Thermodynamic cost for precision of general counting observables

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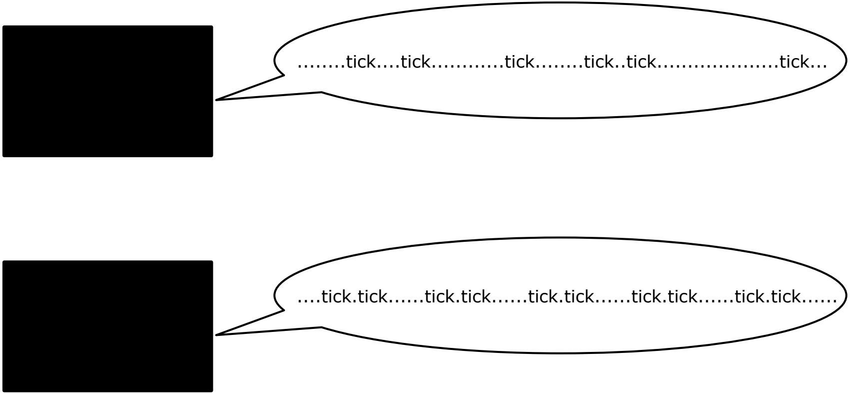
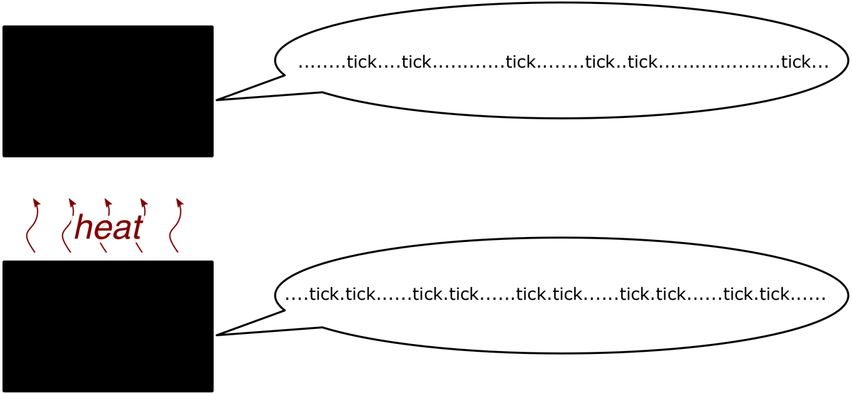
with Francesco Coghi (NORDITA Stockholm)

4 August 2023

Gedankenexperiment



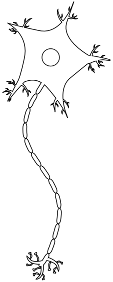
Gedankenexperiment

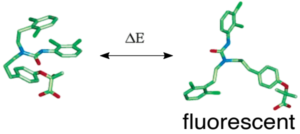


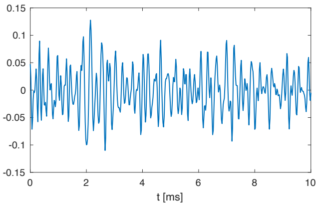
Fundamental question

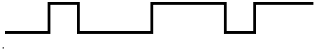
Can we tell from the statistics of ticks whether the system is driven?
If so, how much energy does it cost?

Examples for "ticking" processes

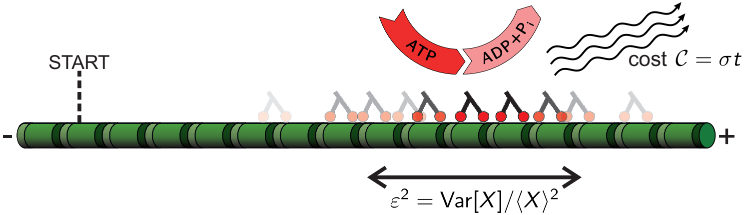








Cost of precision



Thermodynamic uncertainty relation (TUR)

$$\sigma t \frac{\mathrm{Var}[X]}{\langle X \rangle^2} \geq 2 \qquad (\text{set } k_\mathrm{B} = 1)$$

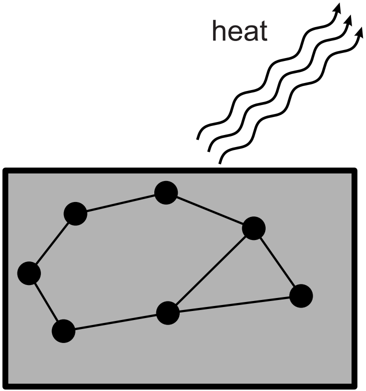
- The more precisely an autonomous system operates, the more energy it must dissipate
- Holds for any **time-antisymmetric** current $X(t)$

[AC Barato, U Seifert, PRL **114**, 158101 (2015), proof by Gingrich *et. al.*, PRL **116**, 120601 (2016)]

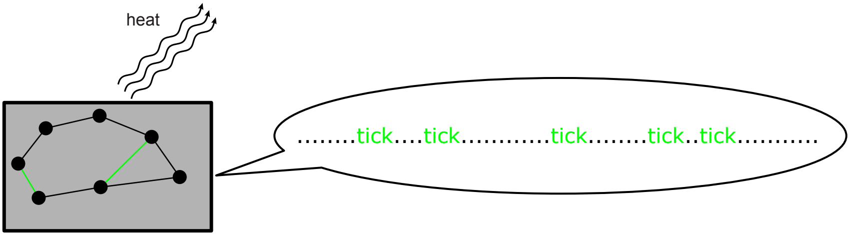
Markovian description

- Transition rate k_{ij} from state i to j
- Stationary state p_i^s
- Entropy production rate

$$\sigma = \sum_{ij} p_i^s k_{ij} \ln \frac{k_{ij}}{k_{ji}}$$

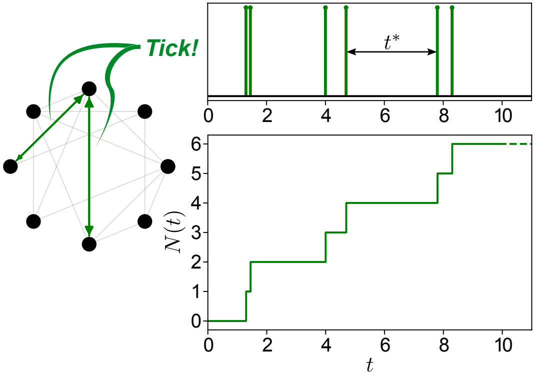


Counting observables



- Flow $n_{ij}(t) := \#(\text{transitons from } i \text{ to } j \text{ during time } t)$
- Transition producing a tick $b_{ij} = 1$, otherwise $b_{ij} = 0$
- Counting observable $N(t) = \sum_{ij} b_{ij} n_{ij}(t)$, tick rate $\langle \dot{N} \rangle = \sum_{ij} b_{ij} p_i^s k_{ij}$
  - Time-symmetric case: $b_{ij} = 1 \Leftrightarrow b_{ji} = 1$, "traffic-like" (traffic $u_{ij} := n_{ij} + n_{ji}$)
  - Generic case: "flow-like"

Precision



Waiting time fluctuations

$$\varepsilon^2 = \frac{\mathrm{Var}[t^*]}{\langle t^* \rangle^2}$$

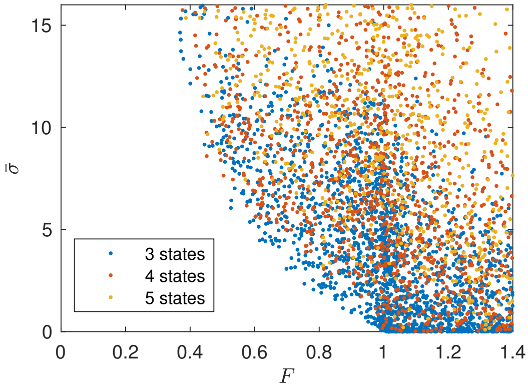
Counting fluctutations ("Fano factor")

$$F = \lim_{t \to \infty} \frac{\mathrm{Var}[N(t)]}{\langle N(t) \rangle}$$

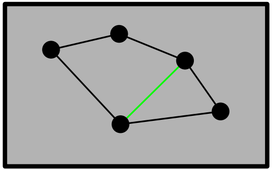
When successive ticks are uncorrelated ("renewal process"): $F = \varepsilon^2$

Trade-off

Example: Fano factor traffic of a single edge
Cost per tick: $\bar{\sigma} := \sigma / \langle \dot{N} \rangle$



random networks



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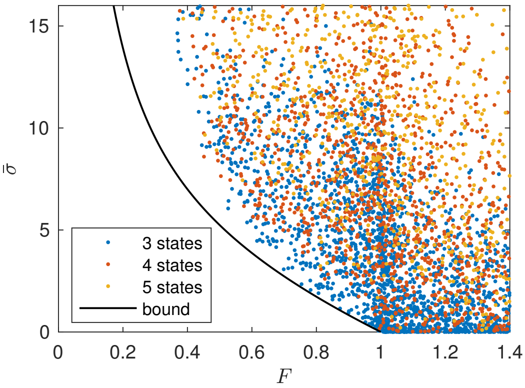

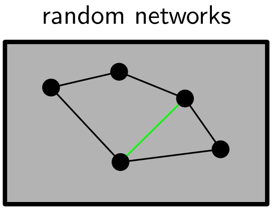

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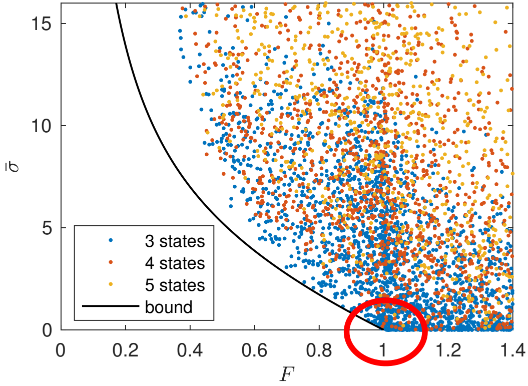
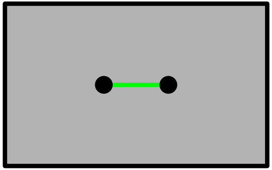
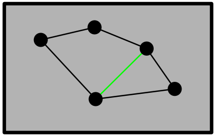


Equal rates $k_{12} = k_{21}$



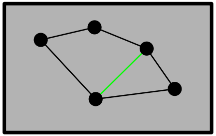
Poisson process: $F = \mathrm{Var}(N)/\langle N \rangle = 1$
Equilibrium: $\sigma = 0$

Mathematical form: Bound on Fano factor of traffic

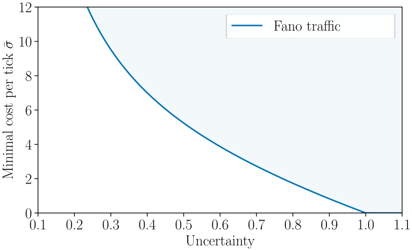


- Given stationary current and traffic of observed edge: $x = j^{\mathrm{s}}_{12}/u^{\mathrm{s}}_{12}$
- Analytic bound $F \geq 1 - x^2 + \dfrac{x^4}{\bar{\sigma}/2 - x\,\mathrm{artanh}(x) + x^2}$

Mathematical form: Bound on Fano factor of traffic



- Unknown stationary current and traffic of observed edge: $x = j^{\mathrm{s}}_{12}/u^{\mathrm{s}}_{12}$
- Analytic bound $F \geq \min_x \left[1 - x^2 + \frac{x^4}{\bar{\sigma}/2 - x\,\mathrm{artanh}(x) + x^2}\right]$



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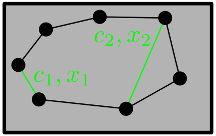
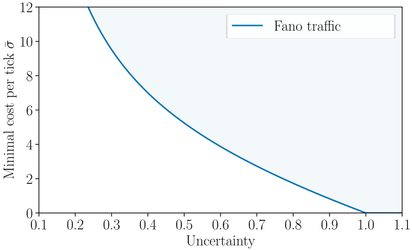


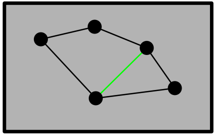
- stationary current and traffic of observed edge: $x = j^s_{12}/u^s_{12}$
- Analytic bound $F \geq 1 - x^2 + \dfrac{x^4}{\bar{\sigma}/2 - x\,\mathrm{artanh}(x) + x^2}$

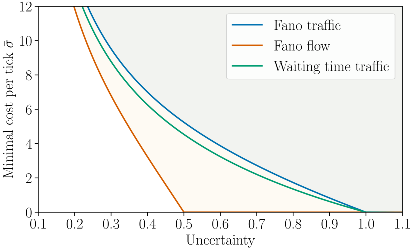


- Several edges: No better precision at same cost

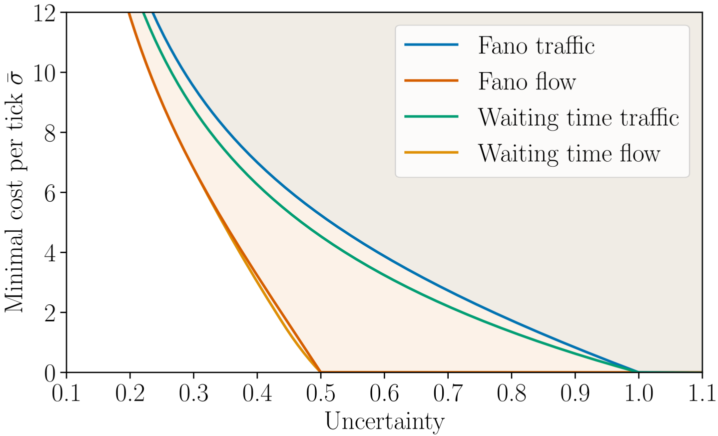
Mathematical form: all bounds (single observed edge)

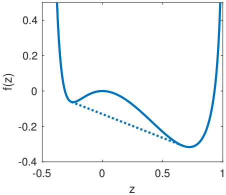
	traffic ($x = j^{s}_{12}/u^{s}_{12}$)	flow ($y = j^{s}_{12}/n^{s}_{12}$)
Fano	$F \geq \min_x \left[1 - x^2 + \frac{x^4}{\bar{\sigma}/2 - x\,\mathrm{artanh}(x) + x^2}\right]$	$F \geq \min_y \left[1 - \frac{1}{2 - y + \frac{2y^2}{\bar{\sigma} + y\ln(1-y)}}\right]$
waiting time	$\varepsilon^2 \geq \min_x \frac{\bar{\sigma}(1-x^2) + 4x^2 - 2(1-x^2)x\,\mathrm{artanh}(x)}{\bar{\sigma}(1+x^2) + 4x^2 - 2(1+x^2)x\,\mathrm{artanh}(x)}$	$\varepsilon^2 \geq \min_y \left[1 - \frac{1}{2 - y + \frac{2y^2}{\bar{\sigma} + y\ln(1-y)}}\right]$





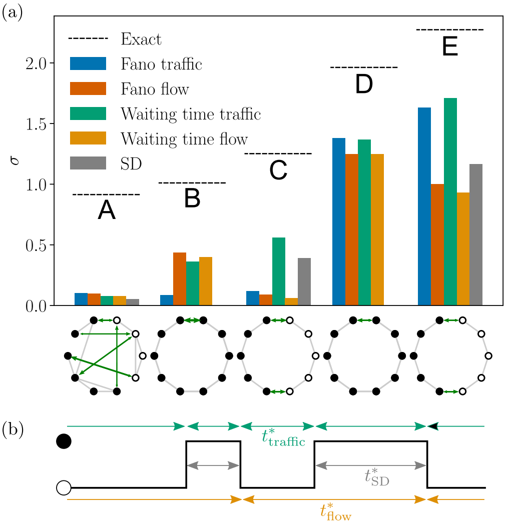
Summary of bounds





Our preprint: arXiv:2305.15392

Inference of entropy production



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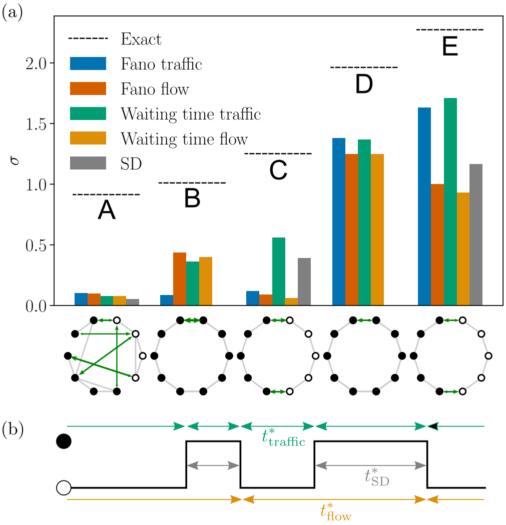
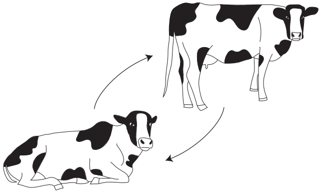
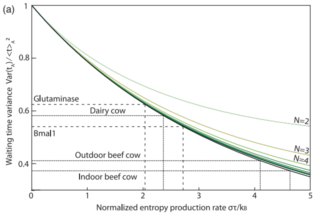




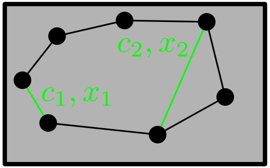


[Skinner, Dunkel PRL **127**, 198101 (2021)]

Several observed edges

Example: Fano, traffic

- Observed edges labeled by a
- Contribution $c_a = u^s_a / \langle \dot{N} \rangle$ to overall tick rate $(\sum_a c_a = 1)$
- Edge dependent $x_a = j^s_a / u^s_a$
- $F \geq -\sum_a c_a x_a^2 + \frac{\left(\sum_a c_a x_a^2\right)^2}{\bar{\sigma}/2 - \sum_a c_a [x_a \, \mathrm{artanh}(x_a) + x_a^2]}$



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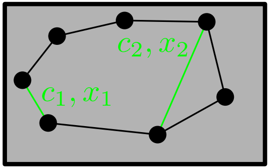
- Substitute $x_a^2 = z_a$



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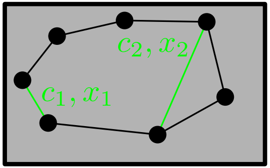

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- Substitute $x_a^2 = z_a$

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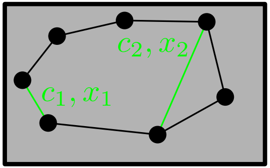
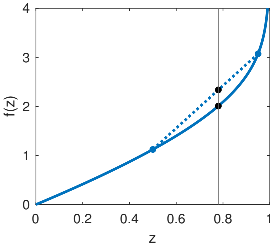
- Substitute $x_a^2 = z_a$
- Jensen's inequality: $\sum_a c_a f(z_a) \geq f(\sum_a c_a z_a)$ with convex $f(z) = \sqrt{z}\, \mathrm{artanh}\, \sqrt{z} + z$
- Result: same bound as for single observed edge

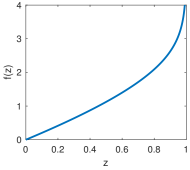




Several observed edges

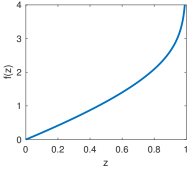
Fano, traffic:

$$f(z) = \sqrt{z}\,\mathrm{artanh}\,\sqrt{z} + z \text{ is convex}$$



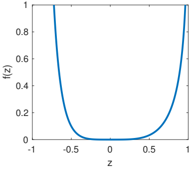
waiting time, traffic:

$$f(z) = \sqrt{z}\,\mathrm{artanh}\,\sqrt{z} + z \text{ is convex}$$



Fano, flow:

$$f(z) = -\frac{2z}{1+z}\left(\frac{1}{2}\ln\frac{1-z}{1+z} + z\right) \text{ is convex}$$



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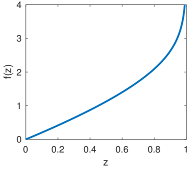
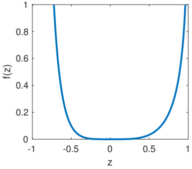
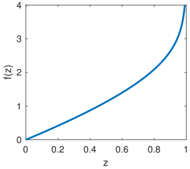
Fano, traffic:
$f(z) = \sqrt{z}\,\text{artanh}\,\sqrt{z} + z$ is convex



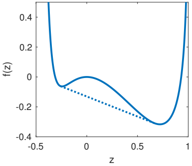
Fano, flow:
$f(z) = -\frac{2z}{1+z}\left(\frac{1}{2}\ln\frac{1-z}{1+z} + z\right)$ is convex



waiting time, traffic:
$f(z) = \sqrt{z}\,\text{artanh}\,\sqrt{z} + z$ is convex



waiting time, flow:
$f(z) = -\frac{3z}{1+2z}\left(\frac{1}{2}\ln\frac{1-z}{1+2z} + 2z\right)$ is *not* convex



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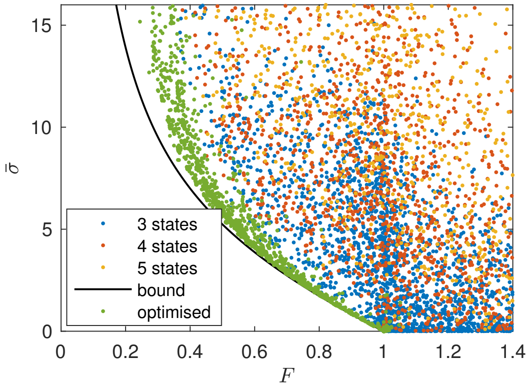
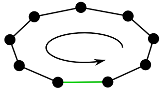
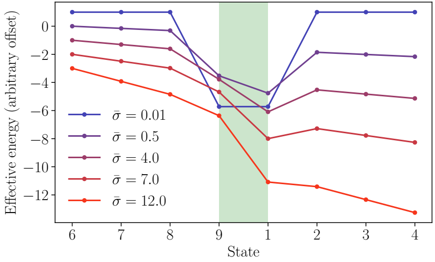


Optimal networks (unicyclic)





Effective energy difference $\Delta E_{ij} = \ln \frac{k_{ij}}{k_{ji}}$