Scaling of Mean First Passage Times of Random Walks in Fractal Media

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Heterogeneous Mean First Passage Time Scaling in Fractal Media

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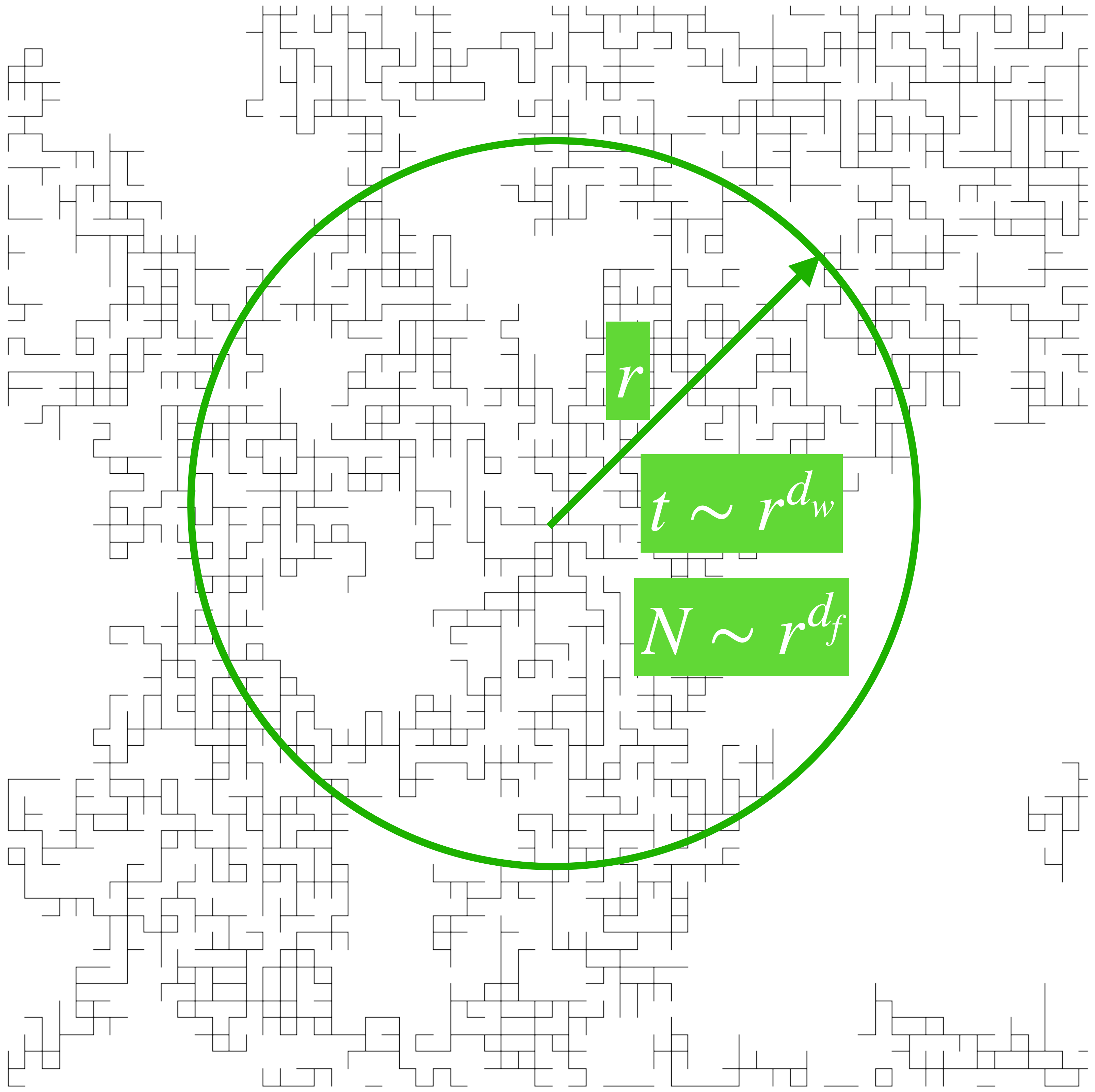
The mean first passage time (MFPT) of random walks is a key quantity characterizing dynamic processes on disordered media. In a random fractal embedded in the Euclidean space, the MFPT is known to obey the power law scaling with the distance between a source and a target site with a universal exponent. We find that the scaling law for the MFPT is not determined solely by the distance between a source and a target but also by their locations. The role of a site in the first passage processes is quantified by the random walk centrality. It turns out that the site of highest random walk centrality, dubbed as a hub, intervenes in first passage processes. We show that the MFPT from a departure site to a target site is determined by a competition between direct paths and indirect paths detouring via the hub. Consequently, the MFPT displays a crossover scaling between a short distance regime, where direct paths are dominant, and a long distance regime, where indirect paths are dominant. The two regimes are characterized by power laws with different scaling exponents. The crossover scaling behavior is confirmed by extensive numerical calculations of the MFPTs on the critical percolation cluster in two dimensional square lattices.

arXiv:2304.14940

Random Walks on Fractals

Havlin and Ben-Avraham (2002)

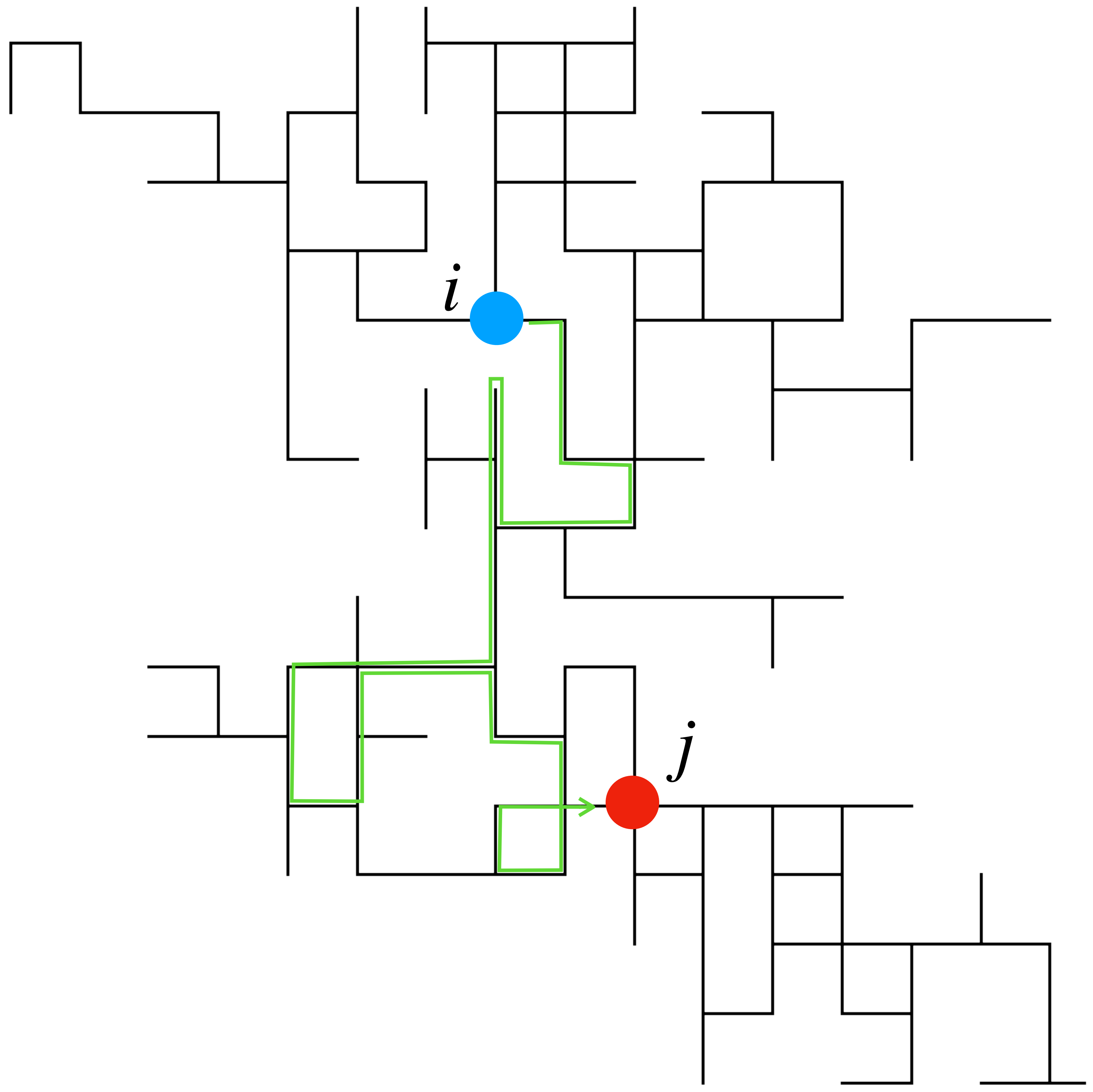


critical percolation cluster

fractal dimension $N \sim r^{d_f}$ with $d_f = \dfrac{91}{48}$ in 2D

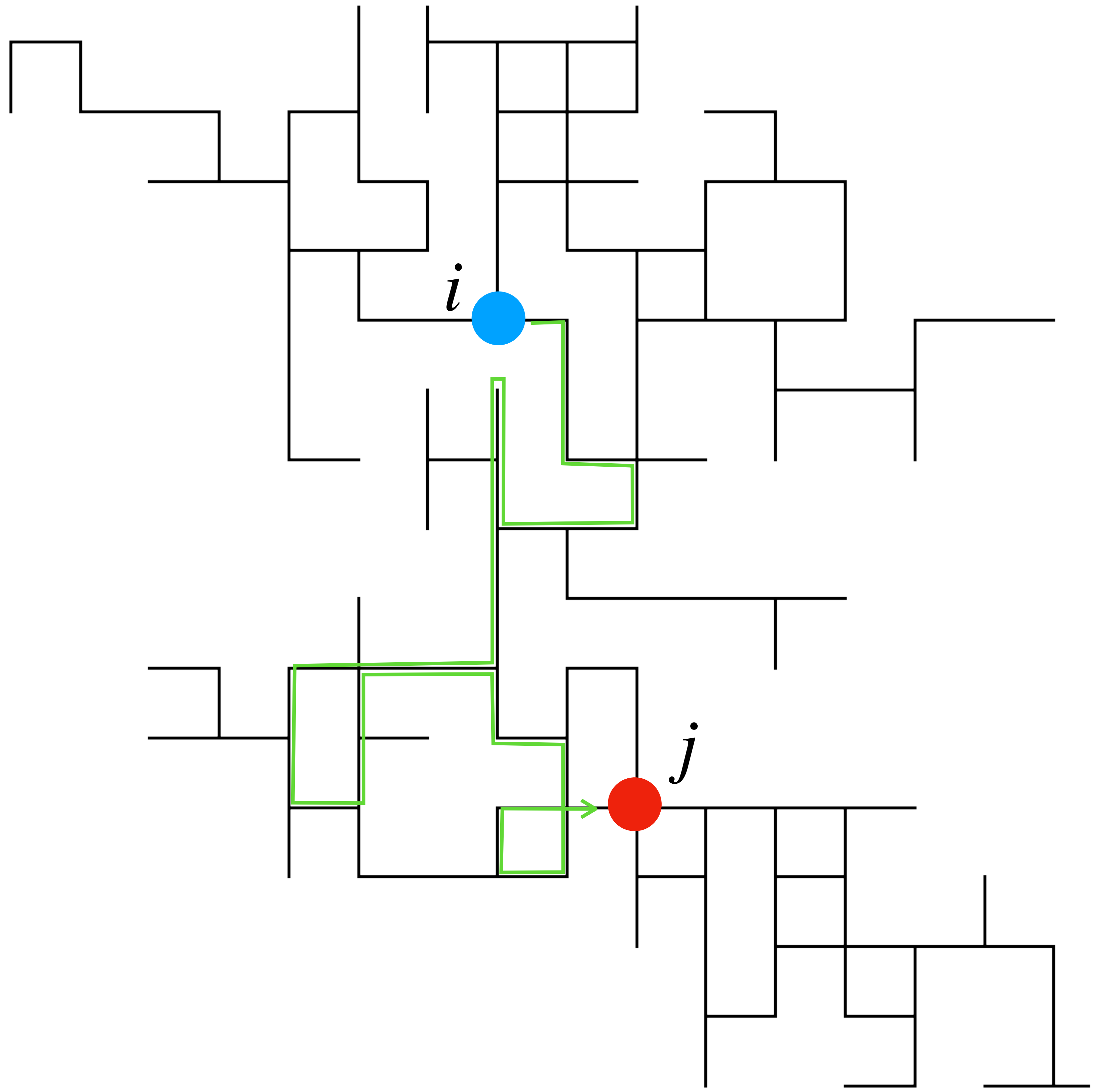
random walk dimension $r \sim t^{1/d_w}$ with $d_w \simeq 2.87$ in 2D

Discrete Time Random Walks on a Finite Graph



adjacency matrix	$A_{ij} = 1$ (edge) or 0
undirected graph	$A = A^T$
transition probability	$W_{ij} = \frac{A_{ij}}{k_i} \quad \left(k_i = \sum_j A_{ij} \right)$
propagator	$P_{ij}(t) = \langle i \mid W^t \mid j \rangle$
	$P_{ij}(t+1) = \sum_l P_{il}(t) W_{lj}$
detailed balance	
steady state	$\pi_i = \lim_{t \to \infty} P_{si}(t) = \frac{k_i}{\sum_j k_j}$

First Passage Time



first passage probability $F_{ij}(t)$

$$P_{ij}(t) = \delta_{t0}\delta_{ij} + \sum_{t'=0}^{t} F_{ij}(t')P_{jj}(t-t')$$

mean first passage time (MFPT) $T_{ij} \equiv \sum_{t=0}^{\infty} tF_{ij}(t)$

$$T_{ij} = \frac{R_{jj} - R_{ij} + \delta_{ij}}{\pi_j}$$ [Noh and Rieger 2004]

where $R = (1-W)^{\#}$ with $R_{ij} = \sum_{t=0}^{\infty} \left(P_{ij}(t) - \pi_j\right)$

generalized group inverse [Meyer 1975]

pseudo Green function [Condamin et al 2007]

MFPT vs Distance

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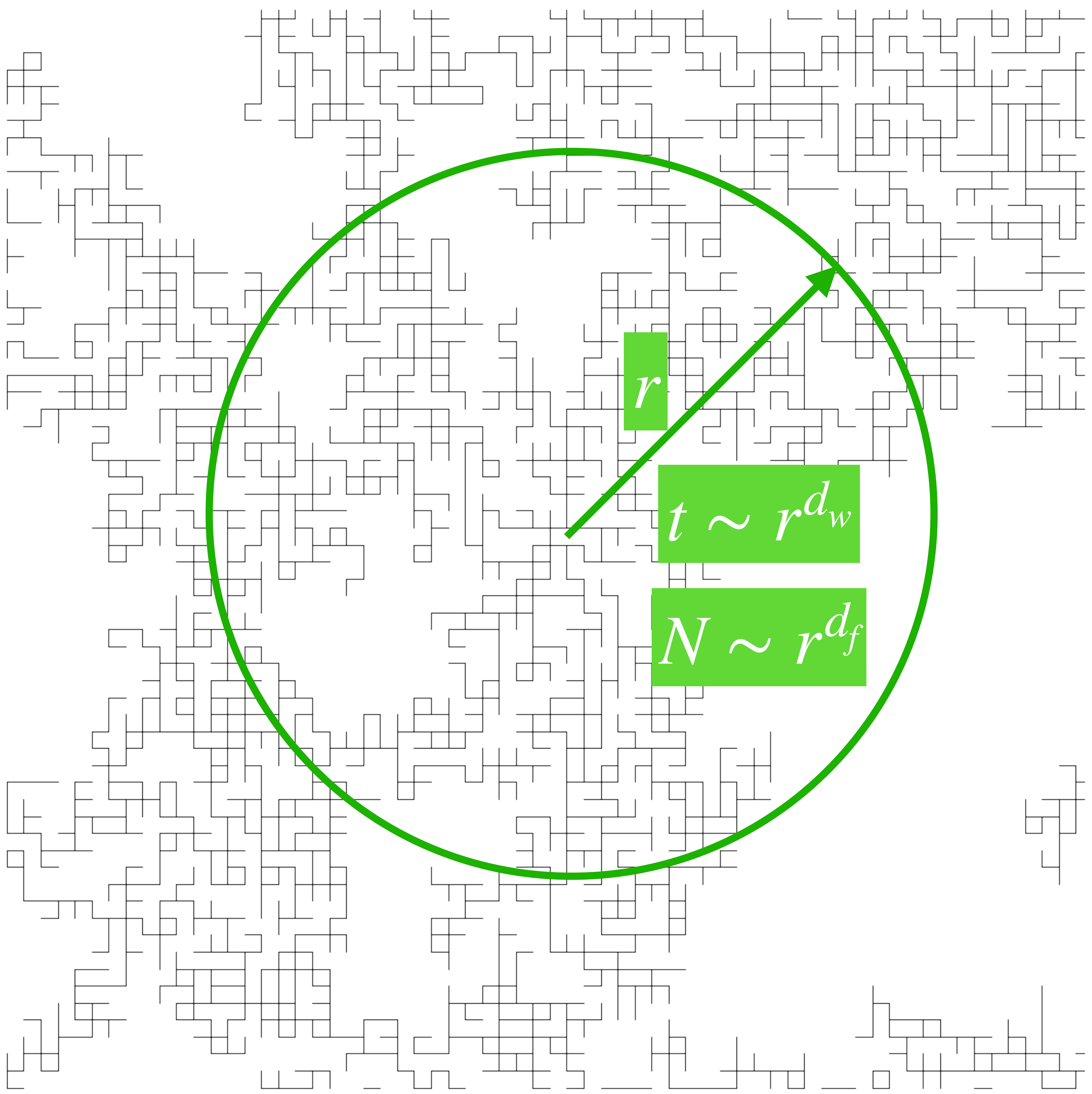
First-passage times in complex scale-invariant media

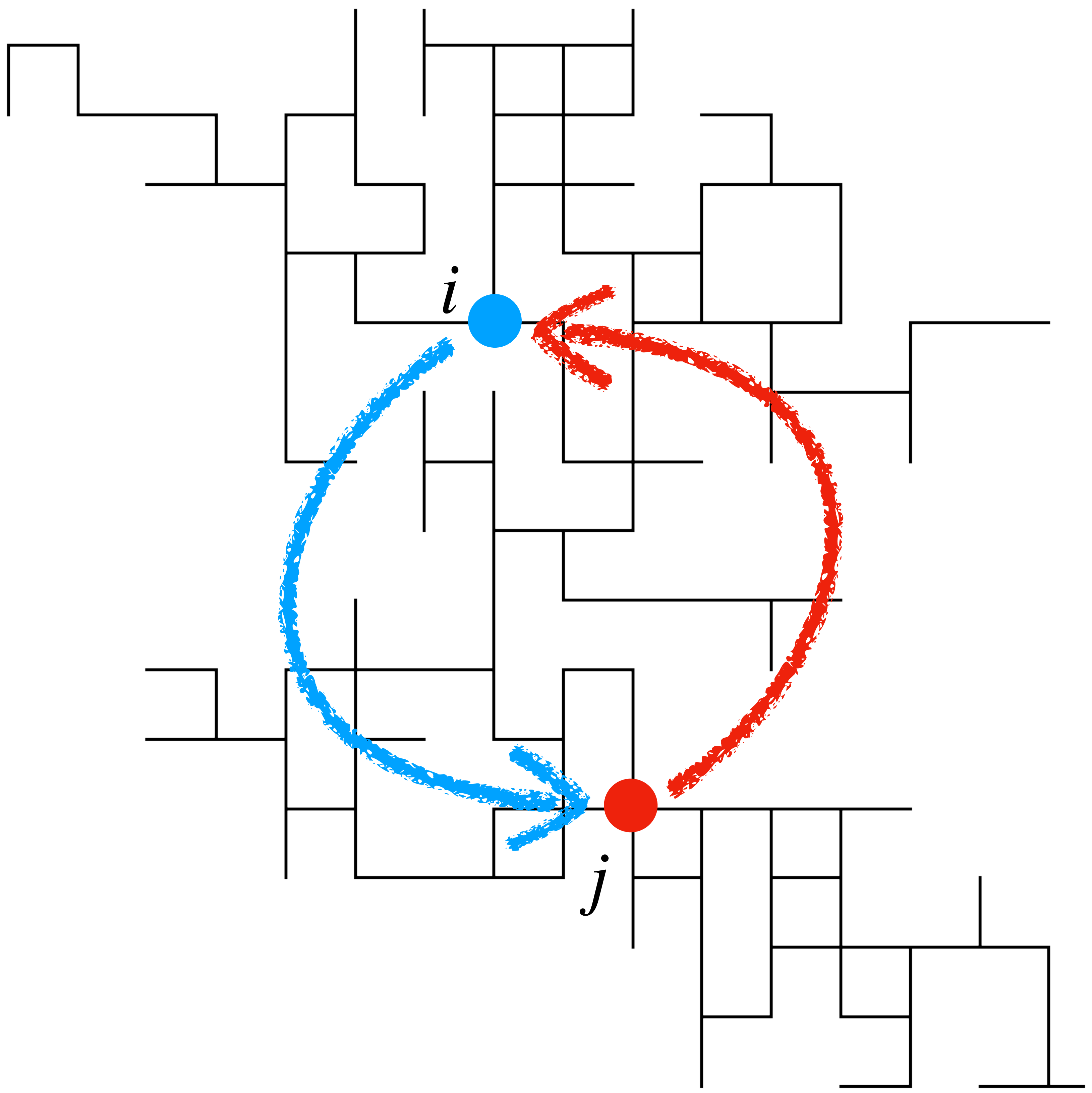
S. Condamin[1], O. Bénichou[1], V. Tejedor[1], R. Voituriez[1] & J. Klafter[2]

$$T(r) \sim N r^{d_w - d_f} \sim L^{d_f} r^{d_w - d_f}$$

when $d_w > d_f$ (compact exploration)



Random Walk Centrality



$$T_{ij} = \frac{R_{jj} - R_{ij} + \delta_{ij}}{\pi_j}$$

$$T_{ij} - T_{ji} = \left(\frac{R_{jj}}{\pi_j} - \frac{R_{ii}}{\pi_i} \right) - \left(\frac{R_{ij}}{\pi_j} - \frac{R_{ji}}{\pi_i} \right)$$

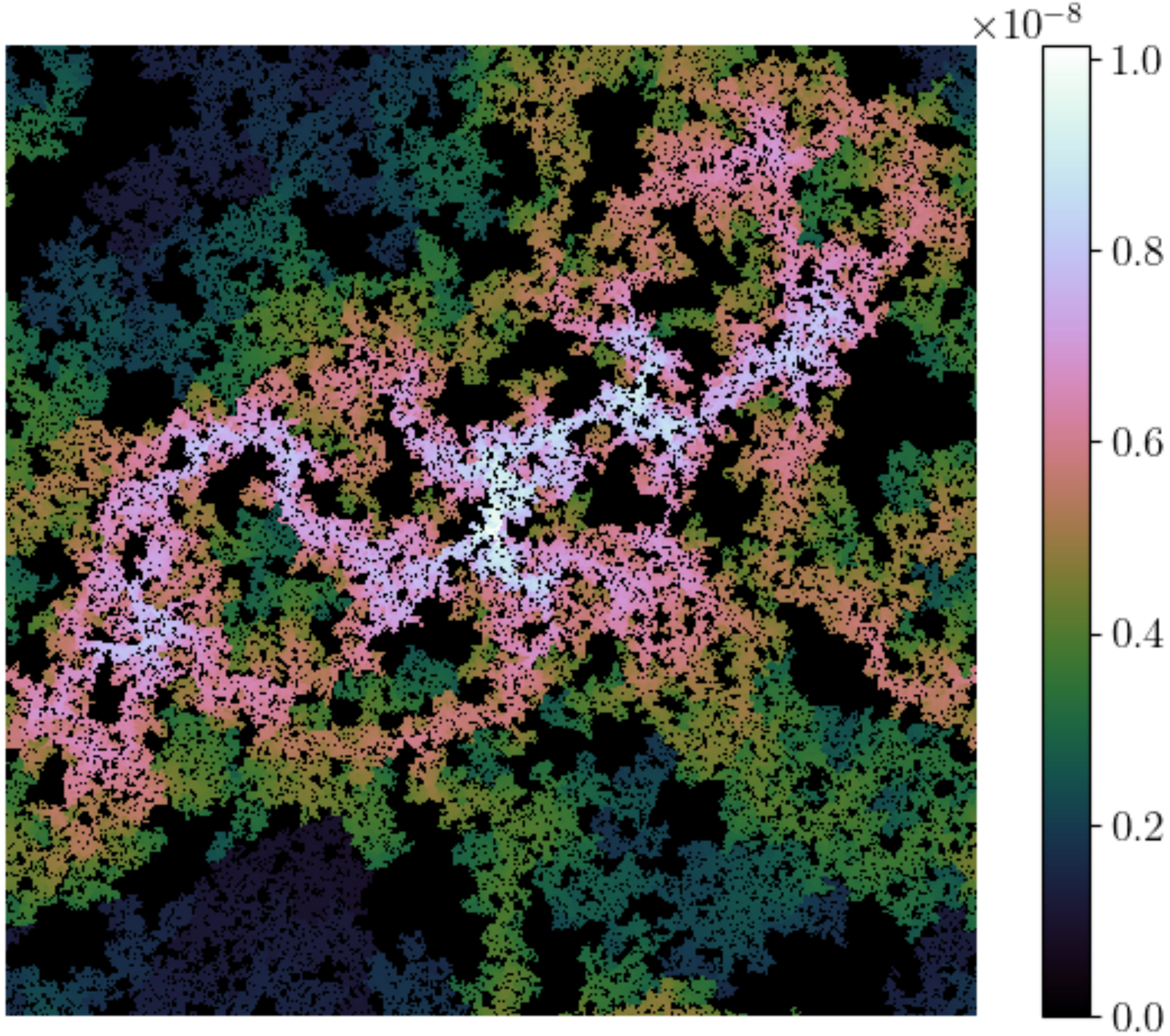
$$= \frac{R_{jj}}{\pi_j} - \frac{R_{ii}}{\pi_i} = C_j^{-1} - C_i^{-1}$$

RWC as a scalar potential $C_i = \frac{\pi_i}{R_{ii}}$ ~ attractiveness

[Noh and Rieger, PRL (2004)]

efficient numerical algorithm

[Hwang, Lee, and Khang PRE (2014)]

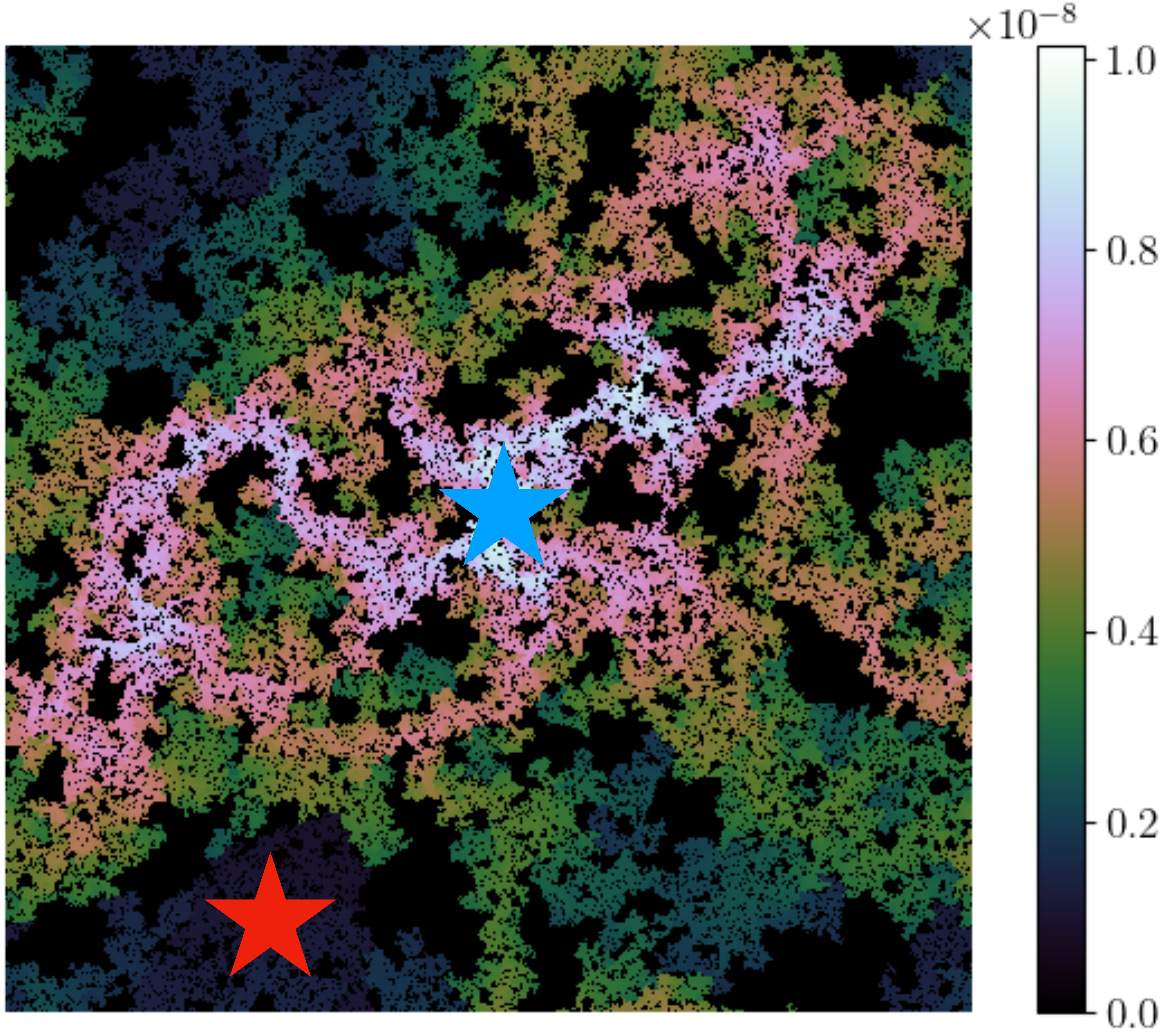


critical bond percolation cluster on 1024 x 1024 square lattice

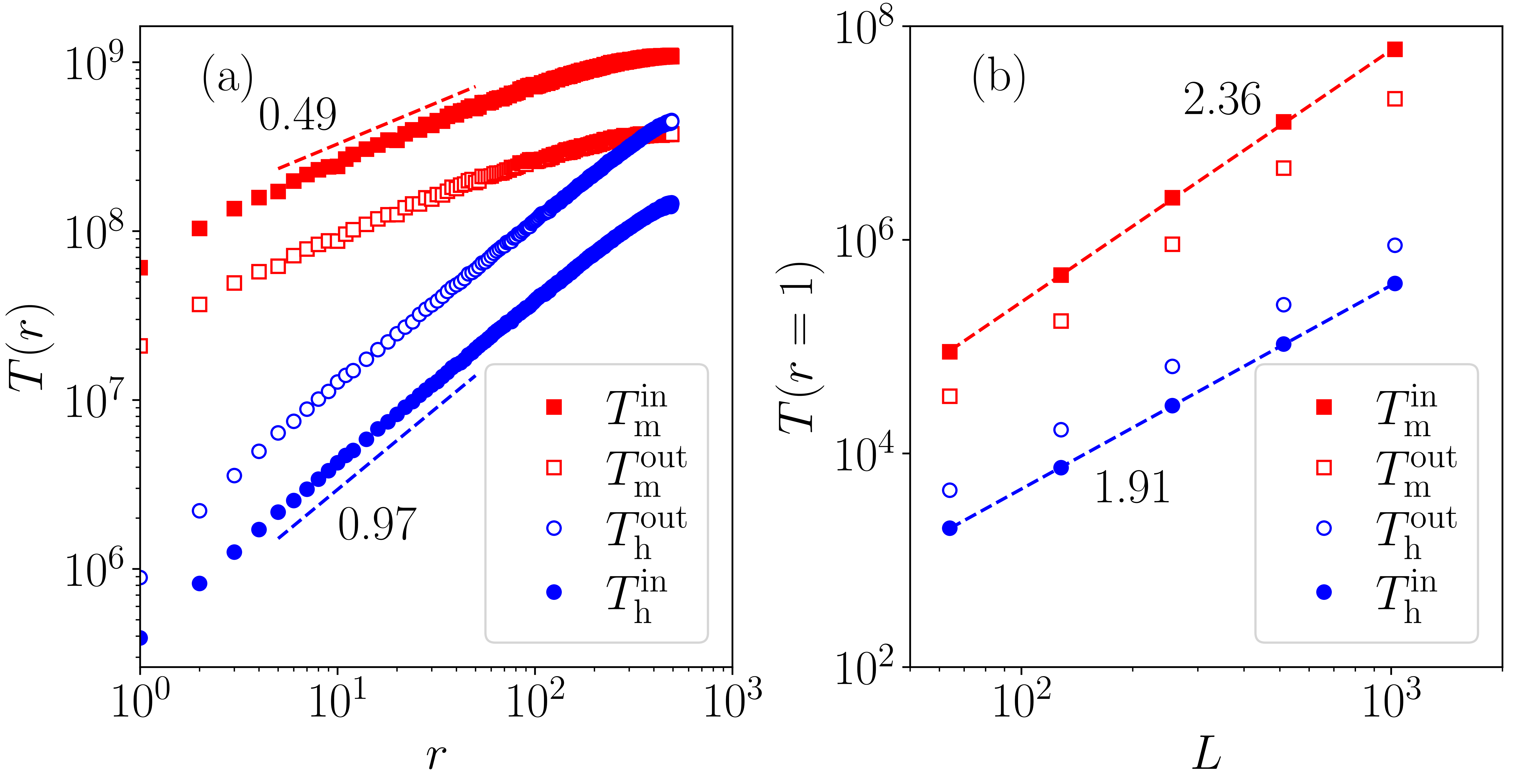
Broad RWC Distribution

hub (H) of highest RWC

marginal site (M) of lowest RWC

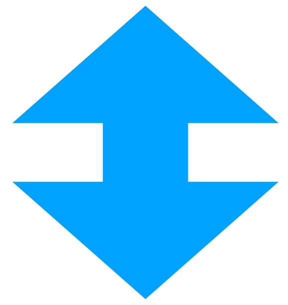


Heterogeneous Scaling



$T(r) \sim L^{\Delta_s} \, r^{\theta_s}$

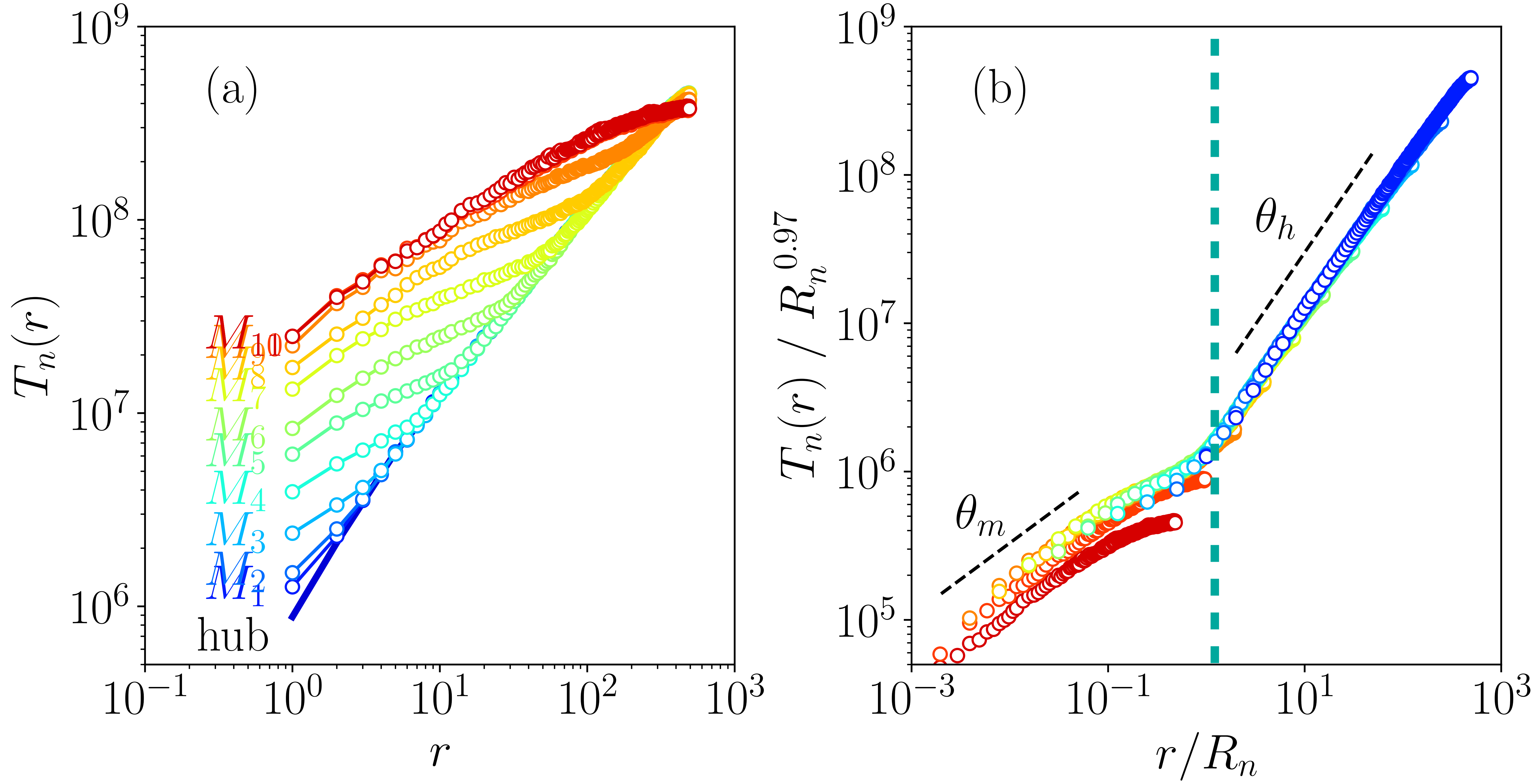
: site-dependent exponents

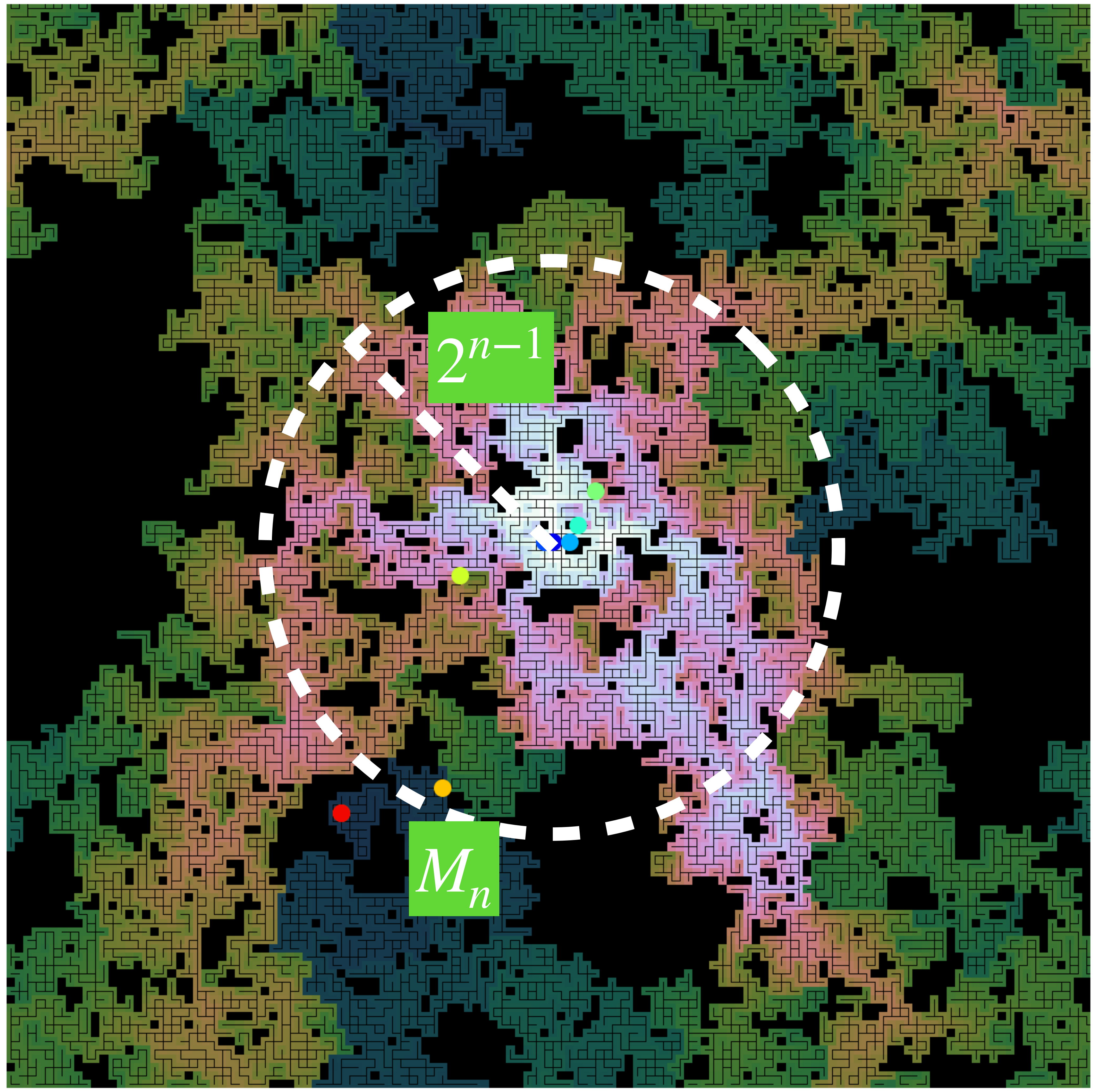


$T(r) \sim L^{d_f} \, r^{d_w - d_f}$

: universal scaling exponents

Crossover Scaling



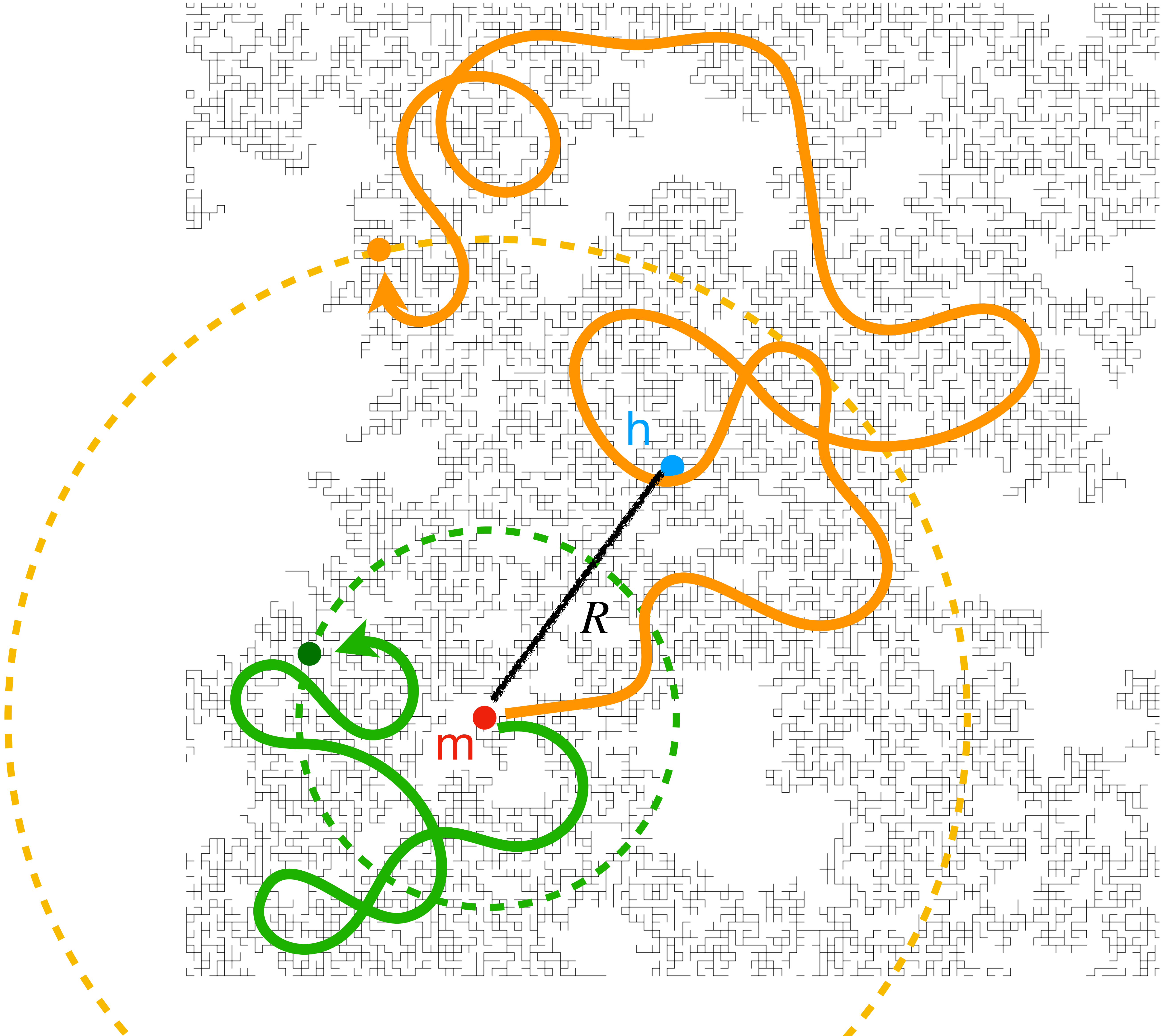


M_n: local minimum RWC site

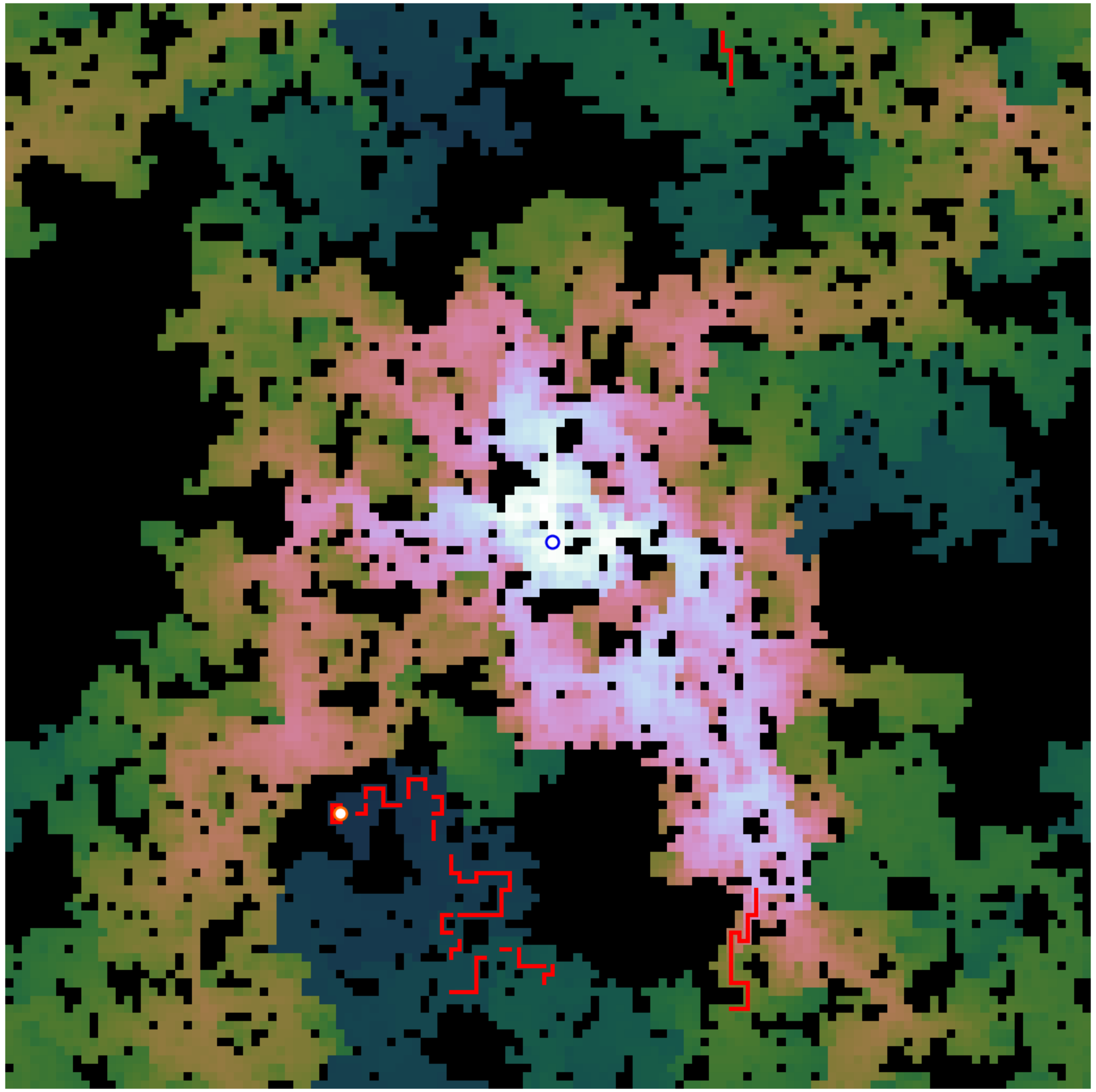
crossover scaling form

$$T_n(r) \;=\; L^{d_f}\, R_n^{\theta_h}\, \mathscr{F}\left(\frac{r}{R_n}\right)$$

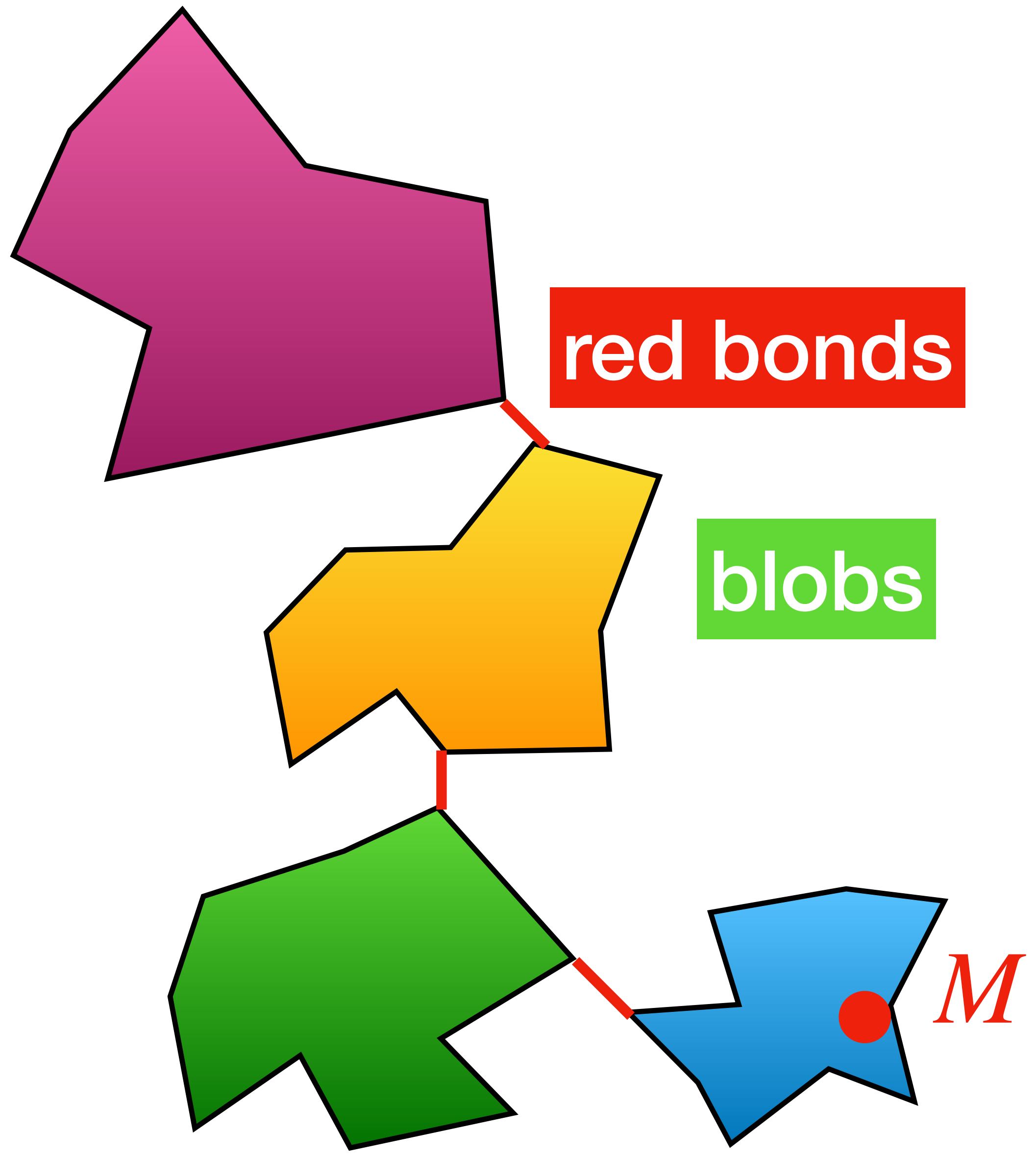
Implication of the Crossover Scaling



Origin(?) of Crossover

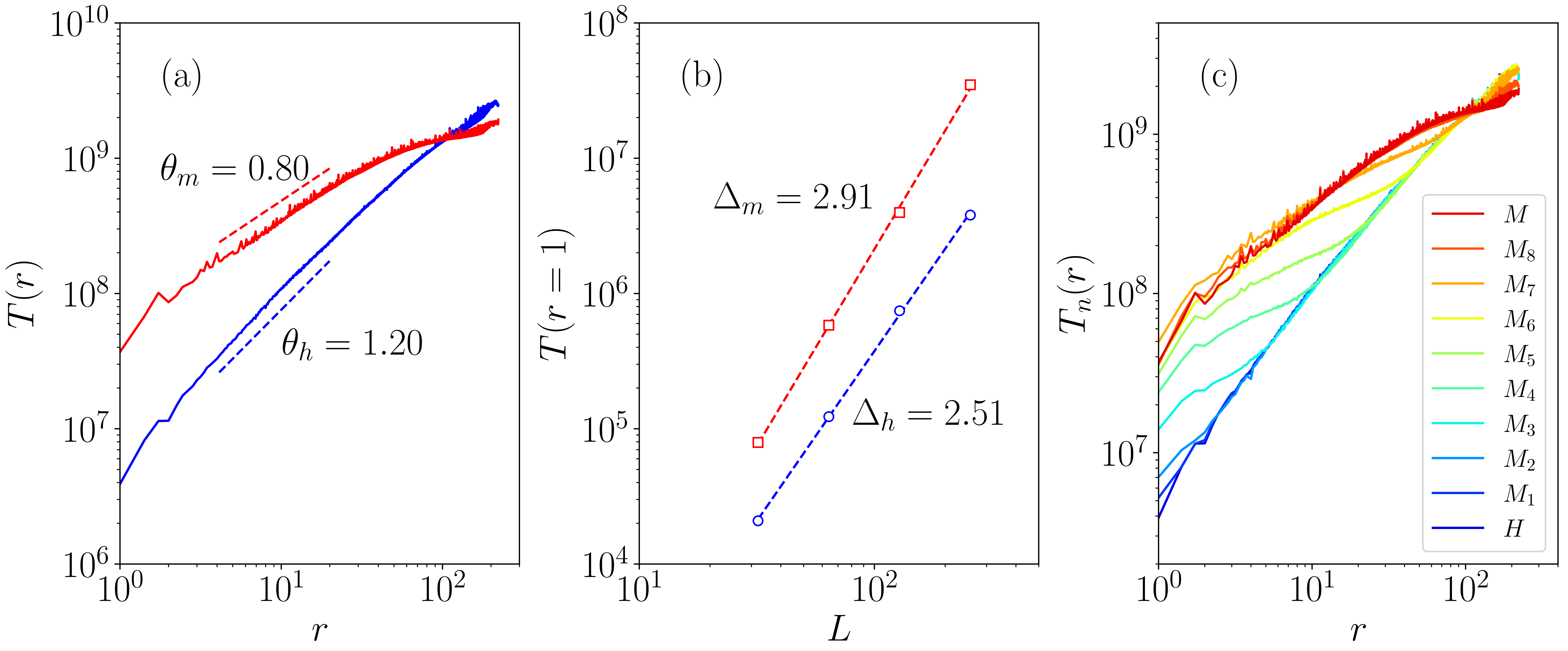


- red bonds
- quasi one-dimensional transport near the marginal sites

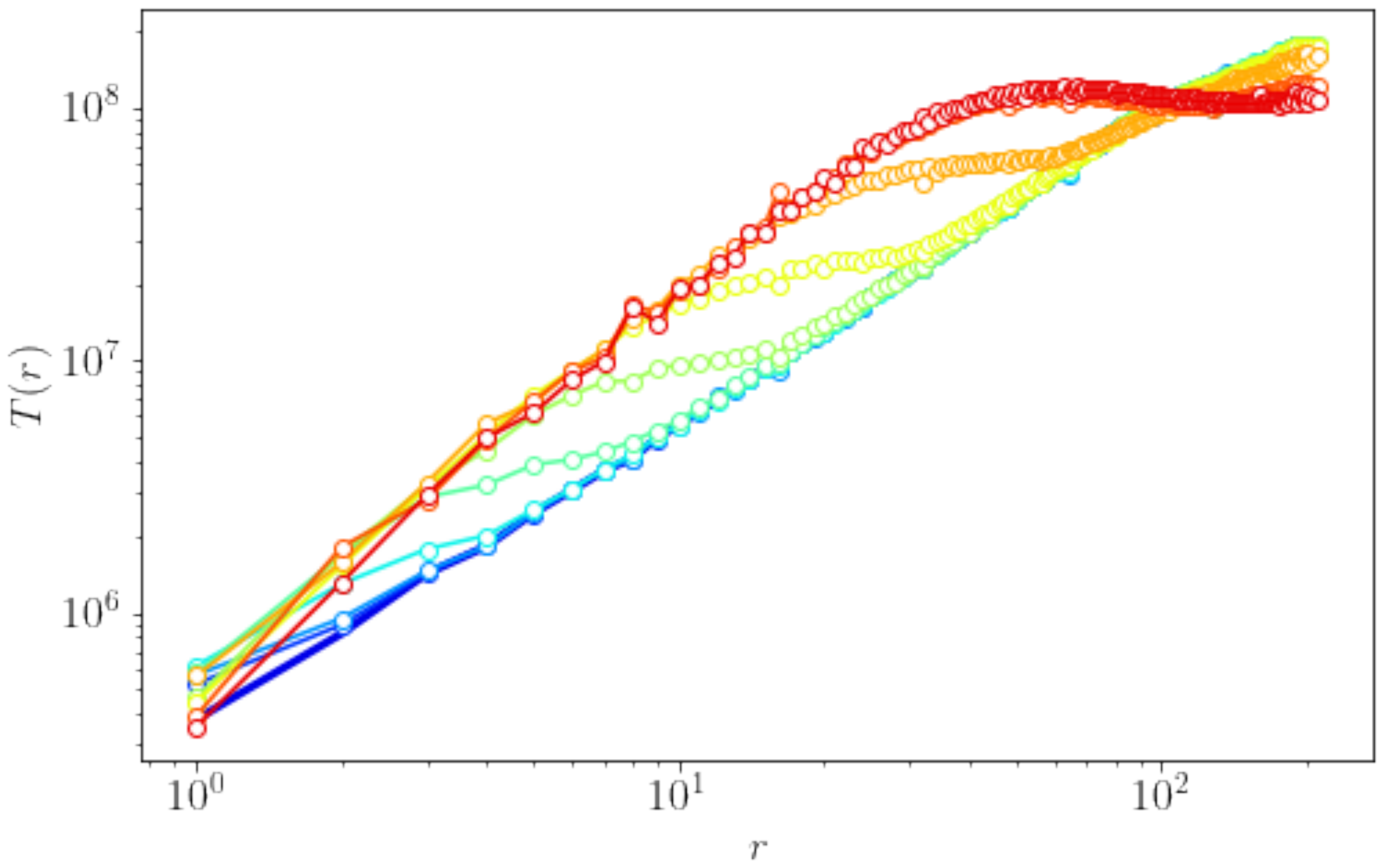


Ubiquity

3D critical percolation cluster



RW trail in 3D



Summary

- Random walks are highly heterogeneous even on random fractals in the Euclidean space.
- Heterogeneous scaling and crossover of the MFPT
- The RWC is a good indicator of the heterogeneity.
- Complex systems
- For more detail, arXiv:2304.14940