# Thermodynamic Geometry for Quantum Thermal Machines

Perspectives on Non-Equilibrium Statistical Mechanics:
The 45th Anniversary Symposium of Yamada Science Foundation

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Kay Brandner

School of Physics and Astronomy, University of Nottingham

KB, K. Saito; Phys. Rev. Lett. **124**, 040602 (2020)

J. Eglinton, KB; Phys. Rev. E **105**, L052102 (2022)

J. Eglinton, T. Pyhäranta, K. Saito, KB; New J. Phys. **25** 043014 (2023)

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# From Horsepowers to Zeptowatts

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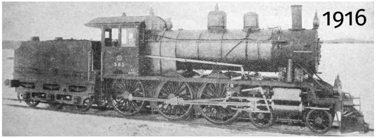
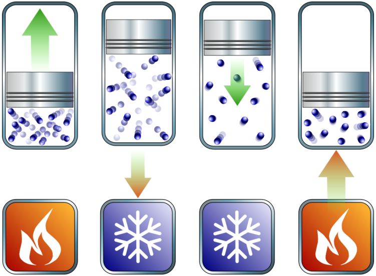
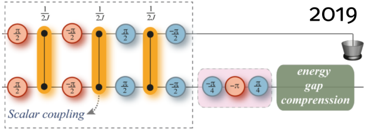
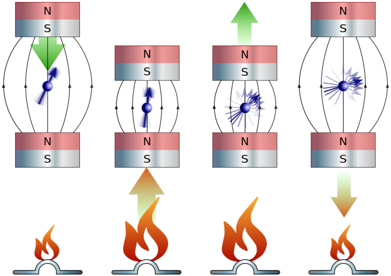
## Macroscopic Engine



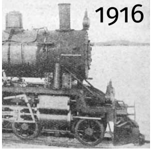



## Microscopic Engine



### Macro Engine

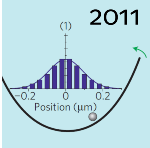



$d \sim 10^{0}\mathrm{m}$

$P \sim 10^{3}\mathrm{W}$

$\eta \simeq 7\% | 0.3\eta_C$

### Micro Engine



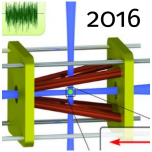

$d \sim 10^{-6}\mathrm{m}$

$P \sim 10^{-22}\mathrm{W}$

$\eta \simeq 14\% | 0.9\eta_C$

➻ V. Blickle, C. Bechinger; Nat. Phys. **8**, 143 (2011).

### Pico Engine



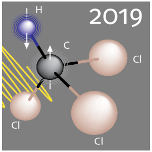

$d \sim 10^{-10}\mathrm{m}$

$P \sim 10^{-22}\mathrm{W}$

$\eta \simeq 0.3\% | 0.07\eta_C$

➻ J. Roßnagel et al.; Science **352**, 325 (2016).

### Quantum Engine

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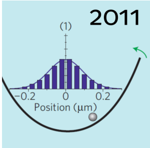
$P \sim 10^{-28}\mathrm{W}$

$\eta \simeq 42\% | 0.5\eta_C$

➻ J. Peterson et al.; PRL **123**, 240601 (2019).

# From Horsepowers to Zeptowatts

## Micro Engine

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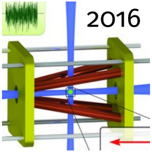
## Pico Engine



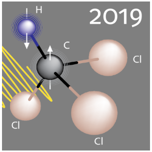

$d \sim 10^{-10}$m

$P \sim 10^{-22}$W

$\eta \simeq 0.3\% | 0.07\eta_C$

➸ J. Roßnagel et al.; Science **352**, 325 (2016).

## Quantum Engine

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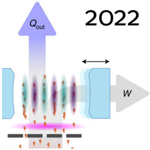
$P \sim 10^{-28}$W

$\eta \simeq 42\% | 0.5\eta_C$

➸ J. Peterson et al.; PRL **123**, 240601 (2019).

## Collective QE

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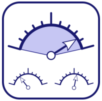
$P \sim 10^{-28}$W

$\eta \simeq 95\% | 0.97\eta_C$

➸ J. Kim et al.; Nat. Phot. **16**, 707 (2022).

## Why going small?

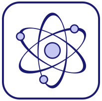


### Fundamental Limits

Are there universal constraints on thermodynamic figures of merit?

### Micro Engineering

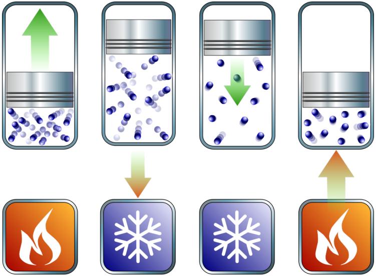
Can we systematically develop optimal design and control strategies?

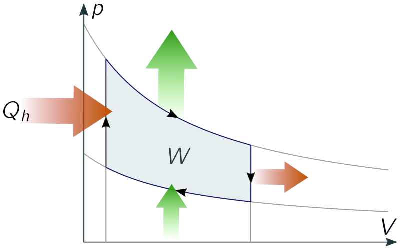


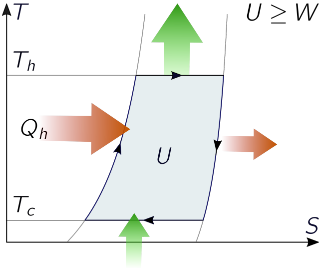
### Quantum World

How can we describe the role of quantum effects and exploit them to improve performance?

# Classical Thermodynamics






Efficiency:

$\eta_{\text{th}} \equiv W/Q_h \leq 1 - T_c/T_h \equiv \eta_{\text{C}}$

Power:

$P = W/\mathcal{T}$

# Thermodynamic Geometry of Microscopic Heat Engines

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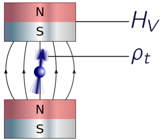




Driving protocol:

$\mathbf{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

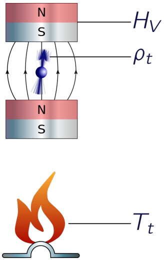
$\gamma : \mathbf{\Lambda} \mapsto \mathbf{\Lambda}_t$

Generalized forces:

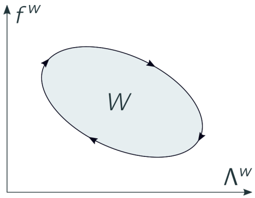
$f_t^u \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

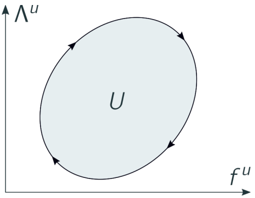
$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

# Thermodynamic Geometry of Microscopic Heat Engines




Generalized $p - V$ and $T - S$ diagrams:




Driving protocol:

$\boldsymbol{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Output and input:

$$W = \oint_\gamma p dV = \int_0^{\mathcal{T}} dt\, f_t^w \dot{\Lambda}_t^w \quad U = \oint_\gamma T dS = -\int_0^{\mathcal{T}} dt\, f_t^u \dot{\Lambda}_t^u$$

Generalized forces:

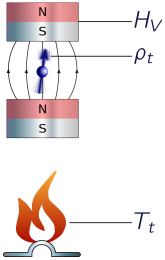
$f_t^u \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

Dissipated availability:

$$A \equiv W - U = \int_0^{\mathcal{T}} dt\, f_t^\mu \dot{\Lambda}_t^\mu = \int_0^{\mathcal{T}} dt\, T_t \Sigma_t \geq 0$$

$$\Sigma_t \equiv \dot{S}_t - \dot{Q}_t / T_t$$

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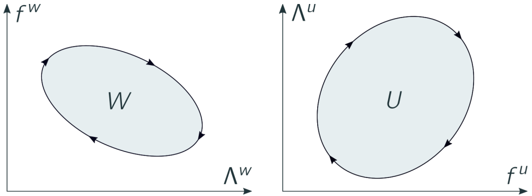


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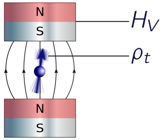
$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

Generalized efficiency:

$$\eta \equiv W/U \leq 1 \qquad (\eta \leq \eta_{\mathrm{th}}/\eta_C)$$

# Thermodynamic Geometry of Microscopic Heat Engines





Output and input:

$$W = \oint_\gamma p dV = \int_0^\mathcal{T} dt\, f_t^w \dot{\Lambda}_t^w \quad U = \oint_\gamma T dS = -\int_0^\mathcal{T} dt\, f_t^u \dot{\Lambda}_t^u$$

Generalized efficiency:

$$\boxed{\eta \equiv W/U \leq 1 \qquad (\eta \leq \eta_{\mathrm{th}}/\eta_{\mathrm{C}})}$$

Driving protocol:

$\boldsymbol{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Generalized forces:

$f_t^u \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

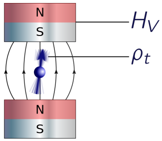
$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

## Quasi-static Limit

$\rho_t = \rho_{\boldsymbol{\Lambda}_t}$ $\qquad$ $\rho_{\boldsymbol{\Lambda}} \equiv \exp[-(H_V - \mathcal{F}_{\boldsymbol{\Lambda}})/T]$

$f_t^\mu = \mathcal{F}_{\boldsymbol{\Lambda}_t}^\mu$ $\qquad$ $\mathcal{F}_\Lambda^\mu \equiv -\partial_\mu \mathcal{F}_{\boldsymbol{\Lambda}}$

$\eta = \eta_{\mathrm{th}}/\eta_{\mathrm{C}} = 1$ $\qquad$ $P = W/\mathcal{T} = A = 0$





Driving protocol:

$\boldsymbol{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Generalized forces:

$f_t^u \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

**Adiabatic Response**

$$f_t^\mu = \mathcal{F}^\mu_{\boldsymbol{\Lambda}_t} + R^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\nu_t \qquad P = \mathcal{W}/\mathcal{T} \qquad \eta = 1 - A/\mathcal{W}$$

Geometric work and dissipated availability:

$$\mathcal{W} \equiv \oint_\gamma \mathcal{A}^\mu_{\boldsymbol{\Lambda}} d\Lambda^\mu \qquad \mathcal{A}^\mu_{\boldsymbol{\Lambda}} \equiv \partial_\mu \mathcal{F}^w_{\boldsymbol{\Lambda}} \Lambda^w$$

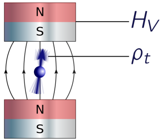
$$A \equiv \int_0^{\mathcal{T}} dt\, g^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\mu_t \dot{\Lambda}^\nu_t \geq 0 \qquad g^{\mu\nu}_{\boldsymbol{\Lambda}} \equiv -(R^{\mu\nu}_{\boldsymbol{\Lambda}} + R^{\nu\mu}_{\boldsymbol{\Lambda}})/2 \succeq 0$$

Thermodynamic length:

$$A \geq \mathcal{L}^2/\mathcal{T} \qquad \mathcal{L} \equiv \oint_\gamma \sqrt{g^{\mu\nu}_{\boldsymbol{\Lambda}} d\Lambda^\mu d\Lambda^\nu}$$

Trade-off relation between power and efficiency:

$$\boxed{(1-\eta)(\mathcal{W}/\mathcal{L})^2 \geq P}$$





Driving protocol:

$\boldsymbol{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Generalized forces:

$f^u_t \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

$f^w_t \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

## Adiabatic Response

$$f^\mu_t = \mathcal{F}^\mu_{\boldsymbol{\Lambda}_t} + R^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\nu_t \qquad P = \mathcal{W}/\mathcal{T} \qquad \eta = 1 - A/\mathcal{W}$$

Geometric work and thermodynamic length:

$$\mathcal{W} \equiv \oint_\gamma \mathcal{A}^\mu_{\boldsymbol{\Lambda}} d\Lambda^\mu \qquad \mathcal{A}^\mu_\Lambda \equiv \partial_\mu \mathcal{F}^w_{\boldsymbol{\Lambda}} \Lambda^w$$

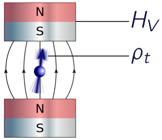
$$A \equiv \int_0^{\mathcal{T}} dt\, g^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\mu_t \dot{\Lambda}^\nu_t \geq \mathcal{L}^2/\mathcal{T} \qquad \mathcal{L} \equiv \oint_\gamma \sqrt{g^{\mu\nu}_{\boldsymbol{\Lambda}} d\Lambda^\mu d\Lambda^\nu}$$

Optimization principle:

$$t \to \phi_t \qquad t \equiv \mathcal{T} \int_0^{\phi_t} ds \sqrt{g^{\mu\nu}_{\boldsymbol{\Lambda}_s} \dot{\Lambda}^\mu_s \dot{\Lambda}^\nu_s} \Big/ \mathcal{L}$$

Optimal efficiency:

$$\boxed{(1 - \eta^*)(\mathcal{W}/\mathcal{L})^2 = P \qquad \eta^* = 1 - \mathcal{L}^2/\mathcal{W}\mathcal{T}}$$





Driving protocol:

$\boldsymbol{\Lambda} \equiv (T, V) \equiv (\Lambda^u, \Lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Generalized forces:

$f_t^u \equiv S_t \equiv -\mathrm{Tr}[\rho_t \ln \rho_t]$

$f_t^w \equiv p_t \equiv -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$

## Adiabatic Response

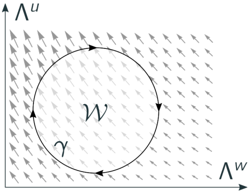
$$f_t^\mu = \mathcal{F}^\mu_{\boldsymbol{\Lambda}_t} + R^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\nu_t \qquad P = \mathcal{W}/\mathcal{T} \qquad \eta = 1 - A/\mathcal{W}$$

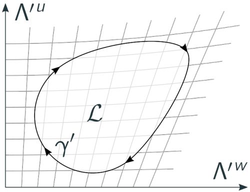
Geometric work and thermodynamic length:

$$\mathcal{W} \equiv \oint_\gamma \mathcal{A}^\mu_{\boldsymbol{\Lambda}} d\Lambda^\mu \qquad \mathcal{A}^\mu_\Lambda \equiv \partial_\mu \mathcal{F}^w_{\boldsymbol{\Lambda}} \Lambda^w$$

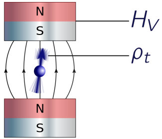
$$A \equiv \int_0^{\mathcal{T}} dt\, g^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\mu_t \dot{\Lambda}^\nu_t \geq \mathcal{L}^2/\mathcal{T} \qquad \mathcal{L} \equiv \oint_\gamma \sqrt{g^{\mu\nu}_{\boldsymbol{\Lambda}} d\Lambda^\mu d\Lambda^\nu}$$

Graphical interpretation:



# Example: Two-Level Engine





**Driving protocols:**

$$T_t = \hbar\Omega\left(1 + \sin^2[\pi\Omega t]\right)$$

$$V_t = 1 + \sin^2[\pi\Omega t + \pi/4]$$

**Hamiltonian:**

$$H_V = -\frac{\hbar\Omega}{2}\left(\varepsilon\sigma_x + \Delta_V\sigma_z\right)$$

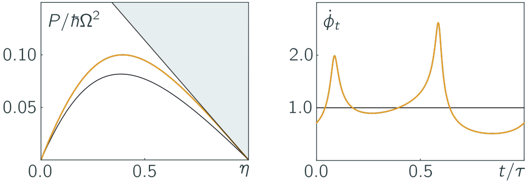
$$\Delta_V \equiv \sqrt{V^2 - \varepsilon^2}$$

**Dynamics:**

$$\dot{\rho}_t = -\frac{i}{\hbar}[H_{V_t}, \rho_t] + \sum_{\sigma=\pm}\left([L^{\sigma}_{\Lambda_t}\rho_t, L^{\sigma\dagger}_{\Lambda_t}] + [L^{\sigma}_{\Lambda_t}, \rho_t L^{\sigma\dagger}_{\Lambda_t}]\right)$$

$$[H_V, L^{\pm}_{\Lambda}] = \pm\hbar\Omega V L^{\pm}_{\Lambda} \qquad \mathrm{Tr}[L^{\pm}_{\Lambda} L^{\pm\dagger}_{\Lambda}] = \frac{\pm\Gamma\Omega V}{1 - \exp[\mp\hbar\Omega V/T]}$$

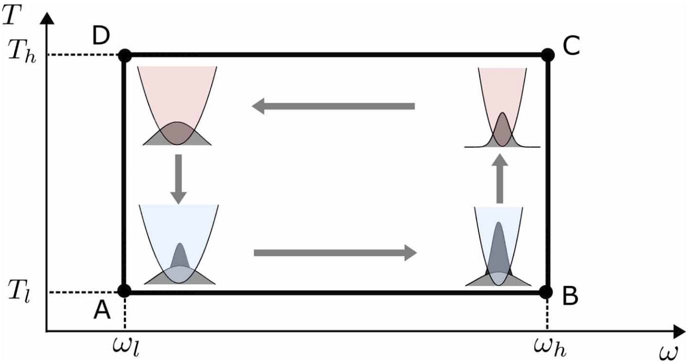
**Results:**



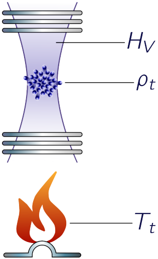

$$(1-\eta)(\mathcal{W}/\mathcal{L})^2 \geq P \qquad t = \mathcal{T}\int_0^{\phi_t} ds\, \sqrt{g^{\mu\nu}_{\Lambda_s}\dot{\Lambda}^{\mu}_s\dot{\Lambda}^{\nu}_s}\Big/\mathcal{L}$$

# Thermodynamic Geometry of Many-Body Quantum Engines

# BEC Engine




- N. M. Myers, F. J. Peña, O. Negrete, P. Vargas, G. D. Chiara, S. Deffner; New J. Phys. **24** 025001 (2022).




Driving protocol:

$$\boldsymbol{\Lambda} \equiv (T, V, \mu) \equiv (\Lambda^u, \Lambda^w, \Lambda^z)$$

$$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$$

Generalized forces:

$$\hat{f}_t^u \equiv S_t = -\mathrm{Tr}[\rho_t \ln \rho_t]$$

$$\hat{f}_t^w \equiv p_t = -\mathrm{Tr}[\rho_t \partial_{V_t} H_{V_t}]$$

$$\hat{f}_t^z \equiv \mathcal{N}_t = \mathrm{Tr}[\rho_t N_t]$$

## Grand Canonical Adiabatic Response

$$\hat{f}_t^\mu = \hat{\mathcal{F}}^\mu_{\boldsymbol{\Lambda}_t} + \hat{R}^{\mu\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\nu_t \qquad \hat{\mathcal{F}}^\mu_{\boldsymbol{\Lambda}} = -\partial_\mu \Phi_{\boldsymbol{\Lambda}}$$

Fix $\Lambda_t^z = \mu_t$ by fixing mean particle number:

$$\mathcal{N}_t = \hat{f}_t^z = \hat{\mathcal{F}}^z_{\boldsymbol{\Lambda}_t} + \hat{R}^{z\nu}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\nu_t \stackrel{!}{=} N$$

$$\Lambda_t^z = \Lambda^{z0}_{\boldsymbol{\lambda}_t} + \Lambda^{za}_{\boldsymbol{\lambda}_t} \dot{\lambda}^a_t \qquad \boldsymbol{\lambda} \equiv (T, V) \equiv (\lambda^u, \lambda^w)$$

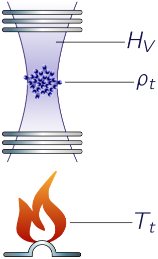
## Effective Canonical Adiabatic Response

$$f_t^a = \mathcal{F}^a_{\boldsymbol{\lambda}_t} + R^{ab}_{\boldsymbol{\lambda}_t} \dot{\lambda}^b_t \qquad a, b = u, w$$

$$g^{ab}_{\boldsymbol{\lambda}} \equiv -(R^{ab}_{\boldsymbol{\lambda}} + R^{ba}_{\boldsymbol{\lambda}})/2 \succeq 0$$

Optimal efficiency:

$$\boxed{(1-\eta^*)(\mathcal{W}/\mathcal{L})^2 \equiv (1-\eta^*)\Psi = P \quad \eta^* = 1 - \mathcal{L}^2/\mathcal{W}\mathcal{T}}$$




Hamiltonian

$$H_\omega = \sum_{\mathbf{n}} E_{\mathbf{n}} A_{\mathbf{n}}^\dagger A_{\mathbf{n}} \quad \mathbf{n} = (n_x, n_y) \quad E_{\mathbf{n}} = \hbar\omega(n_x + n_y + 1)$$

Dynamics:

$$\dot{\rho}_t = -\frac{i}{\hbar}[H, \rho_t] + \kappa \sum_{\mathbf{n}} (\mathfrak{n}_{\mathbf{n}} + 1) \left( A_{\mathbf{n}} \rho_t A_{\mathbf{n}}^\dagger - \frac{1}{2}\{\rho_t, A_{\mathbf{n}}^\dagger A_{\mathbf{n}}\} \right) + \mathfrak{n}_{\mathbf{n}} \left( A_{\mathbf{n}}^\dagger \rho_t A_{\mathbf{n}} - \frac{1}{2}\{\rho_t, A_{\mathbf{n}} A_{\mathbf{n}}^\dagger\} \right)$$

Driving protocol:

$\boldsymbol{\lambda} \equiv (T, \omega) \equiv (\lambda^u, \lambda^w)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

Generalized forces:

$f_t^u \equiv S_t = -\mathrm{Tr}[\rho_t \ln \rho_t]$

$f_t^w \equiv p_t = -\mathrm{Tr}[\rho_t \partial_{\omega_t} H_{\omega_t}]$

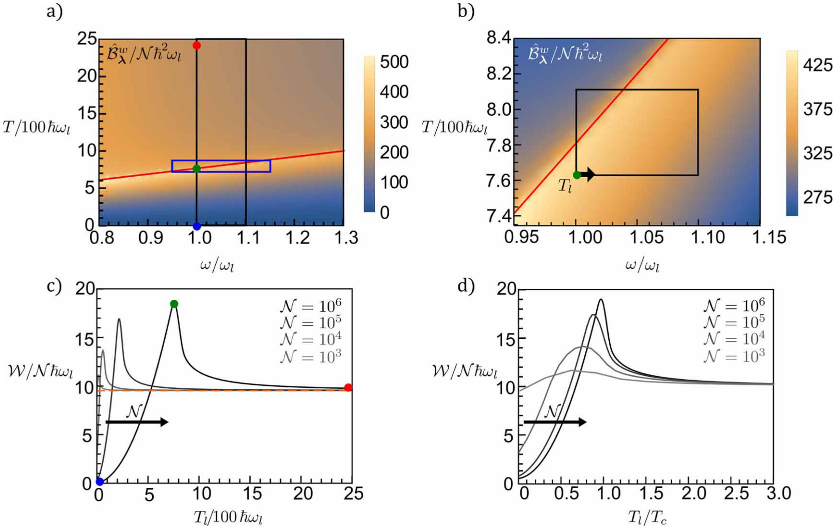
Thermodynamic quantities:

$$\mathcal{W} = \oint_\gamma \mathcal{A}_{\boldsymbol{\lambda}}^a d\lambda^a \equiv \iint_\Gamma \mathcal{B}_{\boldsymbol{\lambda}}^w d\lambda^u d\lambda^w$$

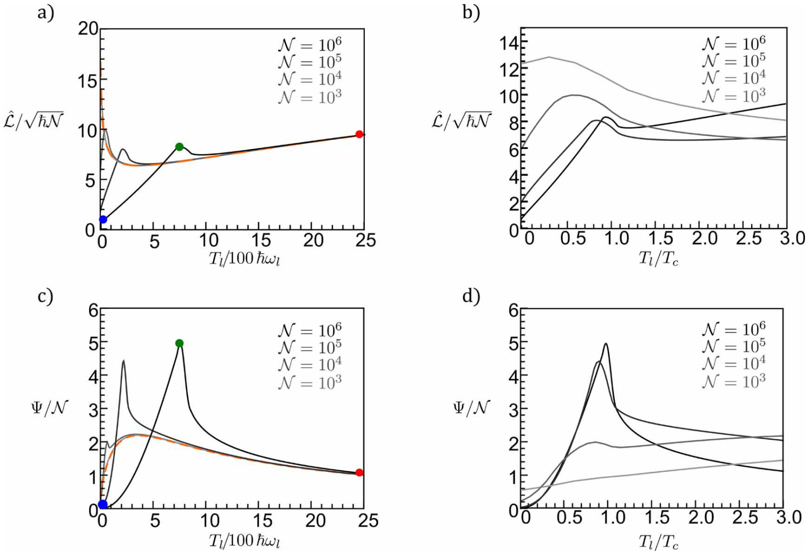
$$\mathcal{L} = \oint_\gamma \sqrt{g_{\boldsymbol{\lambda}}^{ab} d\lambda^a d\lambda^b}$$

$$\Psi \equiv (\mathcal{W}/\mathcal{L})^2$$

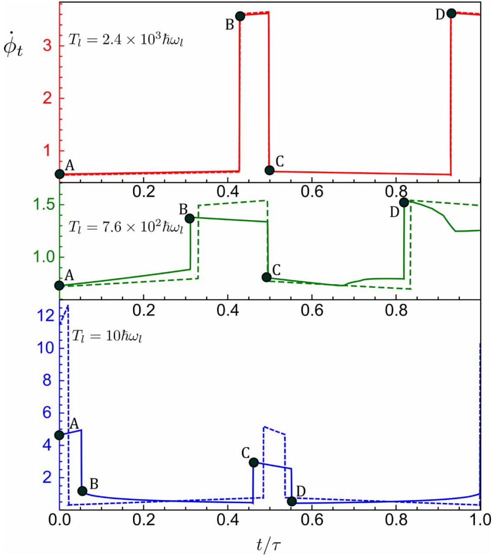
# BEC Engine Revisited: Geometric Work



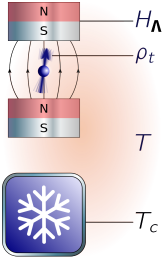
# BEC Engine Revisited: Thermodynamic Length



# BEC Engine Revisited: Optimal Speed Function



# Thermodynamic Geometry of Microscopic Refrigerators




Output and input:

$$Q = \int_0^{\mathcal{T}} dt\, j_t \qquad W = -\oint_\gamma p^\alpha d\Lambda^\alpha = -\int_0^{\mathcal{T}} dt\, f_t^\alpha \dot{\Lambda}_t^\alpha$$

Efficiency:

$$\boxed{\varepsilon \equiv Q/W \leq \varepsilon_C \equiv T_c/(T - T_c)}$$

Driving:

$\mathbf{\Lambda} \equiv (\Lambda^1, \Lambda^2)$

$\gamma : \mathbf{\Lambda} \mapsto \mathbf{\Lambda}_t$

$F_q \equiv 1/T - 1/T_c$

Generalized forces:

$j_t \equiv \mathrm{Tr}[\dot{\rho}_t H_{\mathbf{\Lambda}_t}]$

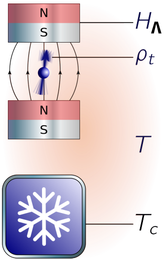
$f_t^\alpha \equiv p_t^\alpha \equiv -\mathrm{Tr}[\rho_t \partial_{\Lambda_t^\alpha} H_{\mathbf{\Lambda}_t}]$

## Adiabatic Response

$$j_t = R_{\mathbf{\Lambda}_t}^{qq} F_q + R_{\mathbf{\Lambda}_t}^{q\alpha} \dot{\Lambda}_t^\alpha / T + R_{\mathbf{\Lambda}_t}^{qq\alpha} \dot{\Lambda}_t^\alpha F/T$$

$$f_t^\alpha = \mathcal{F}_{\mathbf{\Lambda}_t}^\alpha - R_{\mathbf{\Lambda}_t}^{\alpha q} F_q - R_{\mathbf{\Lambda}_t}^{\alpha\beta} \dot{\Lambda}_t^\beta / T - R_{\mathbf{\Lambda}_t}^{\mu qq} F_q^2$$

# Thermodynamic Geometry of Microscopic Refrigerators




## Adiabatic Response

$$j_t = R^{qq}_{\boldsymbol{\Lambda}_t} F_q + R^{q\alpha}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\alpha_t / T + R^{qq\alpha}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\alpha_t F/T$$

$$f^\alpha_t = \mathcal{F}^\alpha_{\boldsymbol{\Lambda}_t} - R^{\alpha q}_{\boldsymbol{\Lambda}_t} F_q - R^{\alpha\beta}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\beta_t / T - R^{\mu qq}_{\boldsymbol{\Lambda}_t} F^2_q$$

Power-efficiency trade-off relation ($R^{qq}_{\boldsymbol{\Lambda}} = 0$):

$$\boxed{\mathcal{Z}(\varepsilon_C - \varepsilon)^2/\varepsilon_C^2 \geq J_q \equiv Q/\mathcal{T}}$$

Driving:

$\boldsymbol{\Lambda} \equiv (\Lambda^1, \Lambda^2)$

$\gamma : \boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}_t$

$F_q \equiv 1/T - 1/T_c$

Generalized forces:

$j_t \equiv \mathrm{Tr}[\dot{\rho}_t H_{\boldsymbol{\Lambda}_t}]$

$f^\alpha_t \equiv p^\alpha_t \equiv -\mathrm{Tr}[\rho_t \partial_{\Lambda^\alpha_t} H_{\boldsymbol{\Lambda}_t}]$

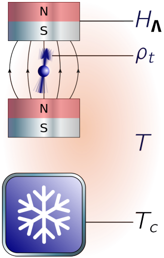
Optimization principle:

$$t = \mathcal{T} \int_0^{\phi_t} ds \sqrt{g^{\alpha\beta}_{\boldsymbol{\Lambda}_s} \dot{\Lambda}^\alpha_s \dot{\Lambda}^\beta_s} \Big/ \mathcal{L} \qquad A_q = -\sqrt{z/\mathcal{T}}$$

$$\mathcal{Z} \equiv L^3_{qw}/4(L^q_{wq} + L^q_{qw})\mathcal{L}^2 \qquad z \equiv \mathcal{L}^2/(L^q_{wq} + L^q_{qw})T$$

$$\begin{bmatrix} L_{wq} & L_{qw} \\ L^q_{wq} & L^q_{qw} \end{bmatrix} = -\oint_\gamma \Lambda^\beta \partial_\alpha \begin{bmatrix} R^{\beta q}_{\boldsymbol{\Lambda}} & R^{q\beta}_{\boldsymbol{\Lambda}} \\ R^{\beta qq}_{\boldsymbol{\Lambda}} & R^{qq\beta}_{\boldsymbol{\Lambda}} \end{bmatrix} d\Lambda^\alpha$$

# Thermodynamic Geometry of Microscopic Refrigerators




## Adiabatic Response

$$j_t = R^{qq}_{\boldsymbol{\Lambda}_t} F_q + R^{q\alpha}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\alpha_t / T + R^{qq\alpha}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\alpha_t F/T$$

$$f^\alpha_t = \mathcal{F}^\alpha_{\boldsymbol{\Lambda}_t} - R^{\alpha q}_{\boldsymbol{\Lambda}_t} F_q - R^{\alpha\beta}_{\boldsymbol{\Lambda}_t} \dot{\Lambda}^\beta_t / T - R^{\mu qq}_{\boldsymbol{\Lambda}_t} F_q^2$$

Power-efficiency trade-off relation ($R^{qq}_{\boldsymbol{\Lambda}} = 0$):

$$\boxed{\mathcal{Z}(\varepsilon_C - \varepsilon)^2/\varepsilon_C^2 \geq J_q \equiv Q/\mathcal{T}}$$

Driving:

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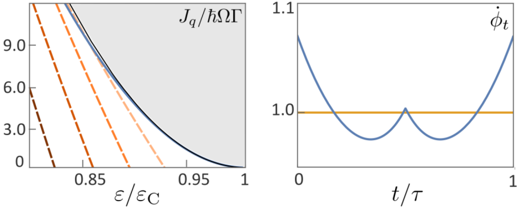
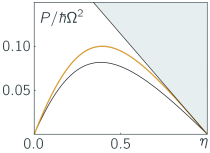


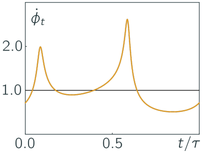
## Why going small?

### Fundamental Limits



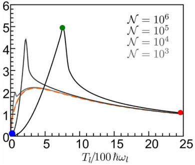

Power-efficiency trade-off

### Micro Engineering




Optimal speed functions

### Quantum World




BEC-enhanced performance