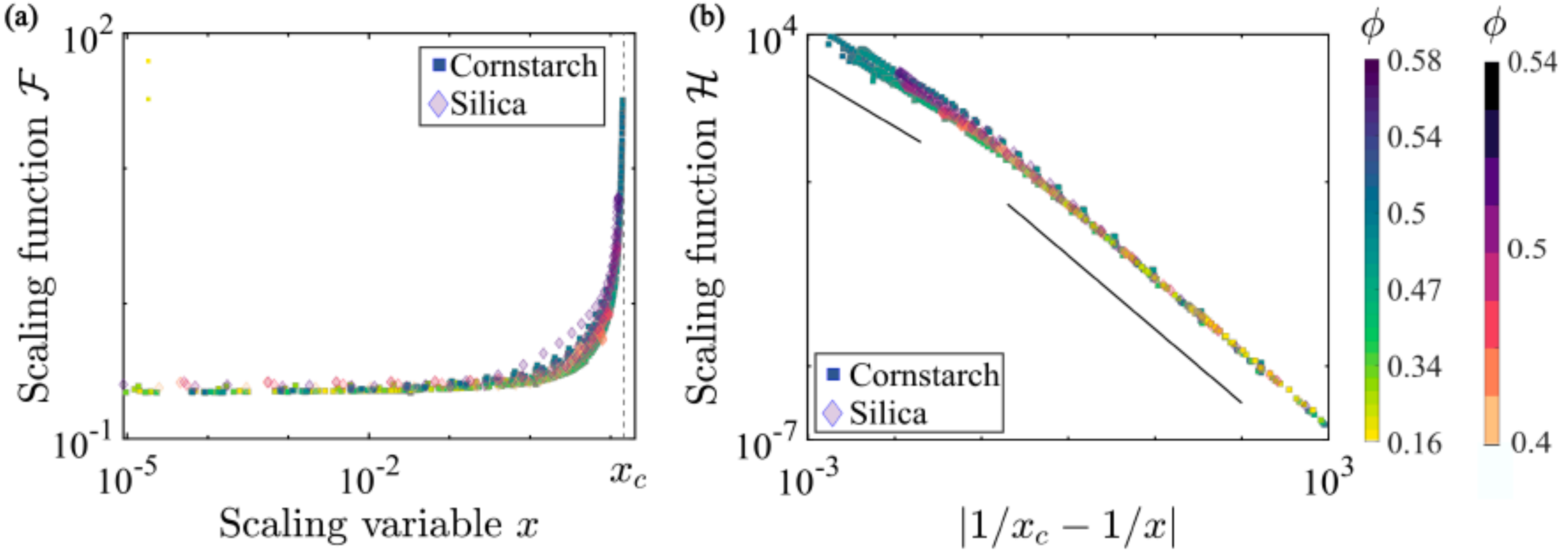

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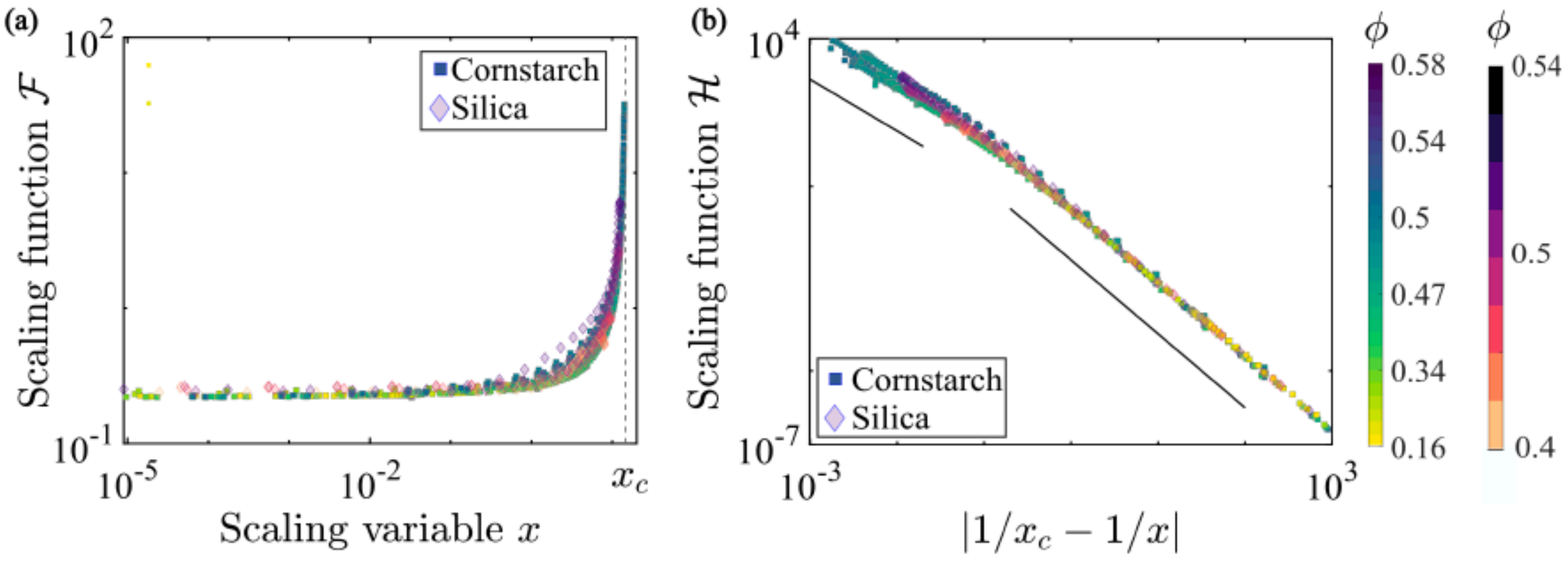

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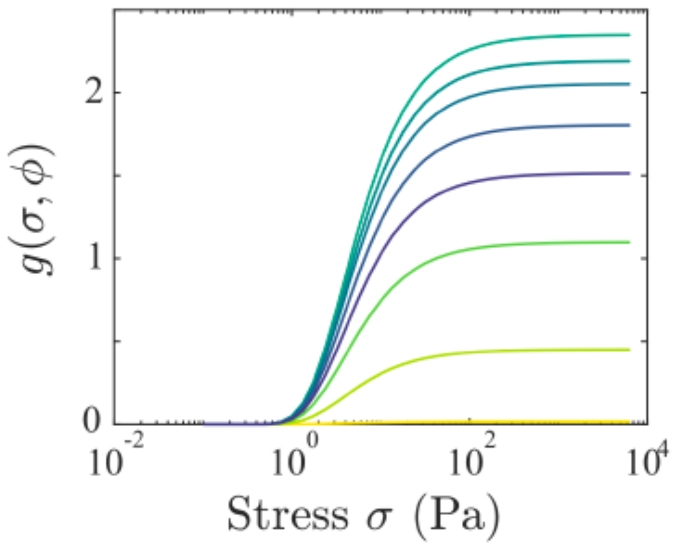

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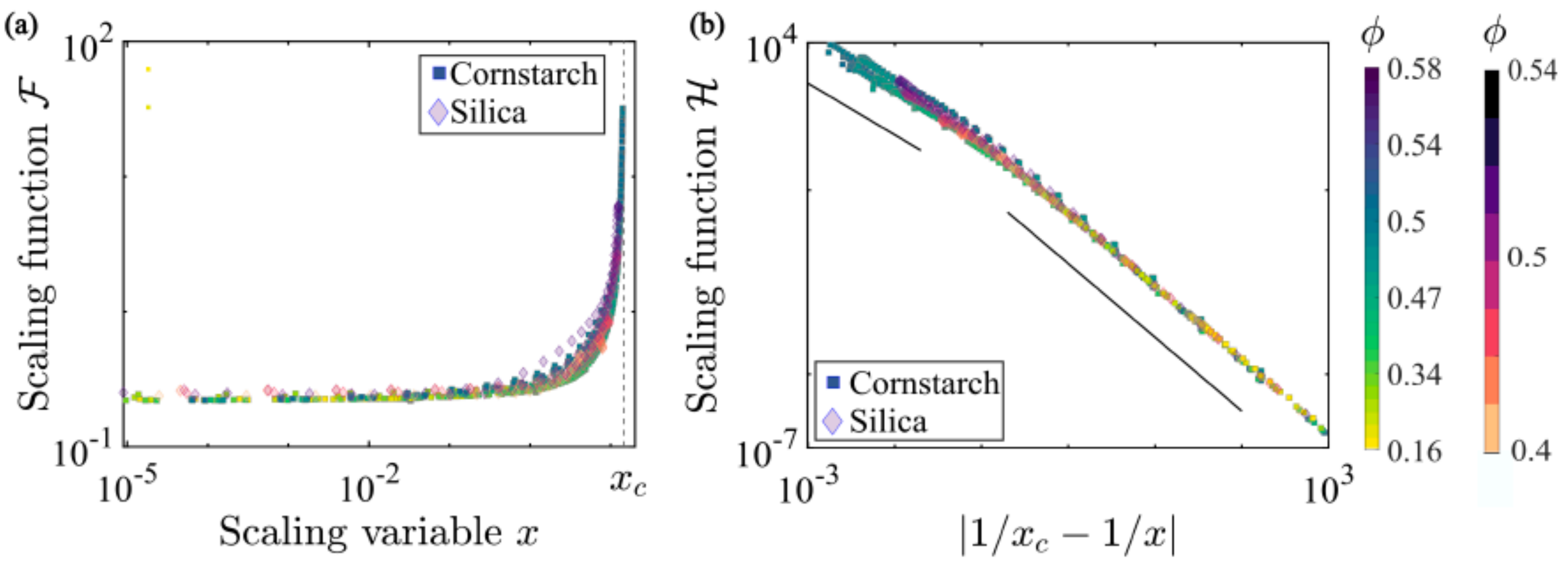

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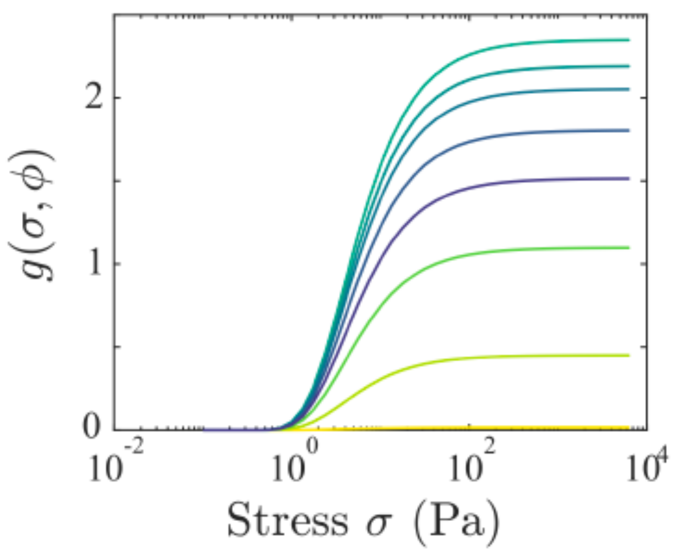
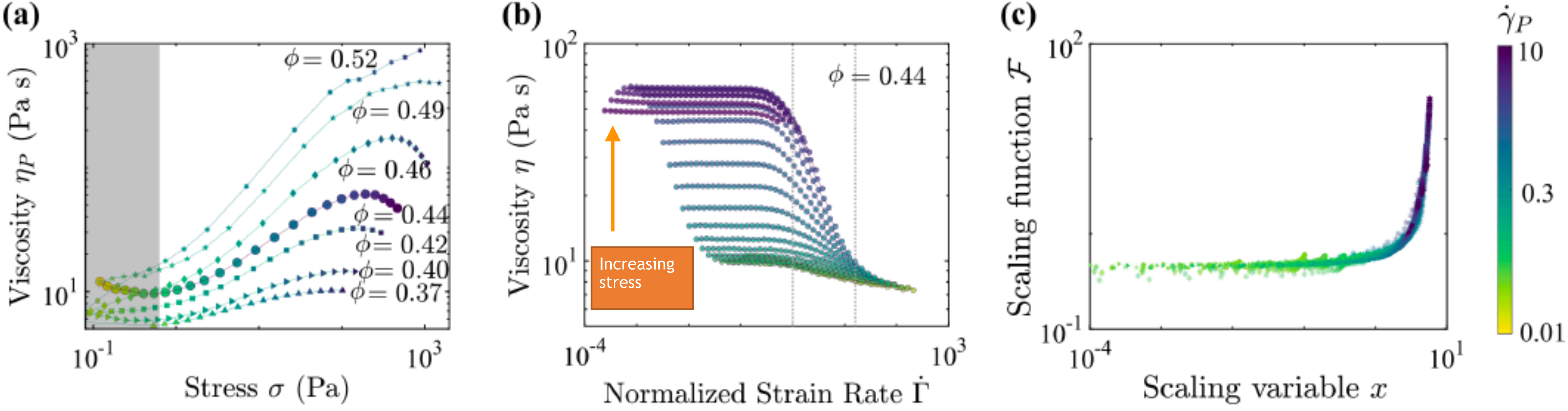


Incorporate other tuning parameters in g ?

Universality ? What determines it ?

Orthogonal Shear

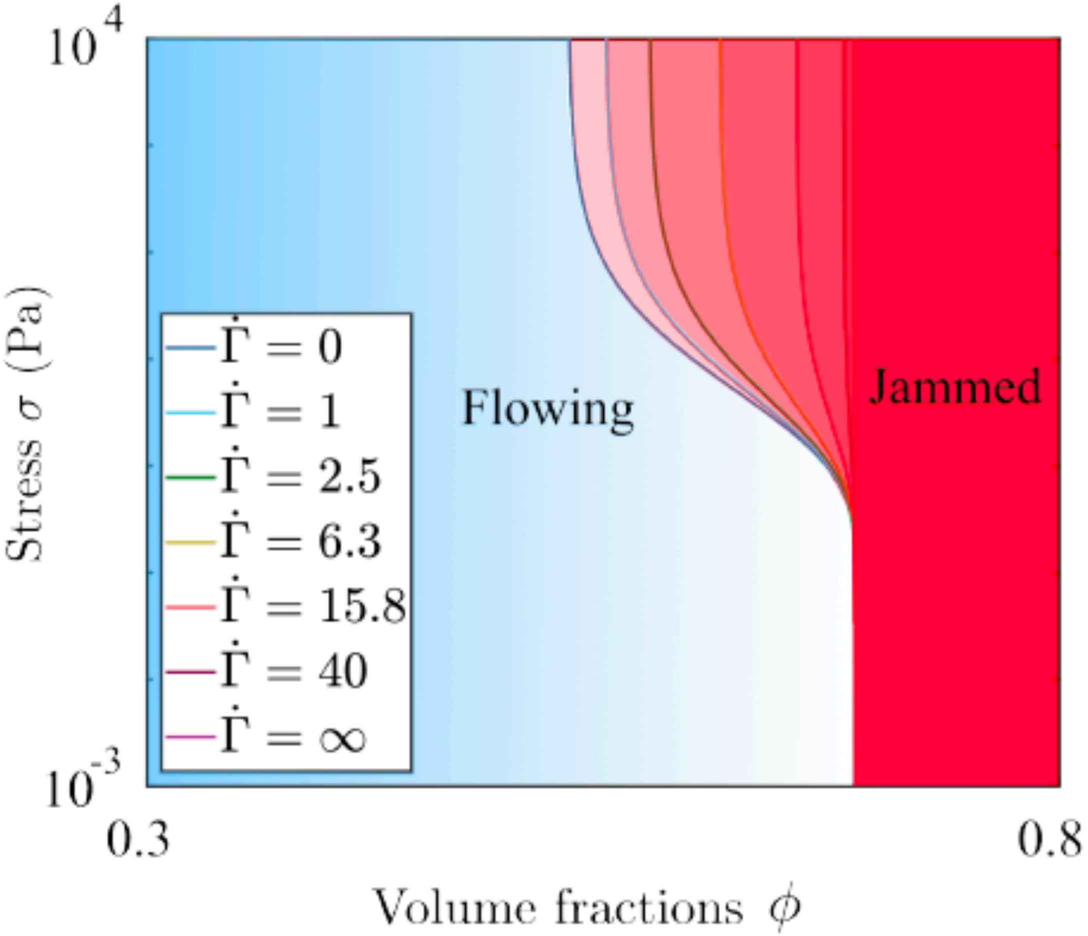


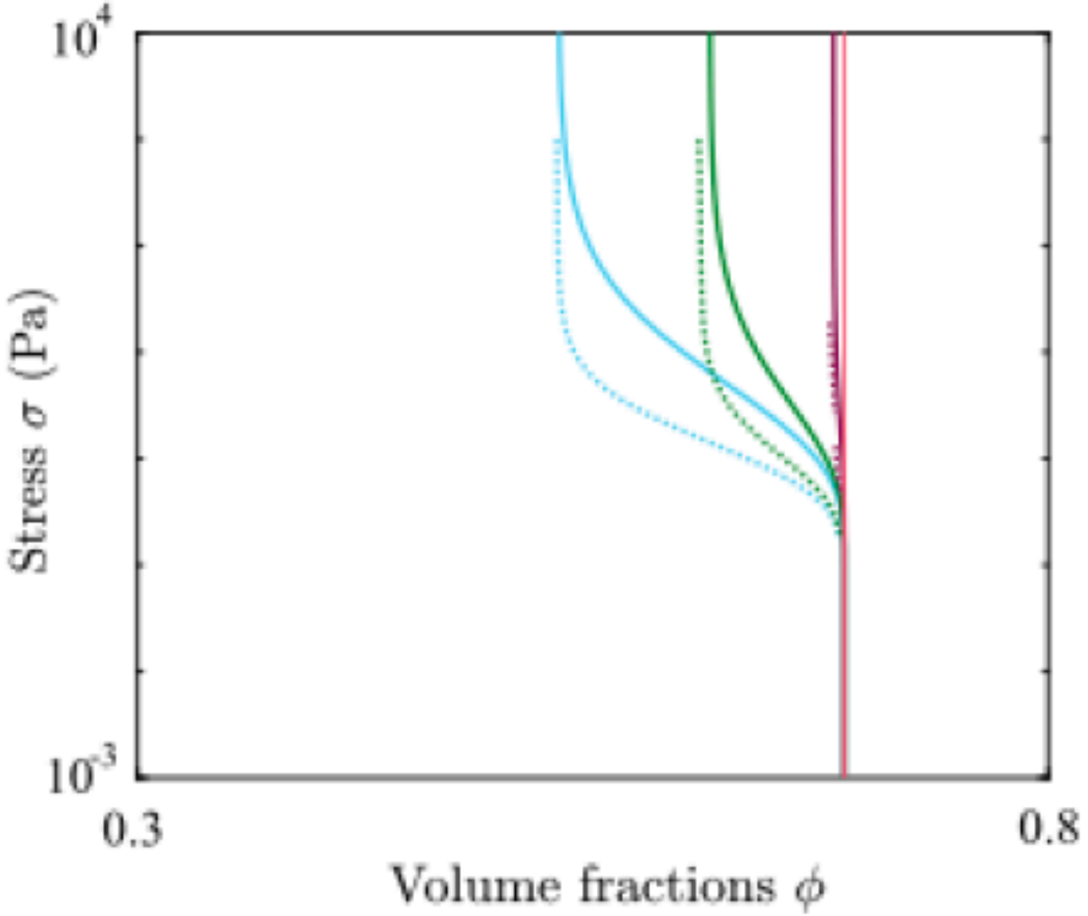
$$x(\phi, \sigma, \dot{\Gamma}) = \frac{f(\sigma)C(\phi)g(\dot{\Gamma})}{(\phi_0 - \phi)}$$

A Universal Scaling Framework for Tunable Shear Thickening

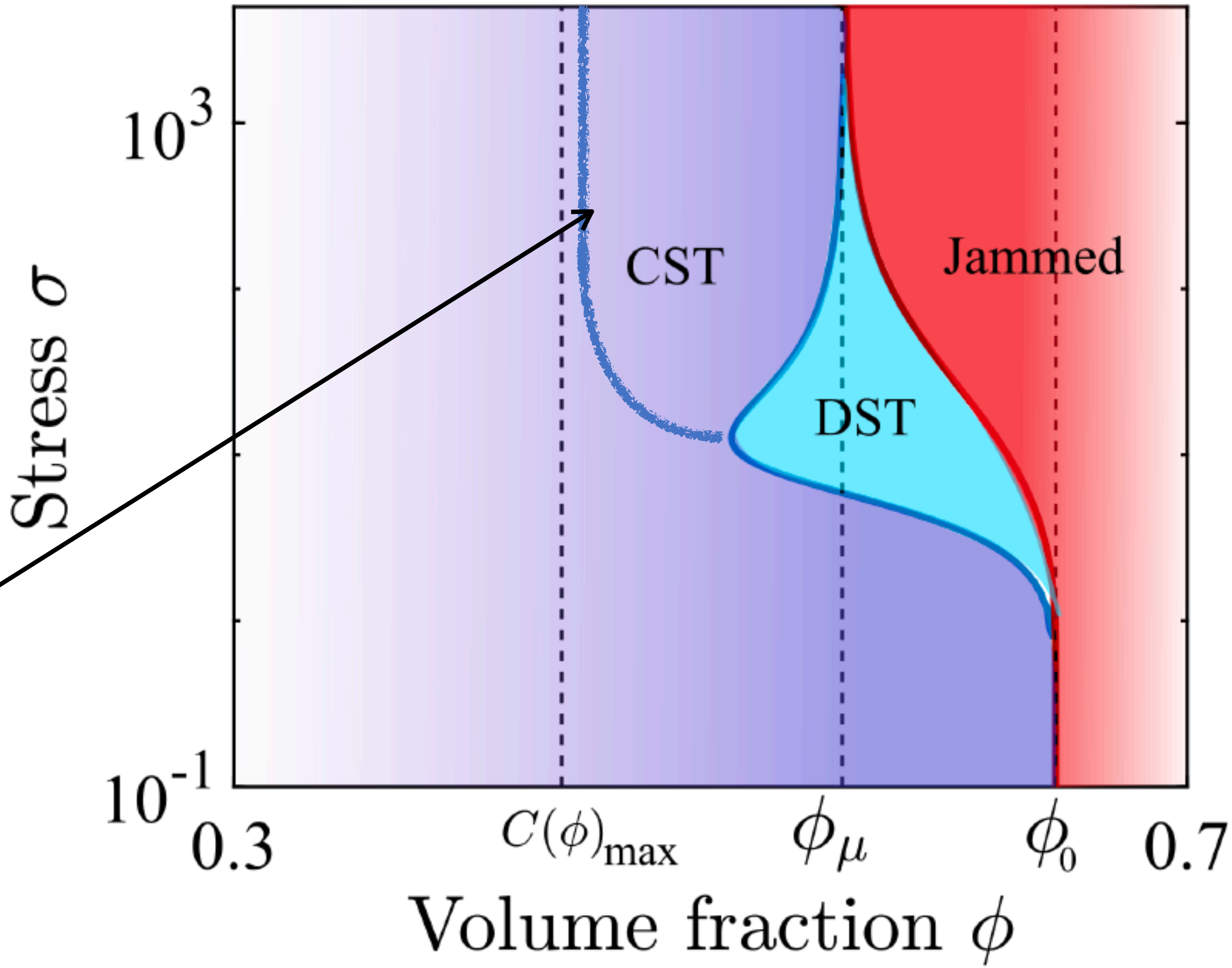
(arXiv: 2205.02184 and 2107.13338)

Jamming Phase Diagram from Scaling





Shadows of a Equilibrium Transition



A line of critical points: rigid clusters appear in the flow

Shadows of a Equilibrium Transition

Aritra Santra, Michel Orsi, & Jeffrey Morris

Simulations in 2D

Mari, R., R. Seto, J. F. Morris, and M. M. Denn, "Shear thickening, frictionless and frictional rheologies in non-Brownian suspensions,"

J. Rheol. 58(6), 1693–1724 (2014).

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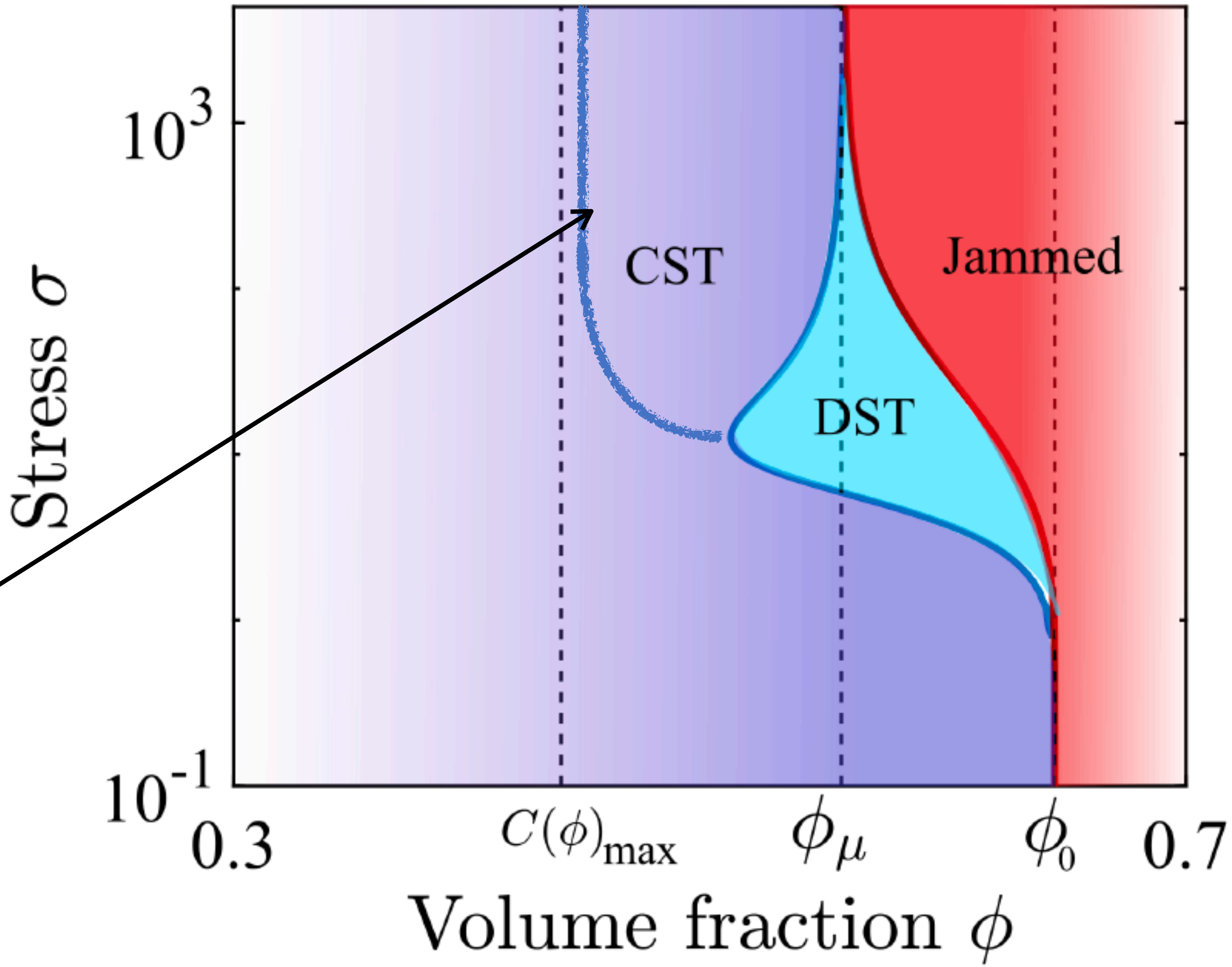

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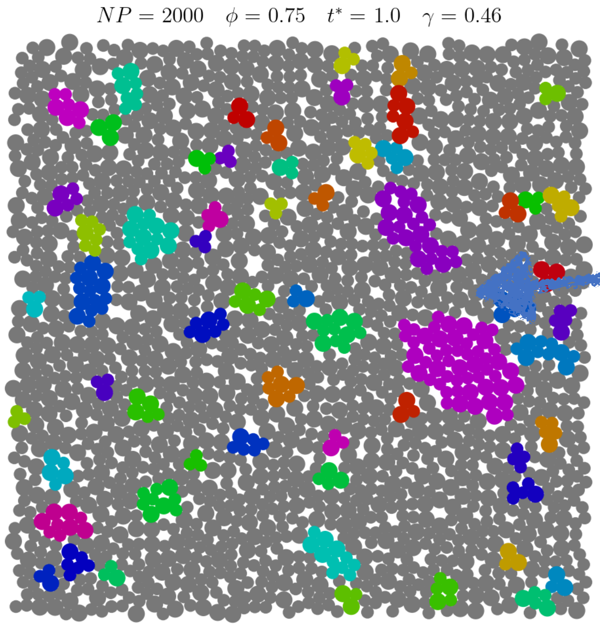
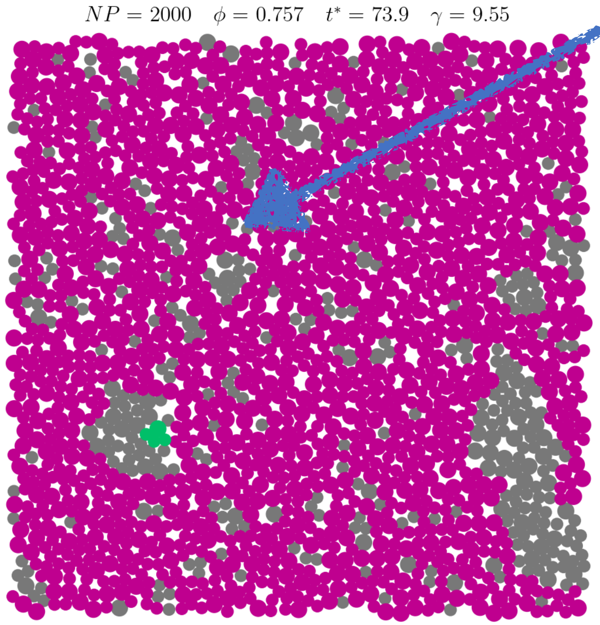
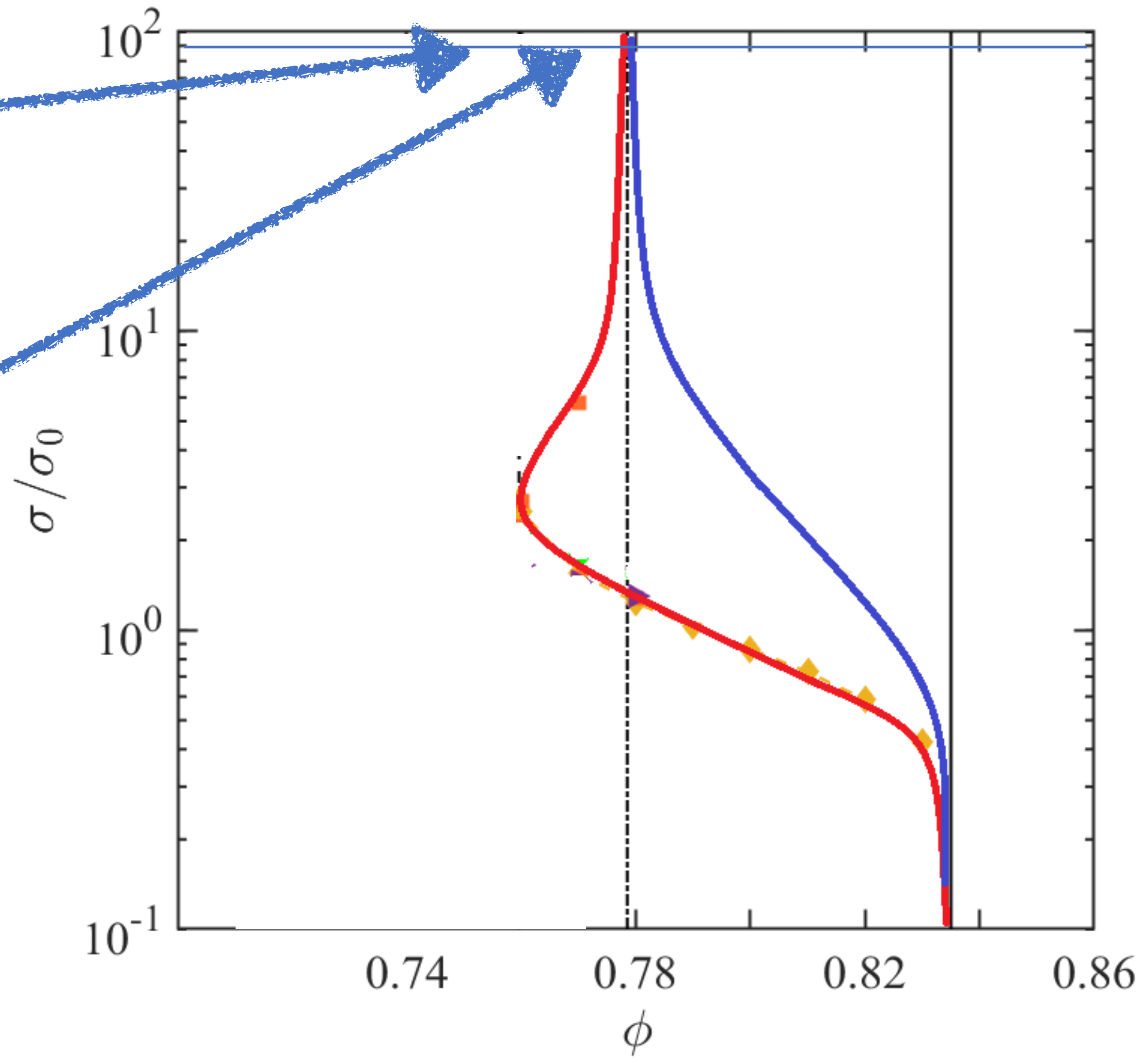


Pebble Game identifies rigid clusters (Silke Henkes)





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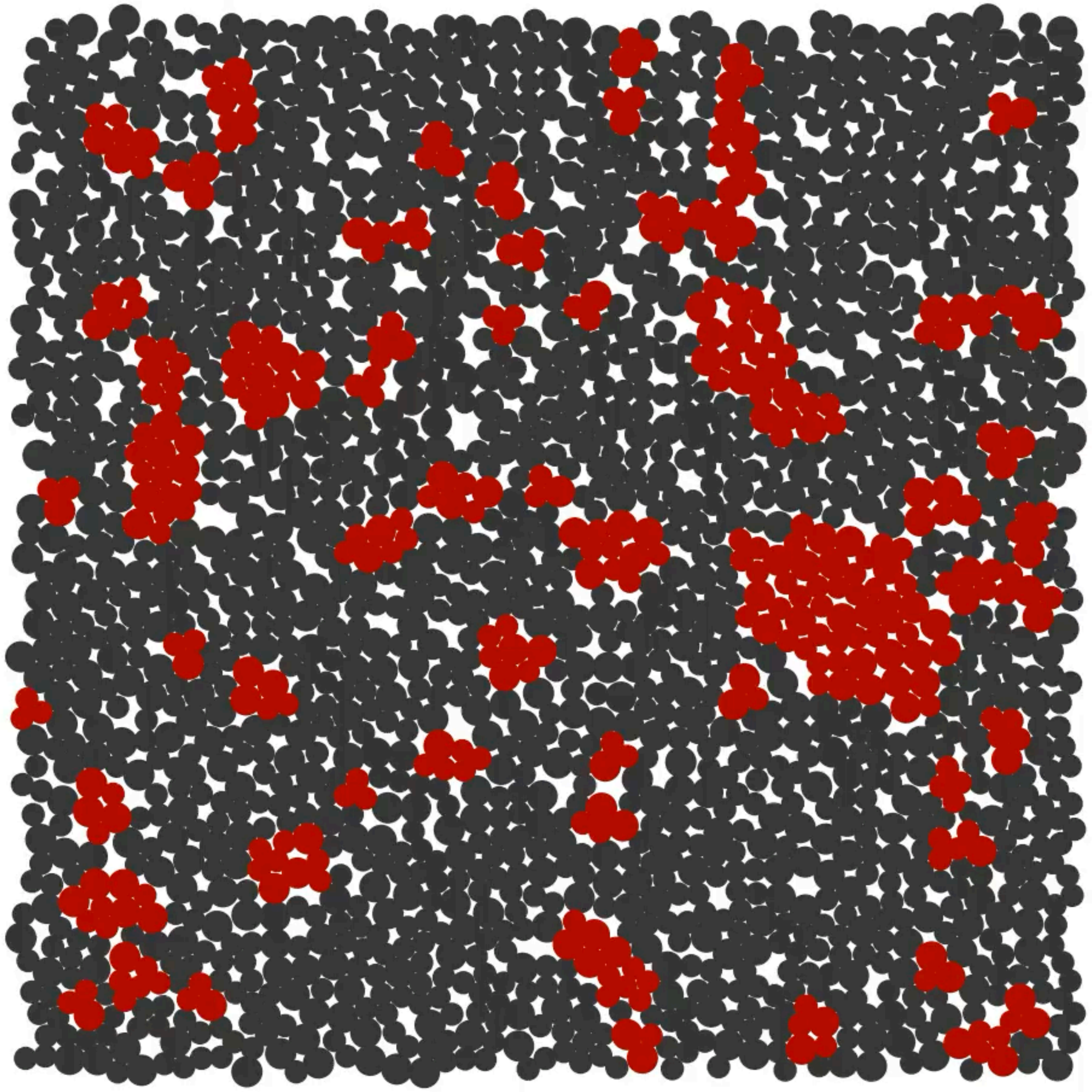
$NP = 2000 \quad \phi = 0.75 \quad t^* = 1.0 \quad \gamma = 0.46$



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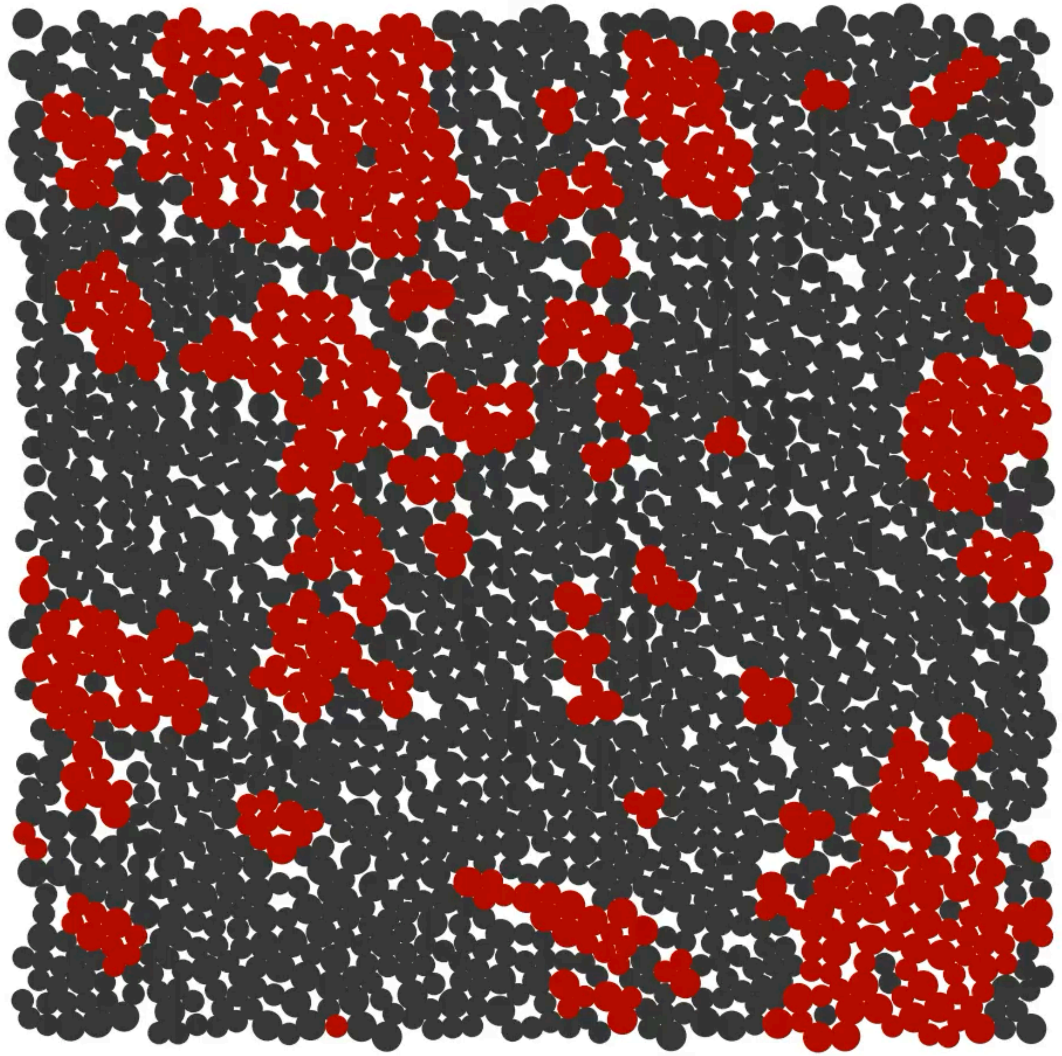

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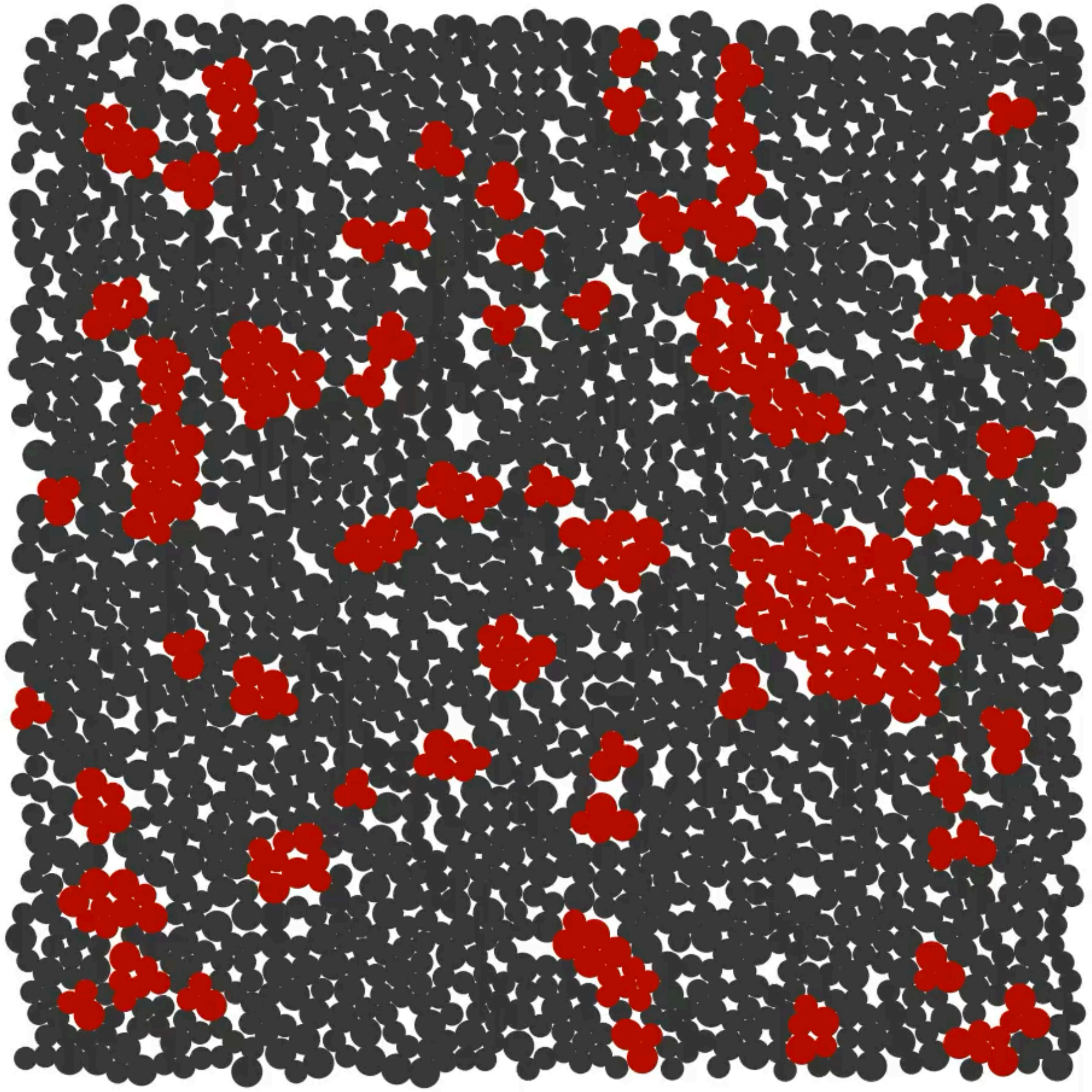

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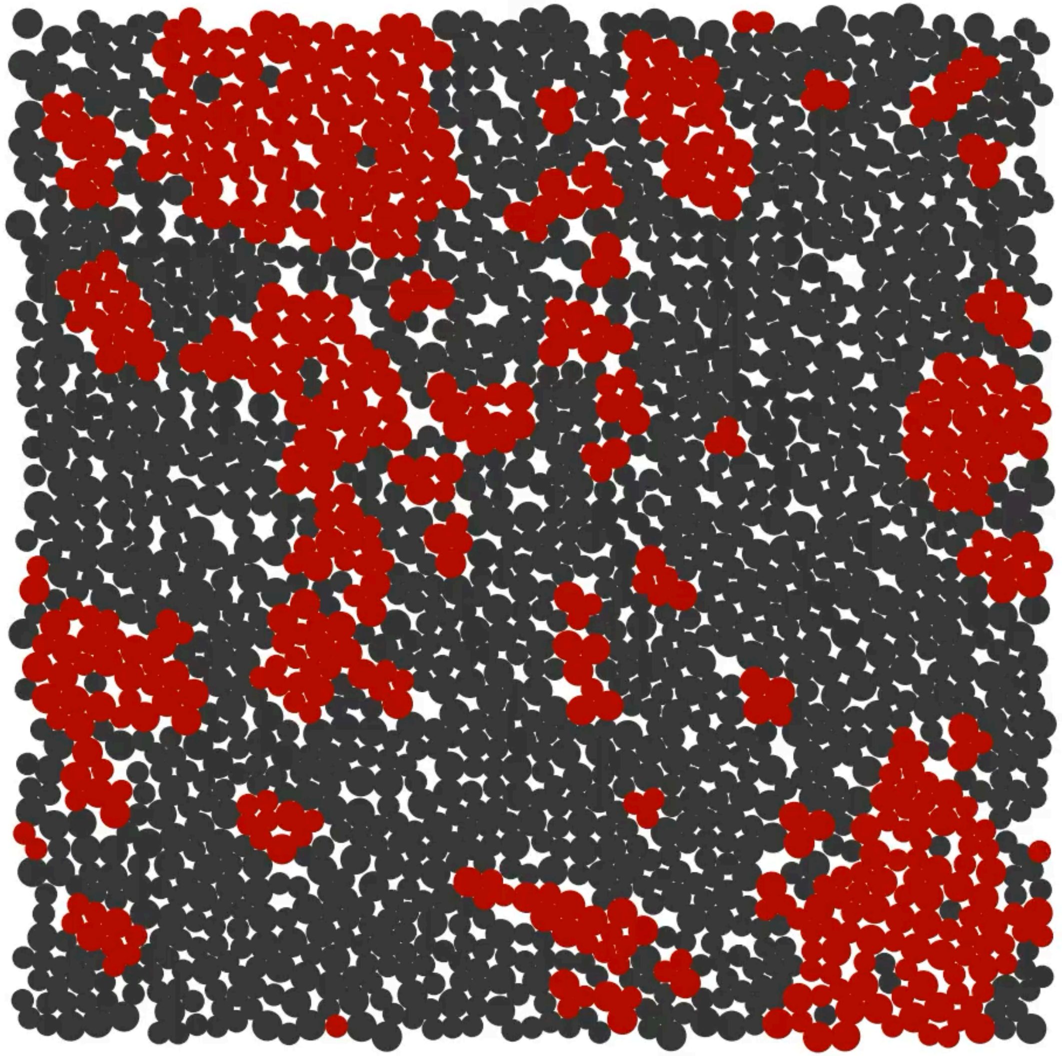

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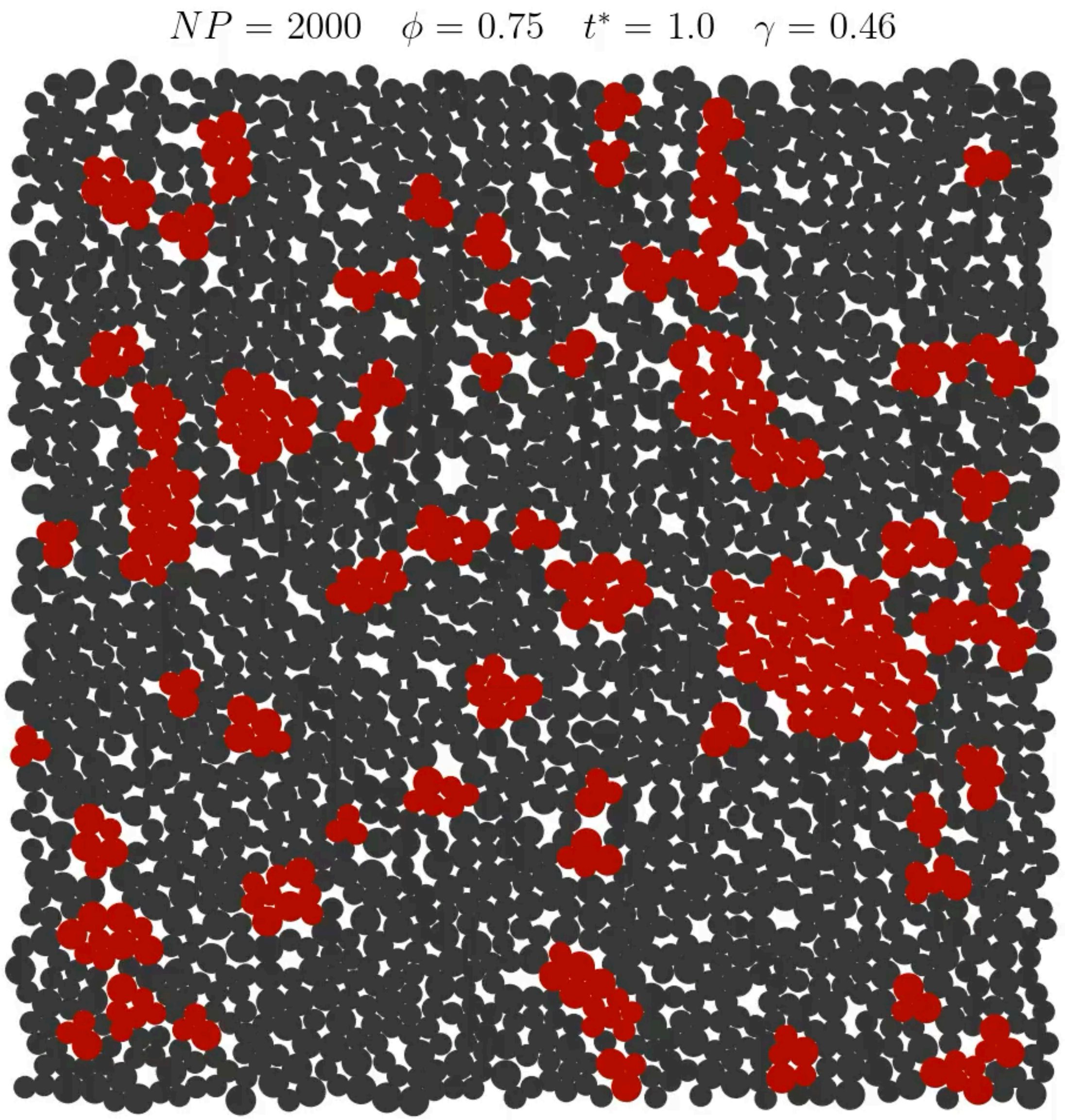

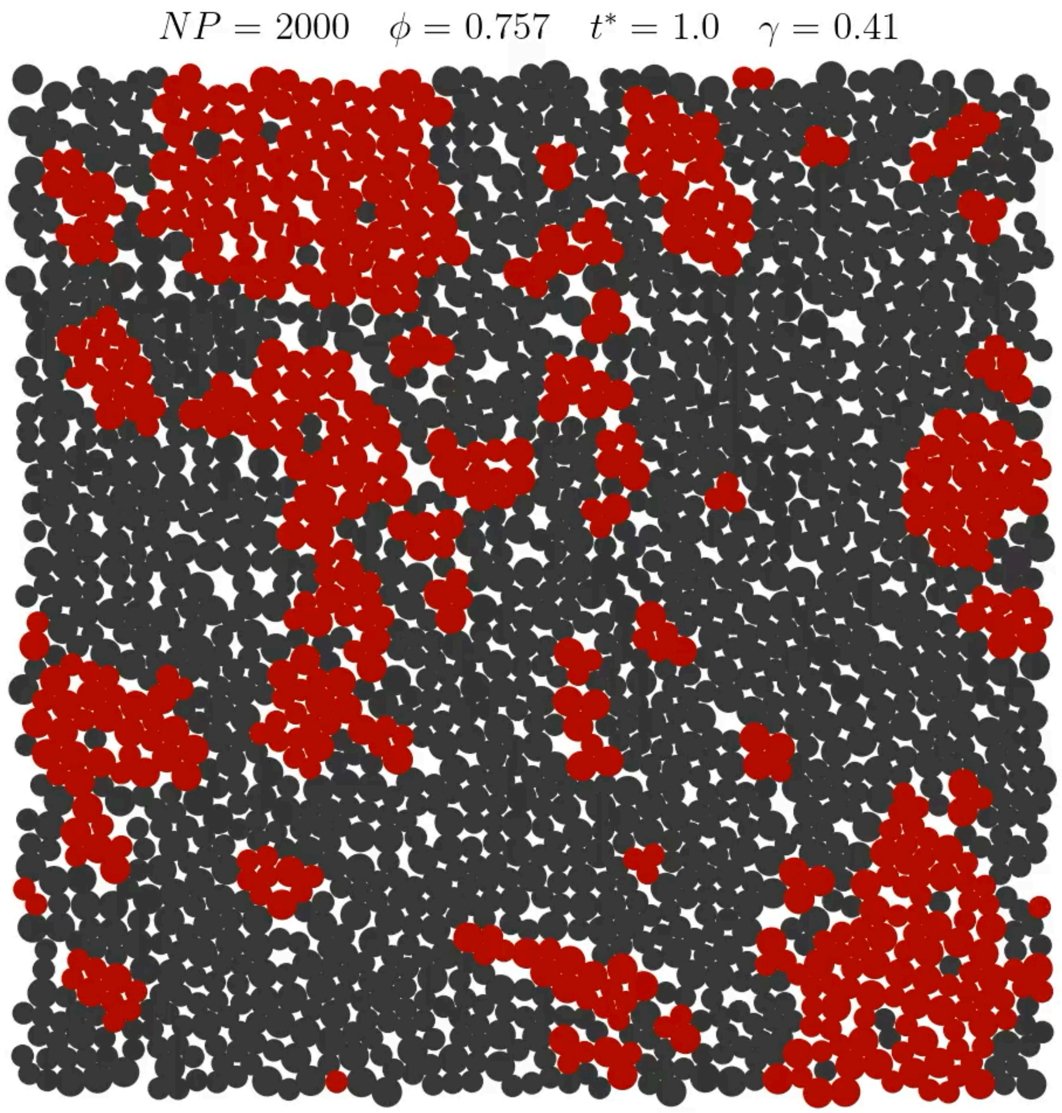

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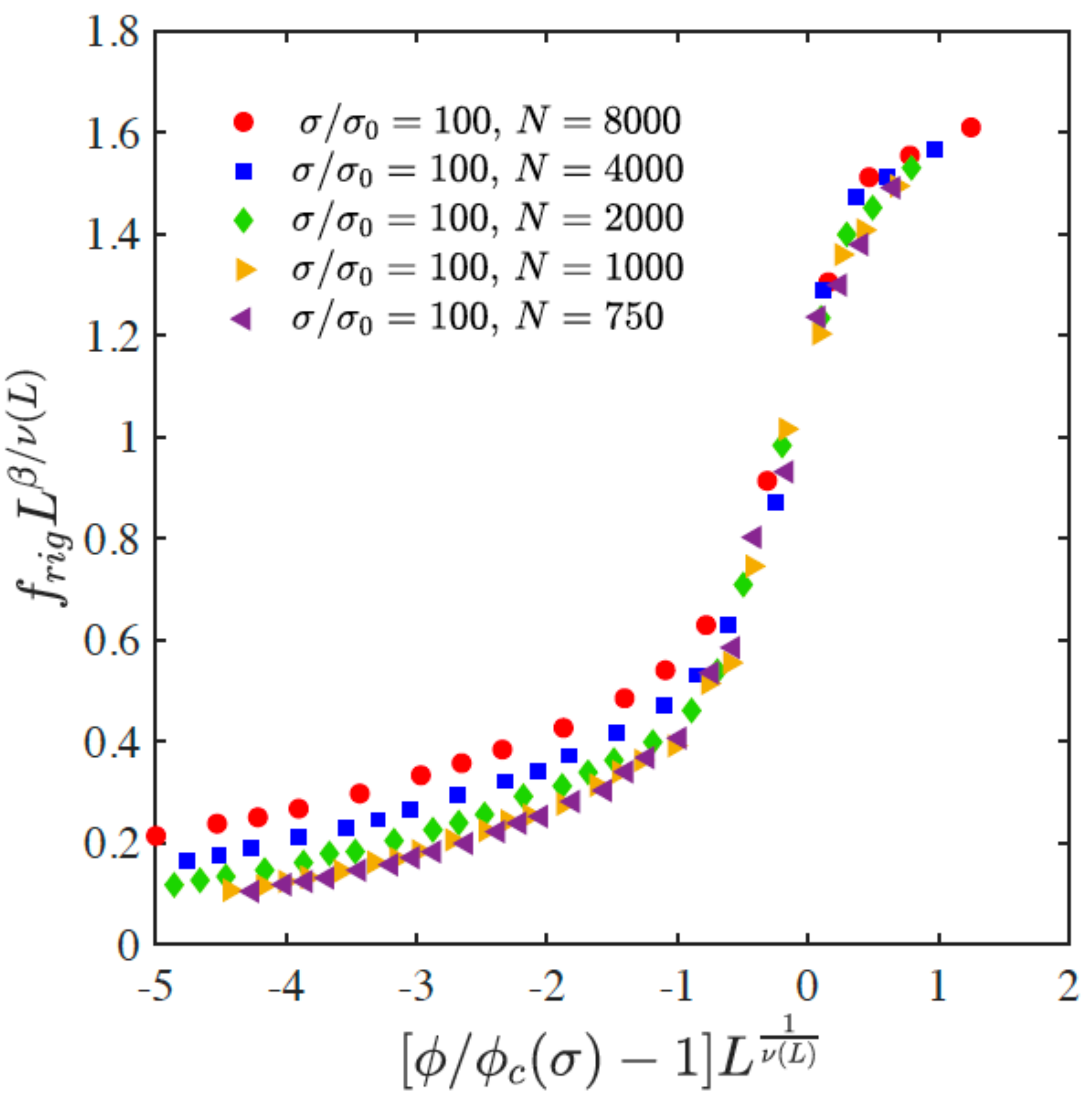

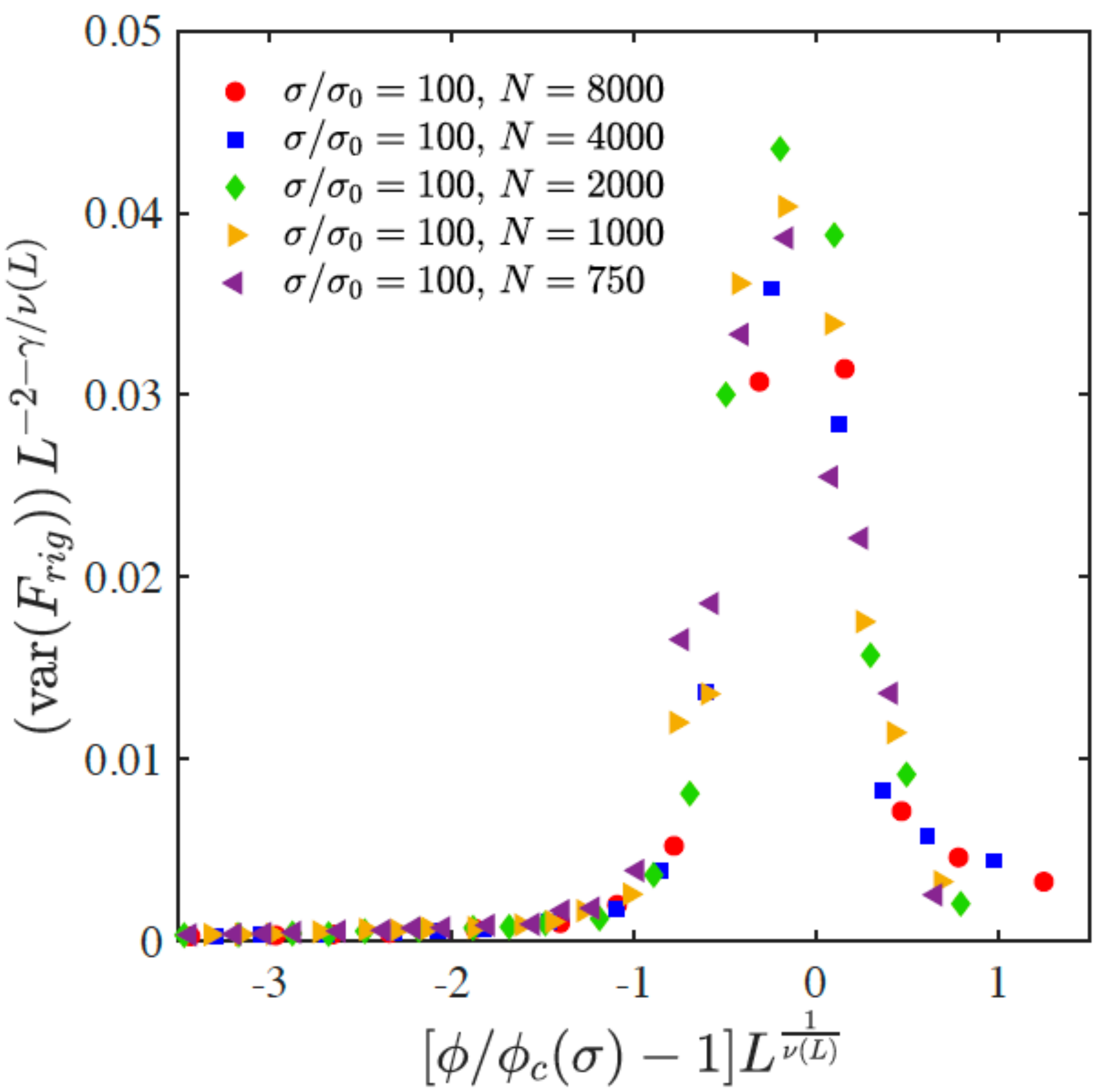

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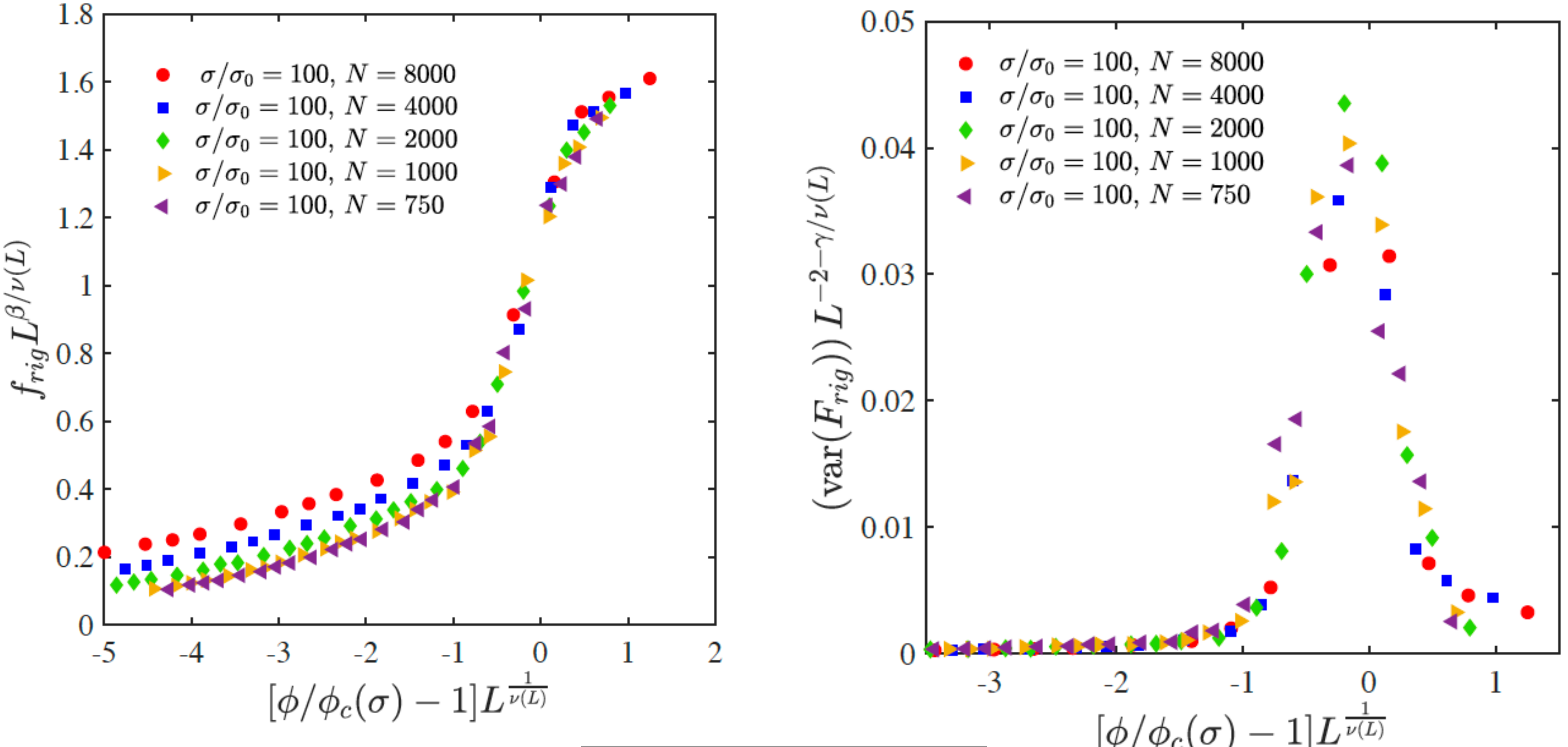
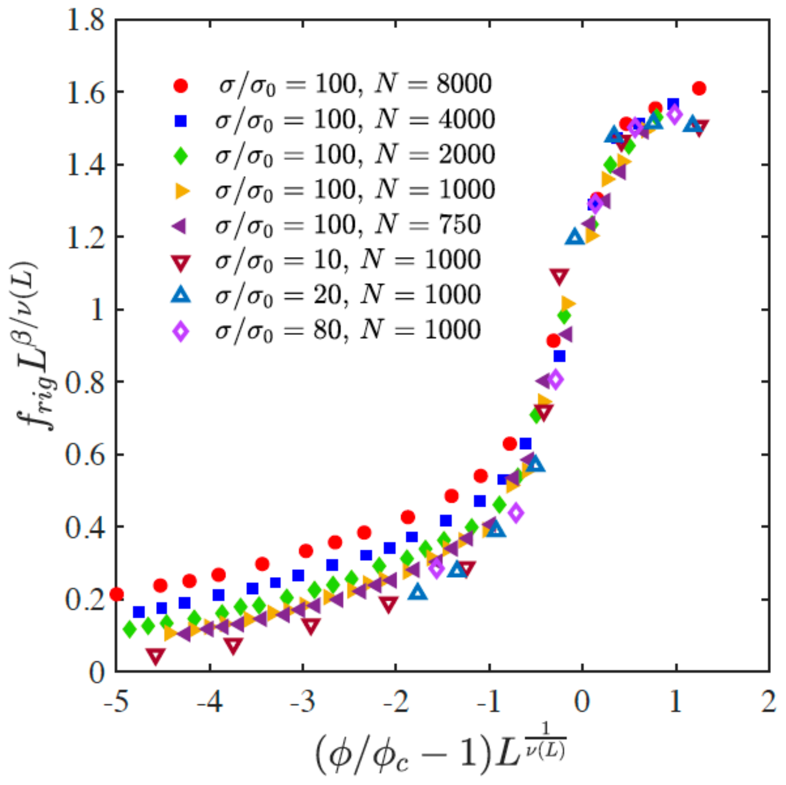
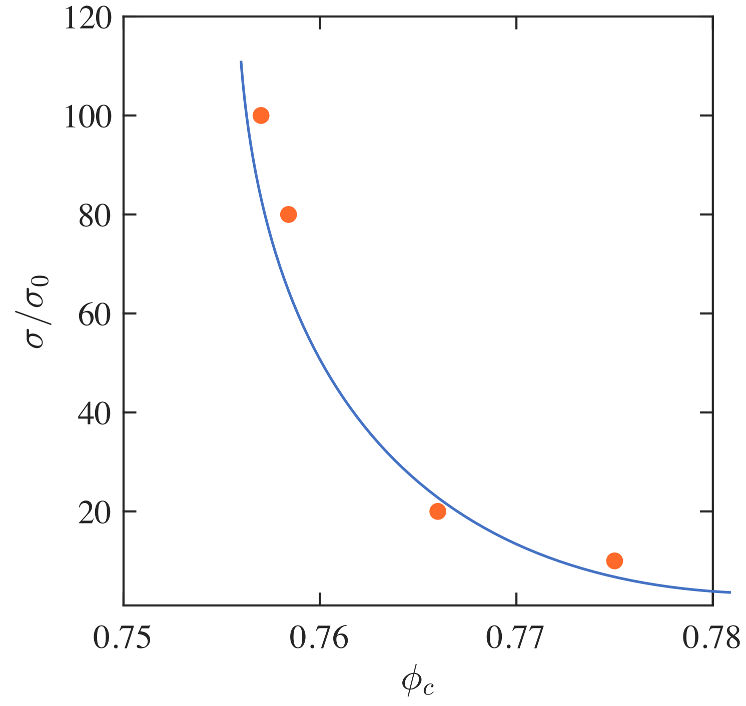
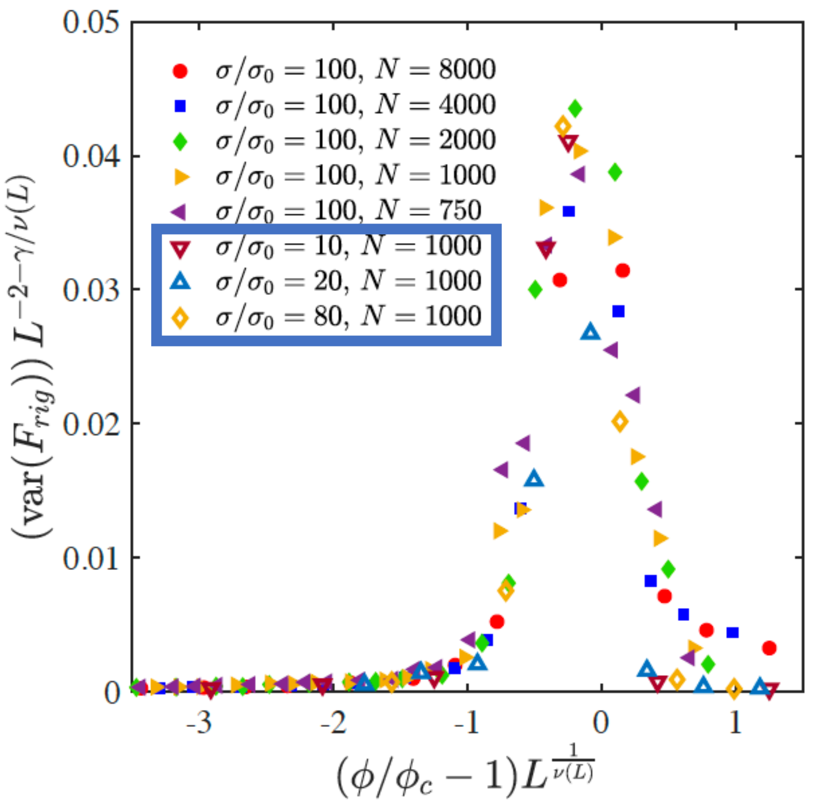


2D Ising Exponents!

Clusters of Rigid Particles





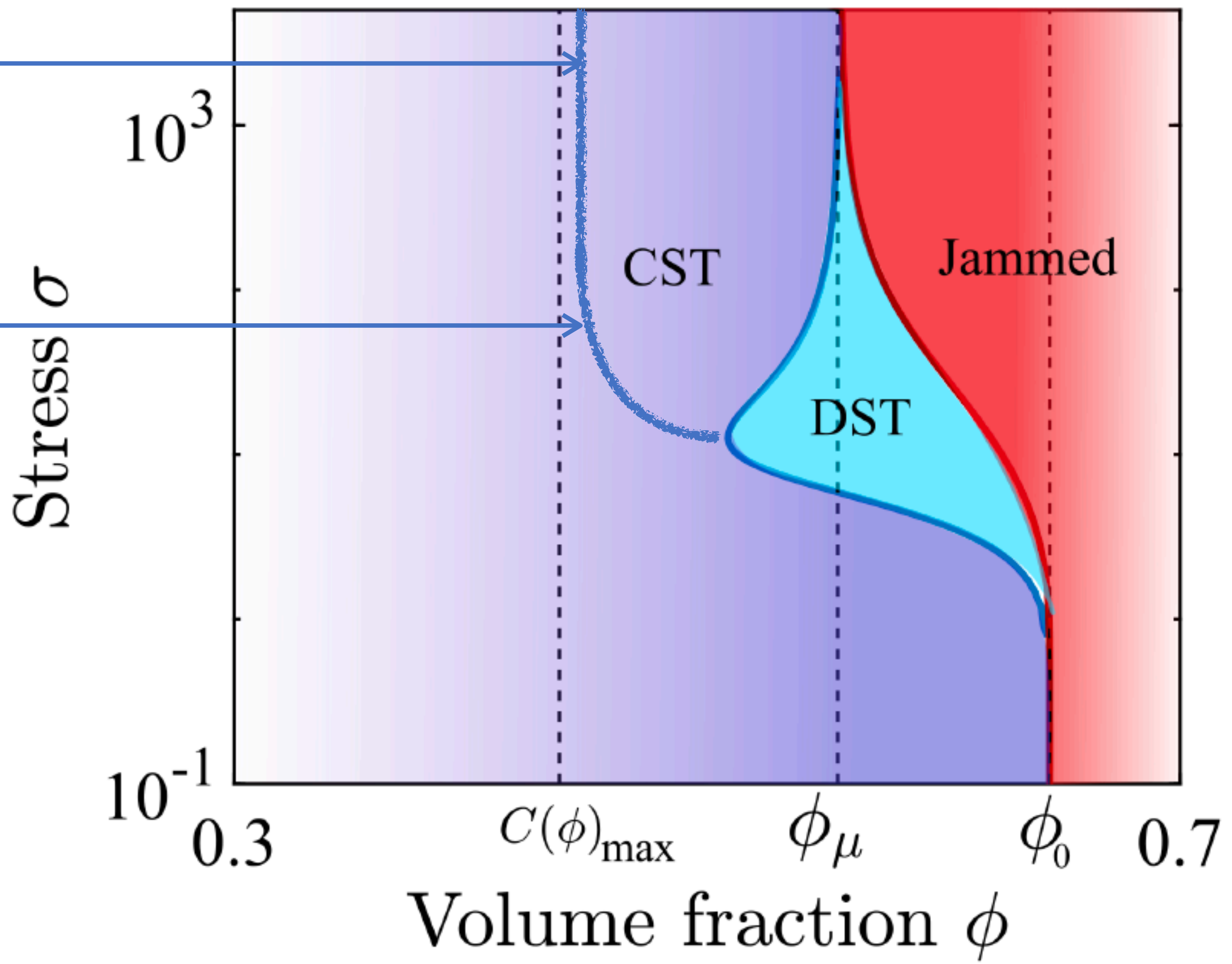


Clusters of Rigid Particles

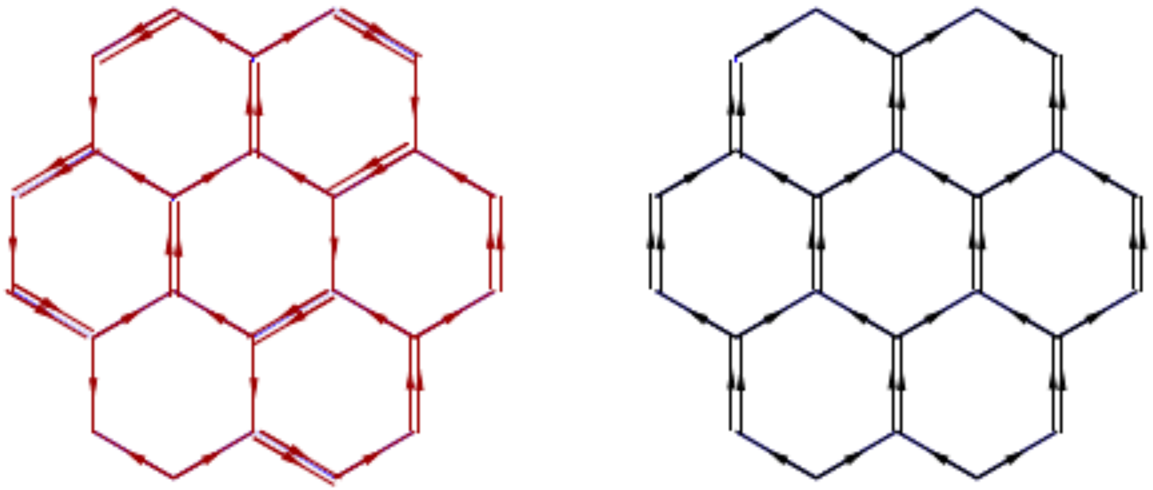
All contacts are frictional
Pure Ising Transition

Frictionless contacts proliferate: vacancies/ spin 0

Tricritical Point ?
Two-Phase coexistence



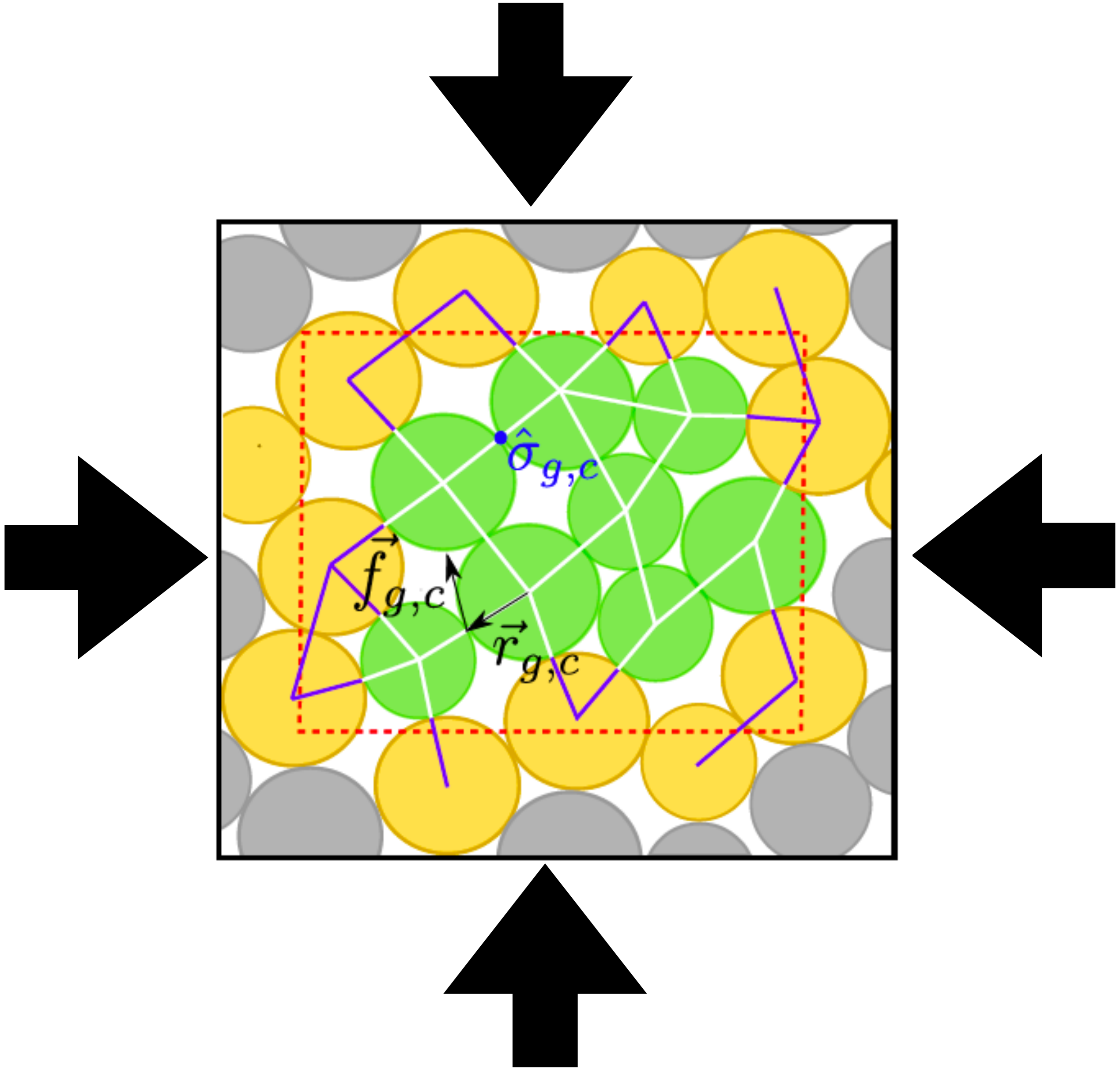
Constraints and Conservation Laws



- Lattice “fluxes” add up to zero at every vertex: divergence free condition satisfied by electric/magnetic fluxes
- Total “flux” in any direction is conserved
- Many microscopic configurations give the same total flux
- Appropriate coarse-graining variable: E field

$\nabla \cdot E = 0$ → Total flux is “conserved” → Constant Flux Ensemble

Force & Torque Balance: Boundary forces



$$\sum_{c \in g} \vec{f}_{g,c} = 0$$
$$= \vec{f}_{body}$$
$$\sum_{c \in g} \vec{r}_{g,c} \times \vec{f}_{g,c} = 0$$

Force-Moment Tensor

$$\hat{\sigma}_{g,c} = \vec{r}_{g,c} \otimes \vec{f}_{g,c}$$

Coarse-grained Stress Tensor

$$\hat{\sigma}(\vec{r}) = \frac{1}{\Omega_r} \sum_{g,c \in \Omega_r} \vec{r}_{g,c} \otimes \vec{f}_{g,c}$$

Mapping to Vector-charge U(1) Gauge Theory

M. Pretko (2018)

Gauss's Law: $\partial_i E_{ij} = \rho_j$

E_{ij} and A_{ij} are conjugate variables

$$A_{ij} = A_{ij} + \partial_i \phi_j + \partial_j \phi_i$$

Charge and Charge angular momentum are conserved

Maxwell's Equations in Vacuum

$$\partial_i E_{ij} = \rho_j$$

$$\partial_i B_{ij} = \tilde{\rho}_j$$

$$\epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c E_{bd} = -\partial_t B_{ij} - \tilde{J}_{ij}$$

$$\epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c B_{bd} = \partial_t E_{ij} + J_{ij}$$

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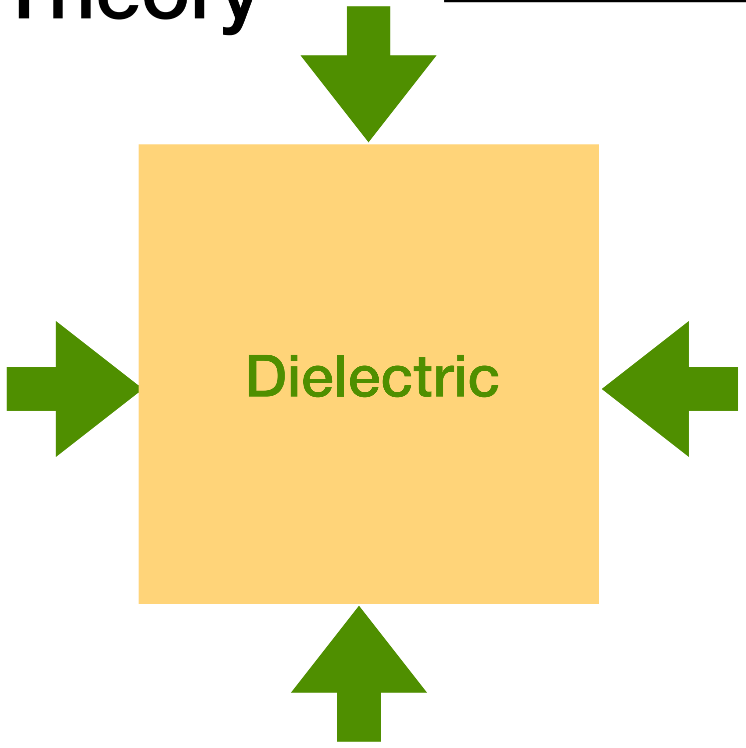

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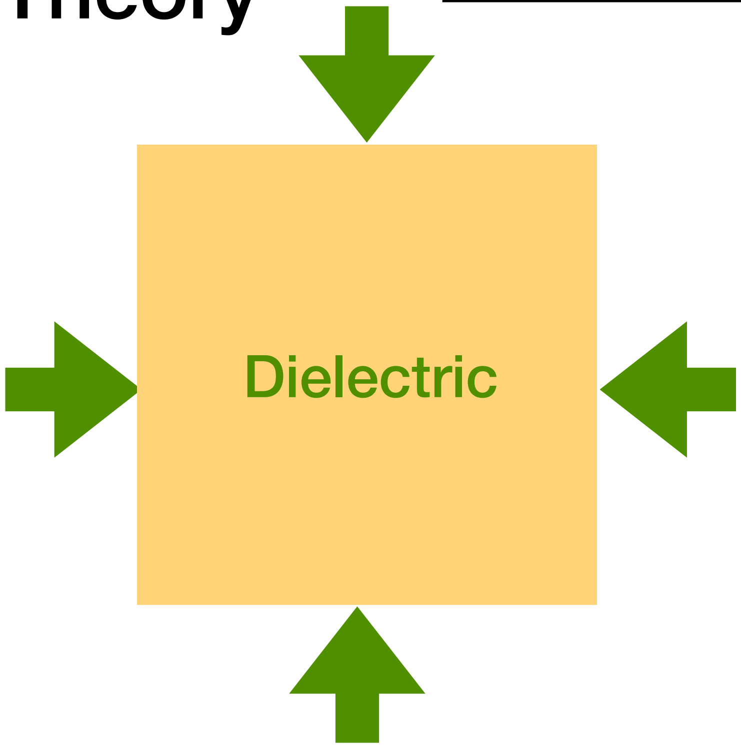


Electrostatics in a Dielectric

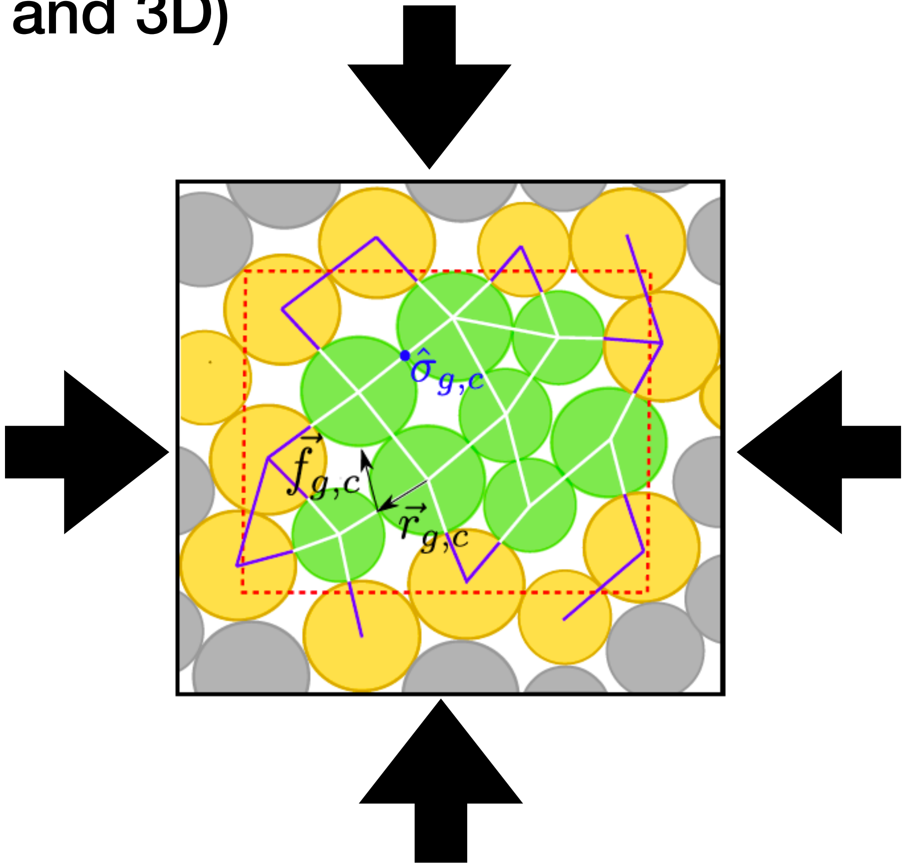
$$\partial_i E_{ij} = \rho_j^{free} + \rho_j^{bound}$$

$$\epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c E_{bd} = 0$$

$$\partial_i D_{ij} = \rho_j^{free}$$

$$D_{ij} = \chi_{ijkl} E_{kl}$$

Mapping of amorphous elasticity to Vector-charge U(1) Gauge Theory (in 2D and 3D)



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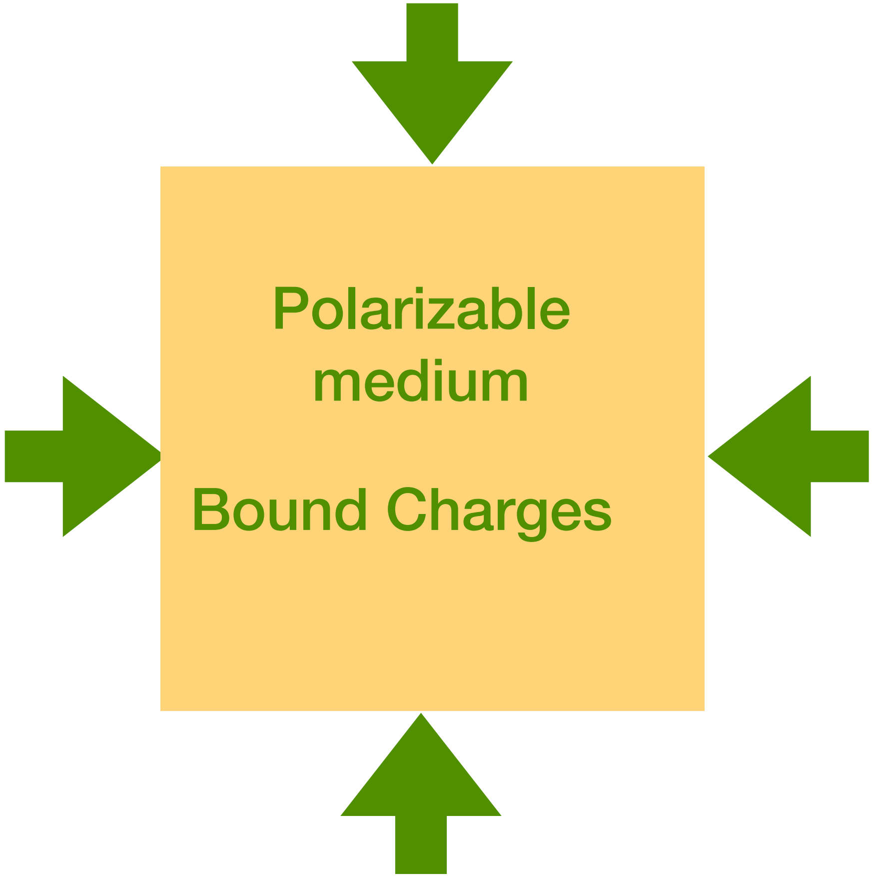

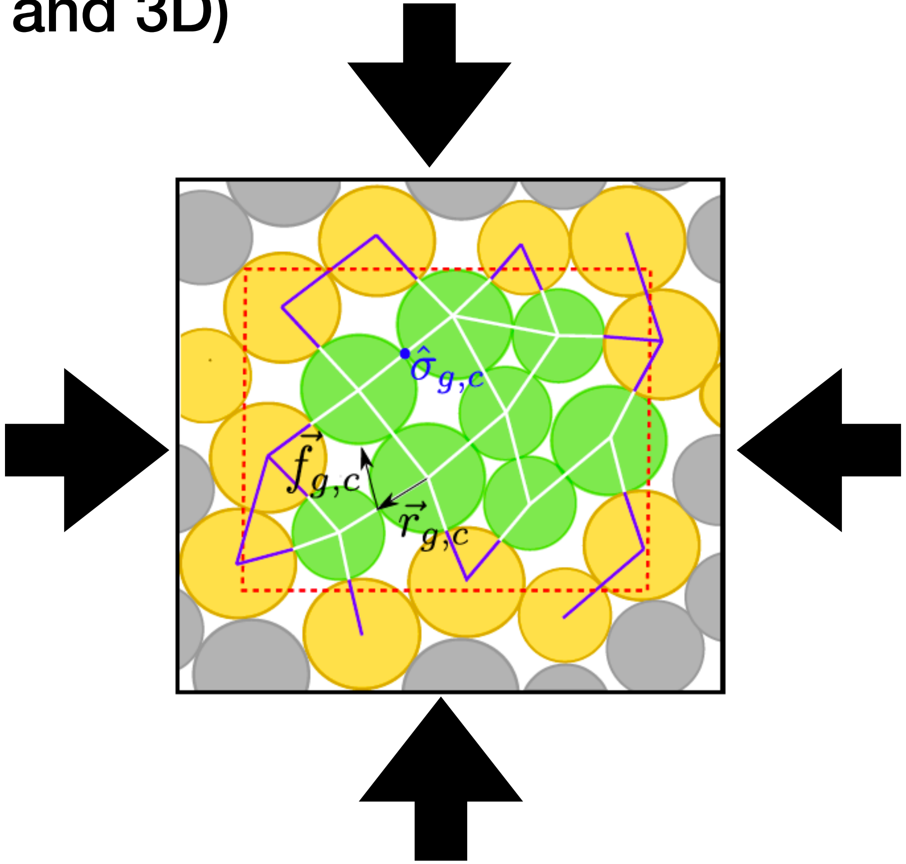

Vectorial “free” Charges/
Tensorial, Symmetric E field

Gauss’s Law: $\partial_i E_{ij} = \rho_j$

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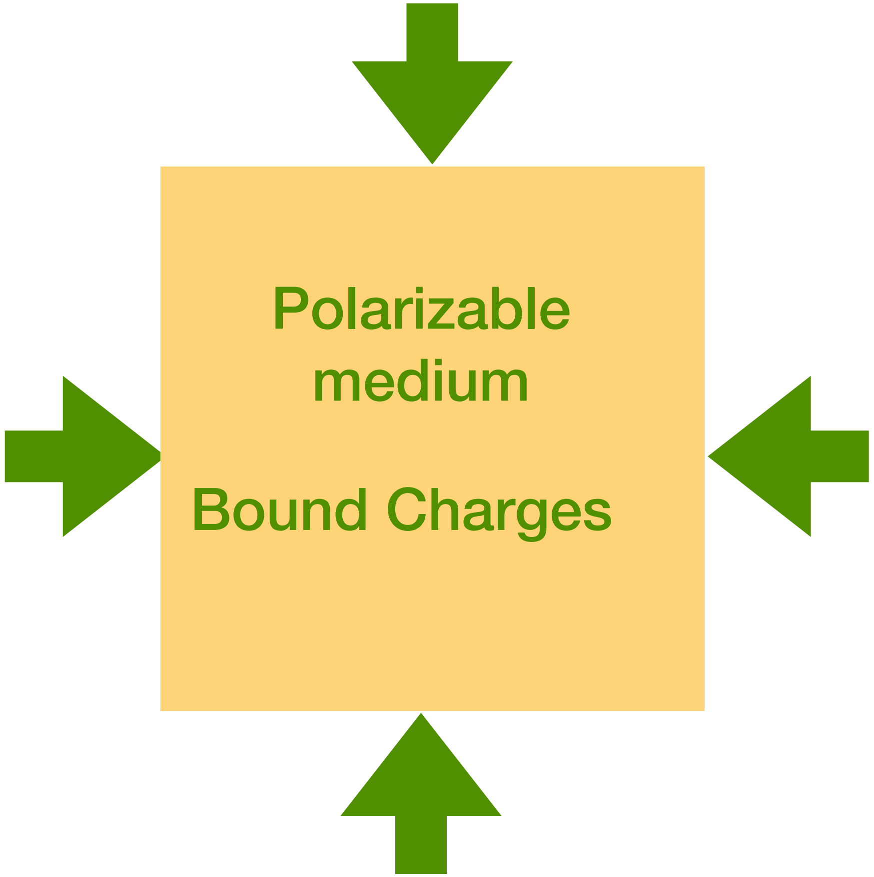

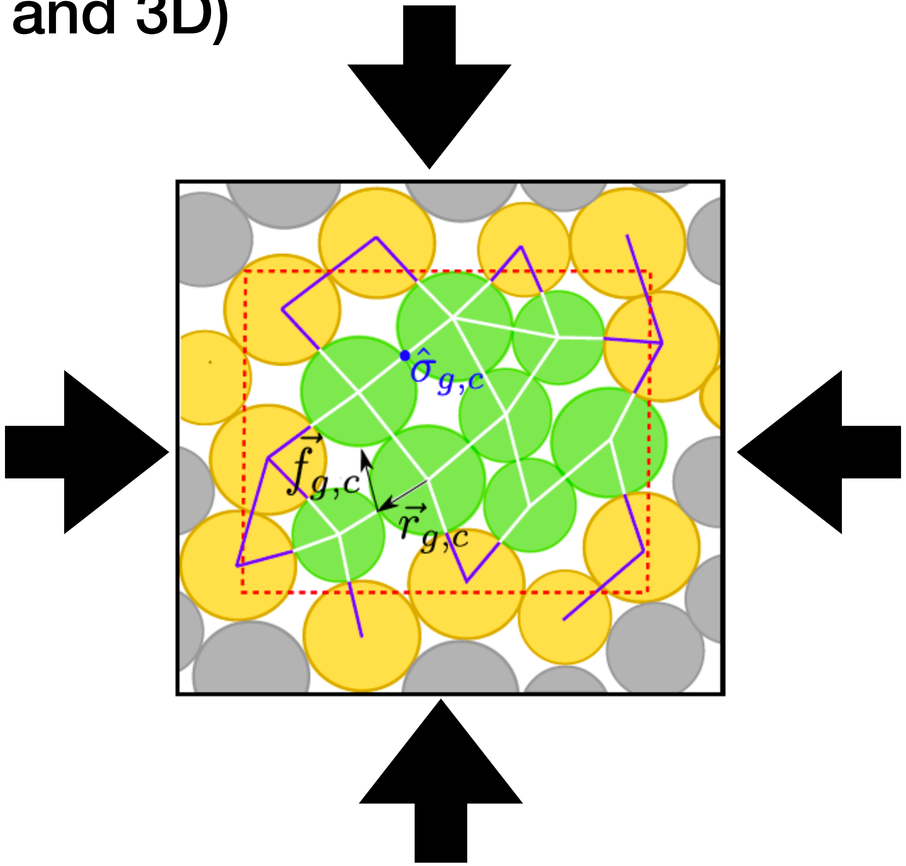

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Charges map to Forces

Bound → Contact Forces

Free → Body/External forces

Stress-only Formulation of the Elasticity of Jammed States

$$\partial_i \sigma_{ij} = f_j^{\text{external}}$$

$$E_{ij} = \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i) \implies \epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c E_{bd} = 0$$

$$\sigma_{ij} = K_{ijkl} E_{kl}$$

J. Nampoothiri et al
PRL (2021), PRE (2023)

Crucial Differences from Classical Elasticity:

- Rigidity is a consequence of constraints not broken symmetry

- Instead of physical displacement fields defining a strain tensor, there are gauge potentials defining a field that looks like the strain tensor

- The elastic moduli are not material properties but emerge from properties of the network created by external stresses (does not have the usual symmetries)

- The elastic moduli do not have to satisfy the symmetry requirements coming from a free-energy.

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J. Nampoothiri et al
PRL (2021), PRE (2023)

Total flux of stress components is conserved
Bi et al: Annual Reviews of Condensed Matter (2015)

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Constraints → Conserved Quantities

$$\frac{dP_\alpha(t)}{dt} = \sum_\beta W(\alpha|\beta)P_\beta(t) - W(\beta|\alpha)P_\alpha(t)$$

If dynamics conserves some quantity (U),

then $P_\alpha = f_\alpha / Z_\mu(U)$

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→ Intensive Variables can be defined

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- Analogs of Microcanonical and Canonical Ensembles
- Edwards "Thermodynamics"
- Field Theories are Emergent Gauge Theories (due to constraints)
- Effective Hamiltonians ==> Universality and Scaling ?