Background

- 1905 "Langevin **func**tion" $\coth x - \frac{1}{x}$ (Paul Langevin)

  — equilibrium response of a classical spin under field, $\beta h \mu_0 = x$.

- 1908 "Langevin **equ**ation" (Paul Langevin)

  — equation for stochastic evolution with *drift* and *noise* . → S.D.E.

- 1940's+ "Martingale" in math. (Lévy, Doob, Kunita-Watanabe, ...)

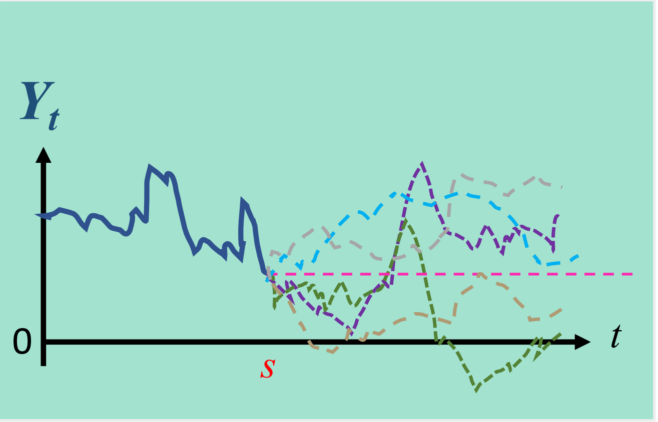
  — [definition below]

- This talk : Langevin **equ**ation + Martingale ⟶ Langevin **func**tion

  — How can a ***spin*** emerge from the Langevin dynamics in $\mathbb{R}^d$?

A **martingale process** is defined *in connection with* a **reference process**

"Process Y_t is martingale associated with the process X_t":



$\Leftrightarrow$ 0) For $\forall t$ $\exists$ $\langle \| \vec{Y}_t \| \rangle < \infty$ (L^1-integrable)

i) $\vec{Y}_t$ is causally determined by $\{\vec{X}_s\}_{0 \le s \le t}$, hereafter $\vec{X}_{[0,t]}$

ii) for $\forall t \ge s \quad \langle \vec{Y}_t | \vec{X}_{[0,s]} \rangle = \vec{Y}_s$

Itô differential
$d\vec{Y}_t \equiv \vec{Y}_{t+dt} - \vec{Y}_t$ $\Rightarrow$ $\langle d\vec{Y}_t | \vec{X}_{[0,s]} \rangle = 0 \quad \text{for } \forall t \ge s$ (*if d* and ⟨ ⟩ are commutative)

"Harmonic function" is a special type of martingale

"$\boldsymbol{h}$ is harmonic function associated with the process $\boldsymbol{x}_t$"

$\Leftrightarrow$ $\vec{Y}_t = \vec{h}(\vec{x}_t)$ is martingale associated with $\vec{x}_{[0,t]}$

$\Rightarrow$ $\langle d\vec{h}(\vec{x}_t) | \vec{x}_{[0,s]} \rangle = 0 \quad \text{for } \forall t \ge s$

A **family of harmonic functions** is defined by a Langevin equation

Case : Reference process $\{x_t\}$ generated by Langevin equation, or, S.D.E.

$$d\vec{x}_t = \vec{a}(\vec{x}_t)dt + d\vec{W}_t \qquad \begin{cases} \vec{a}(\vec{x}) : & d\text{-dimensional drift} \\ \vec{W}_t : & d\text{-dimensional Wiener process} \end{cases}$$

(Itô's theorem)

• Itô ptoduct

$$d\vec{h}(\vec{x}_t) = (d\vec{W}_t \bullet \nabla)\vec{h} + dt\left[(\vec{a}(\vec{x}_t)\cdot\nabla)\vec{h} + \frac{1}{2}\Delta\vec{h}\right]$$

harmonic: $\langle d\vec{h}(\vec{x}_t)|\vec{x}_{[0,s]}\rangle = 0 \quad \text{for } \forall t \geq s \quad \Rightarrow \quad (\vec{a}\cdot\nabla)\vec{h} + \frac{1}{2}\Delta\vec{h} = 0$

$\mathcal{H}_{\vec{a}}$: **Family of harmonic *h*** associated with $d\vec{x}_t = \vec{a}(\vec{x}_t)dt + d\vec{W}_t$

$$\Leftrightarrow \quad \mathcal{H}_{\vec{a}} \equiv \left\{ \vec{h} \,\middle|\, (\vec{a}(\vec{x})\cdot\nabla)\vec{h} + \frac{1}{2}\Delta\vec{h} = 0 \right\}$$

We introduce "*self-harmonic drift*", a kind of self-reference

Definition: "$\vec{a}^*$ is ***self-harmonic drift***"

$\Leftrightarrow$ $\vec{a}^*(\vec{x}_t)$ is martingale associated with the process $\boldsymbol{x_t}$ generated by $d\vec{x}_t = \vec{a}^*(\vec{x}_t)dt + d\vec{W}_t$

$\Leftrightarrow$ $\vec{a}^* \in \mathcal{H}_{\vec{a}^*}$

$$\underset{\vec{h} = \vec{a}^*}{\Leftrightarrow} \quad (\vec{a}^* \cdot \nabla)\vec{a}^* + \frac{1}{2}\Delta \vec{a}^* = \vec{0}$$

2nd order non-linear **P**DE

Isotropic *self-harmonic drift* is **equilibrium response of a single spin**

$$(\vec{a}^* \cdot \nabla)\vec{a}^* + \frac{1}{2}\Delta \vec{a}^* = \vec{0}$$ ⊕ isotropic assumption: $\vec{a}^*(\vec{x}) = L_d(\|\vec{x}\|)\hat{x}$

$\hat{x}$: unit vector along $\vec{x}$

d : dimension

$$\Rightarrow \quad L_d''(x) + 2L_d(x)L_d'(x) + \frac{d-1}{x}\left(L_d'(x) - \frac{L_d(x)}{x}\right) = 0$$

ODE with *scale invariance* : $L_d(x) \to \kappa L_d(\kappa x)$

⇒ first integral, $L_d'(x) + (L_d(x))^2 + \frac{d-1}{x}L_d(x) = \kappa^2$ $(\kappa = 1)$

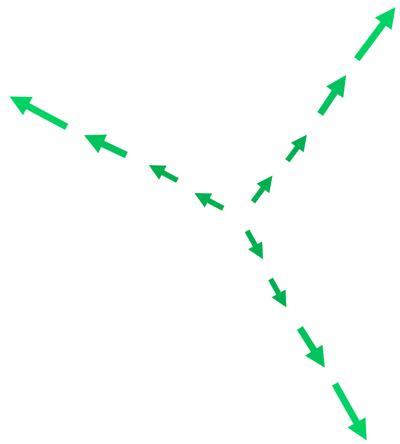
⇒ $L_d \equiv \frac{d}{dx}\log Z_d$ (Ricatti transformation) gives $Z_d'' + \frac{d-1}{x}Z_d' - Z_d = 0$

regular solution ⇒ $Z_d(\|\vec{x}\|) \propto \oint_{\|\hat{S}\|=1} e^{\hat{S}\cdot\vec{x}} d\Omega_S,$ canonical partition function of a **spin** !

$\vec{x} =$ static external field

$$\Rightarrow \quad \vec{a}^*(\vec{x}) = \nabla \log[Z_d(\|\vec{x}\|)]$$

$$\Leftrightarrow \quad \vec{a}^*(\vec{x}) = \langle \hat{S} | \text{field} = \vec{x} \rangle_{eq}$$: equilibrium response of a spin



the farther from x=0, the stronger field

Langevin functions emerge in the *self-harmonic drift*

$$\left.\begin{aligned} \vec{a}^*(\vec{x}) &= \langle \hat{S} | \text{field} = \vec{x} \rangle_{eq} \\ \vec{a}^*(\vec{x}) &= L_d(\|\vec{x}\|)\hat{x} \end{aligned}\right\} \Rightarrow L_d \text{ are Langevin functions}$$

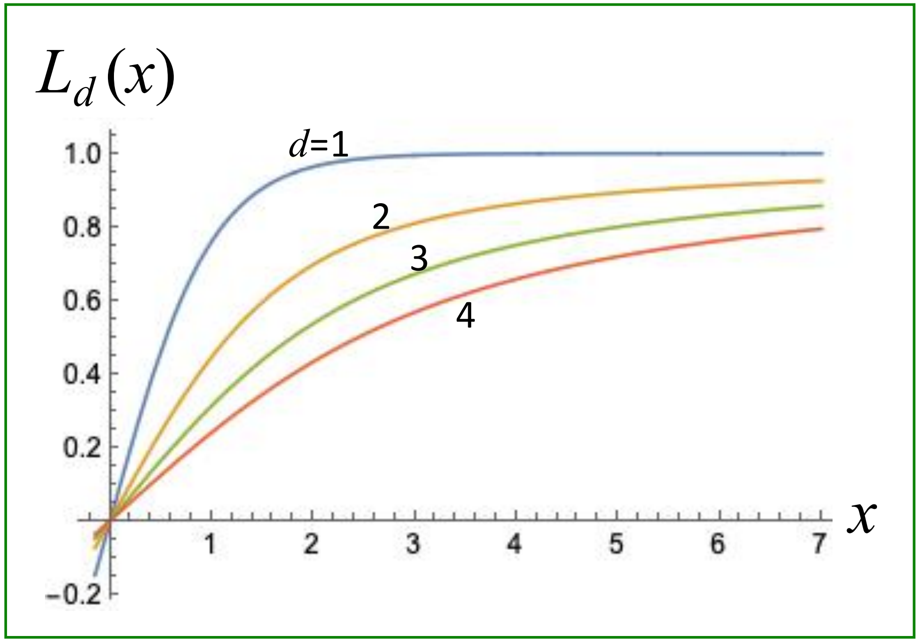
$$L_1(x) = \tanh(x)$$

$$L_2(x) = \frac{I_1(x)}{I_0(x)}$$

$$L_3(x) = \coth x - \frac{1}{x}$$

$$L_4(x) = \frac{x[I_0(x) + I_2(x)] - 2I_1(x)}{2xI_1(x)}$$

I_n : n-th Bessel function of 2nd kind



cf. Drift a^* saturates under strong field

Martingality of ***self-harmonic drift*** was numerically verified

Initial state need not be 0: $\boldsymbol{x}_0=0$ or $\neq 0$

Constraints between drift and noise: $d\vec{x}_\tau = C\kappa\,\vec{a}^{\,*}(\kappa\,\vec{x}_\tau)d\tau + \sqrt{C}d\vec{W}_\tau$

Numerical check in d=2 : $C=\kappa=1$,

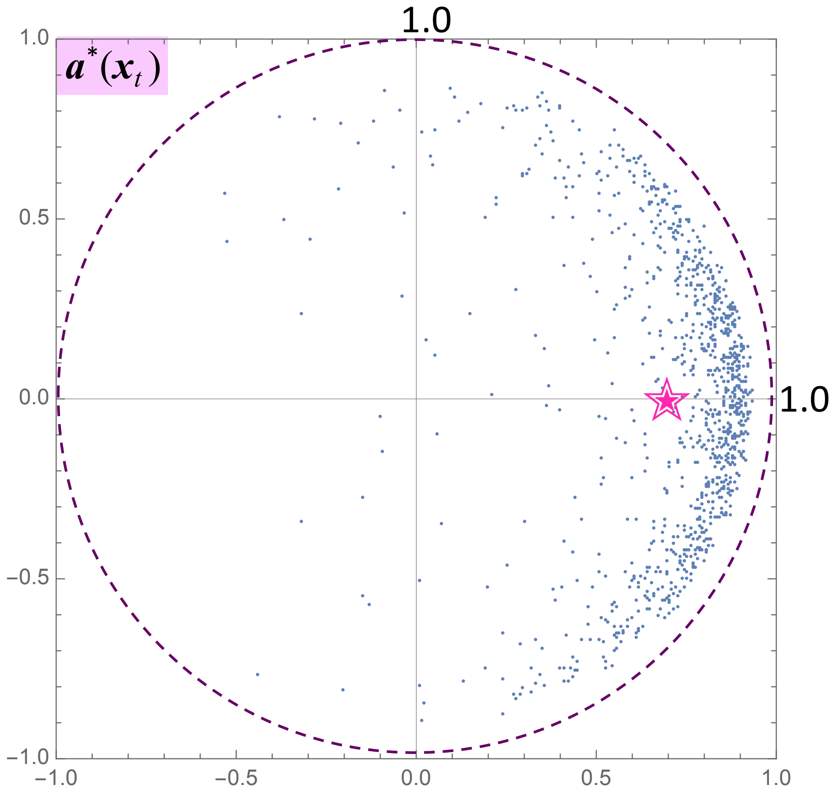
$t=0$; $x_0=(2,0)$: $\boldsymbol{a}^*(x_0) = (0.\mathbf{6978},\ 0.0000)$ ☆

$t=2$; $\{\boldsymbol{a}^*(\boldsymbol{x}_{t=2})\}_{\text{1000 samples}}$ . . . . .

⟶ $\langle \boldsymbol{a}^*(\boldsymbol{x}_{t=2}) \,|\, \boldsymbol{x}_0\rangle = (0.\mathbf{6981},\ 0.0097)$

⇒ $\langle \vec{a}^{\,*}(\vec{x}_{t=2})|\vec{x}_0\rangle = \vec{a}^{\,*}(\vec{x}_0)$

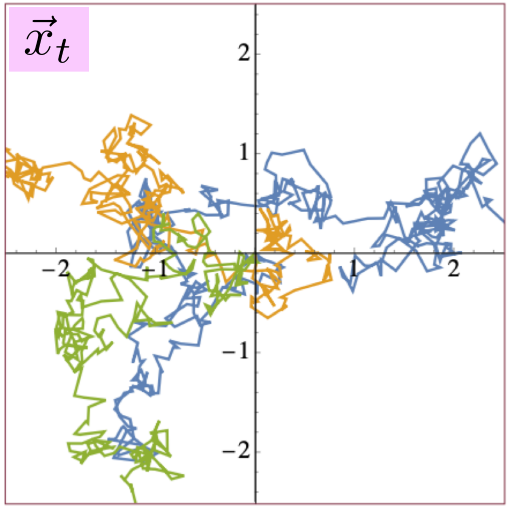
up to rel.error = {**0.05%**, 1.34%}



Reference process $\boldsymbol{x_t}$ has cross-over from ***Random*** to ***Ballistic*** trajectory

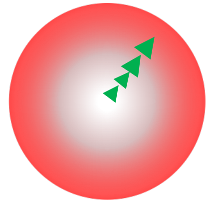
Simulation with $C{=}\kappa{=}1,\ d{=}2,\ \vec{x}_0 = \vec{0}$ $(0 \le t \le 16)$

$\|\vec{x}_t\| \lesssim 1$

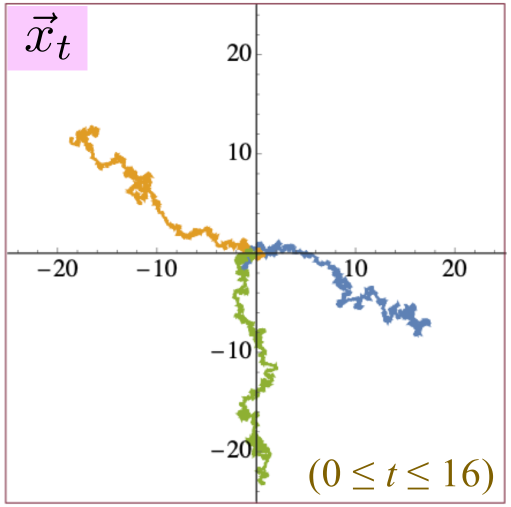


$$d\vec{x}_t = \vec{a}^*(\vec{x}_t)dt + d\vec{W}_t$$

Noise dominates



$\|\vec{x}_t\| \gg 1$



$$d\vec{x}_t = \vec{a}^*(\vec{x}_t)dt + d\vec{W}_t$$

Drift dominates

Furthermore,

Drift **converges**: $\vec{a}^*(\vec{x}_t) \xrightarrow[t \to \infty]{} \vec{a}^*_\infty$: Ballistic direction $\vec{a}^*_\infty = \lim_{t\to\infty} \frac{\vec{x}_t}{t}$

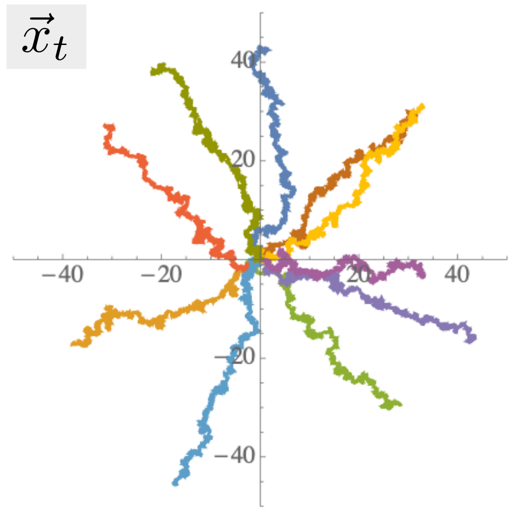
Martingale convergence theorem: "If martingale & bounded ⇒ converges! "

⇒ $\vec{a}^*_\infty$: "spin" $\|\vec{a}^*_\infty\| = 1$ and $\langle \vec{a}^*_\infty | \vec{x}_0 \rangle = \vec{a}^*(\vec{x}_0)$

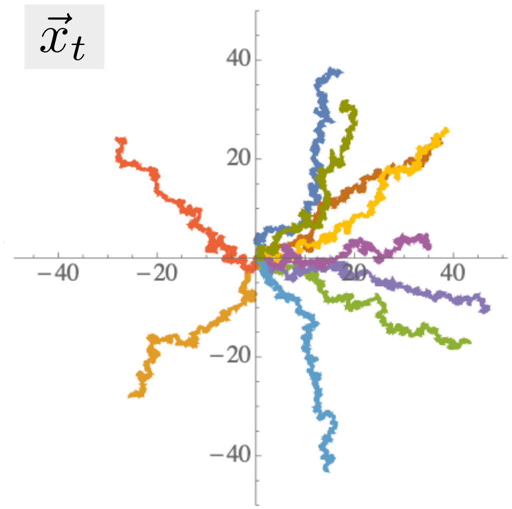
$L_d(\infty)=1$

Distribution of "spin" $\boldsymbol{a}_\infty{}^*$ reflects the initial state $\boldsymbol{x}_0$

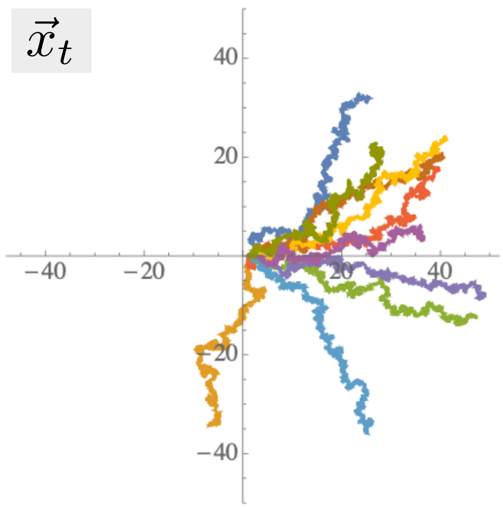
Solutions of $\{\vec{x}_t\}$, $0 \leq t \leq \mathbf{40}$ starting from $\vec{x}_0$ (10 samples, with the same set of Wiener noises)



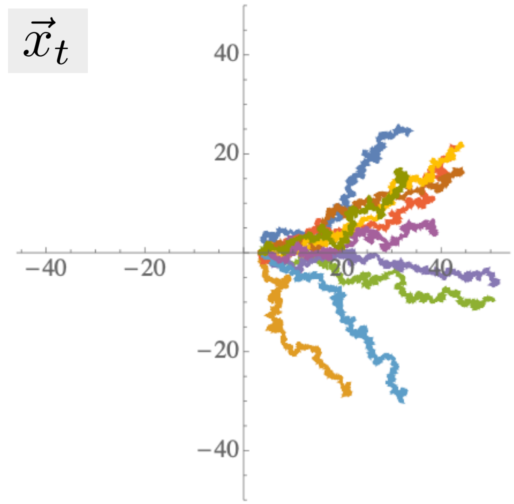
(a) $x_0 = 0$



(b) $x_0 = 1$



(c) $x_0 = 2$



(d) $x_0 = 4$

Ensemble of "spin" $\vec{a}^*_\infty$ reproduces *canonical* spin distribution at $\boldsymbol{x_0}$ ($t = 0$)

$t = 0$ $\quad \vec{a}^*(\vec{x}_0) = \langle \hat{S} | \text{field} = \vec{x}_0 \rangle_{eq}$ $\Rightarrow$ Invisible spin $\hat{S} \in S^{d-1}$ behind $L_d(\|x_0\|)$ obeys $\rho(\hat{S}) = \dfrac{e^{\vec{x}_0 \cdot \hat{S}}}{Z_d(\|\vec{x}_0\|)}$

Cumulative probability ($\cos\theta \equiv \hat{x}_0 \cdot \hat{S}$) :

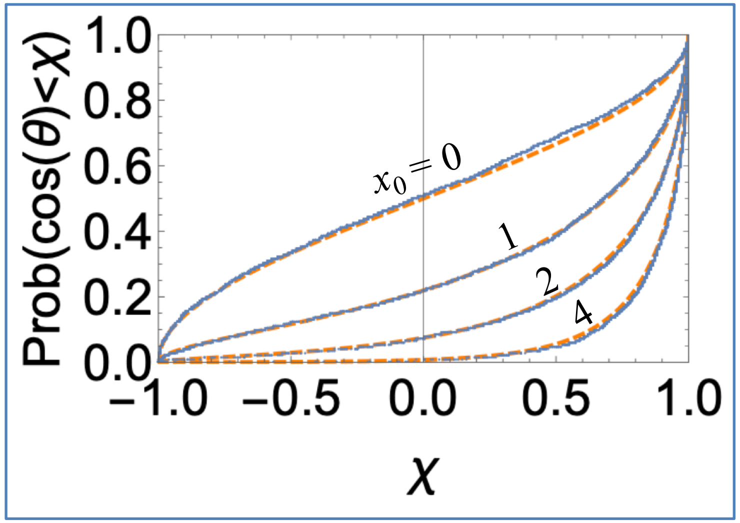
$$Prob^{(\mathrm{can})}(\cos\theta < \chi | \vec{x}_0) = \frac{1}{\pi I_0(x_0)} \int_{-1}^{\chi} \frac{e^{x_0 \xi} d\xi}{\sqrt{1-\xi^2}}$$

$t = \infty$ $\quad$ "spin" of asymptotic trajectory : $\lim_{t\to\infty} \dfrac{\vec{x}_t}{t} = \vec{a}^*_\infty \in S^{d-1}$

Numerical cumulative probability of $\cos\theta_\infty \equiv \hat{x}_0 \cdot \vec{a}^*_\infty$:

"t_∞"=40, 3000 samples/ each x_0

$Prob^{(\vec{a}^*_\infty)}(\cos\theta_\infty < \chi | \vec{x}_0)$:= [normalized rank of $\cos\theta_\infty = \chi$]



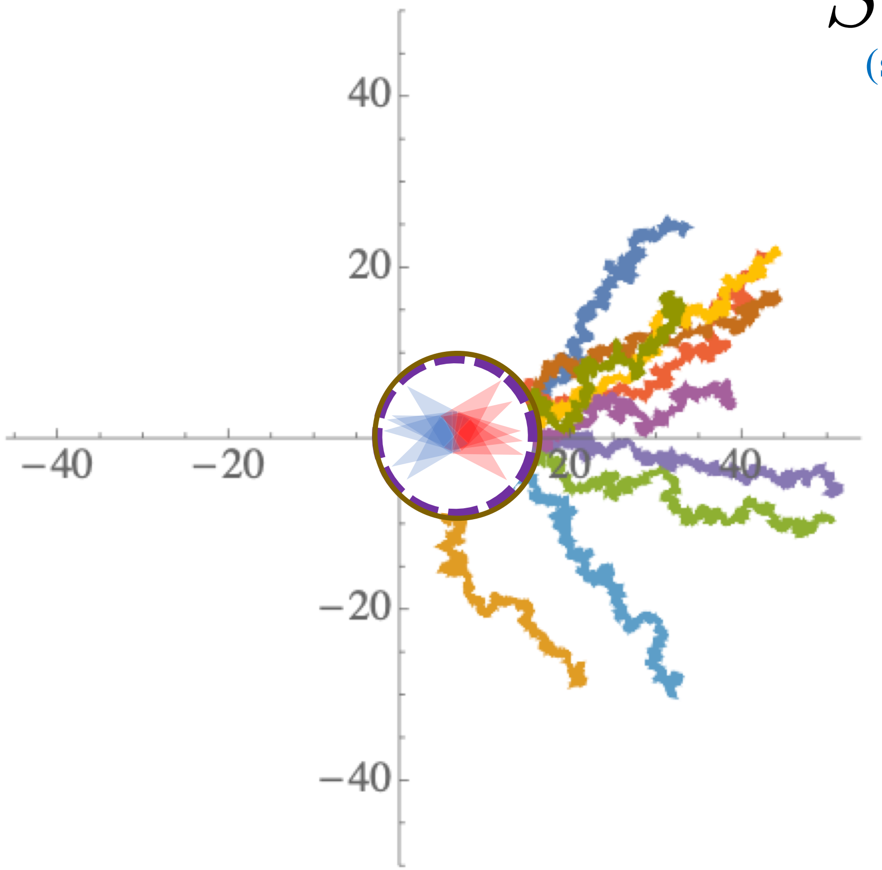
$\Rightarrow$ $\vec{a}^*_\infty \simeq \hat{S}$ (statistically)

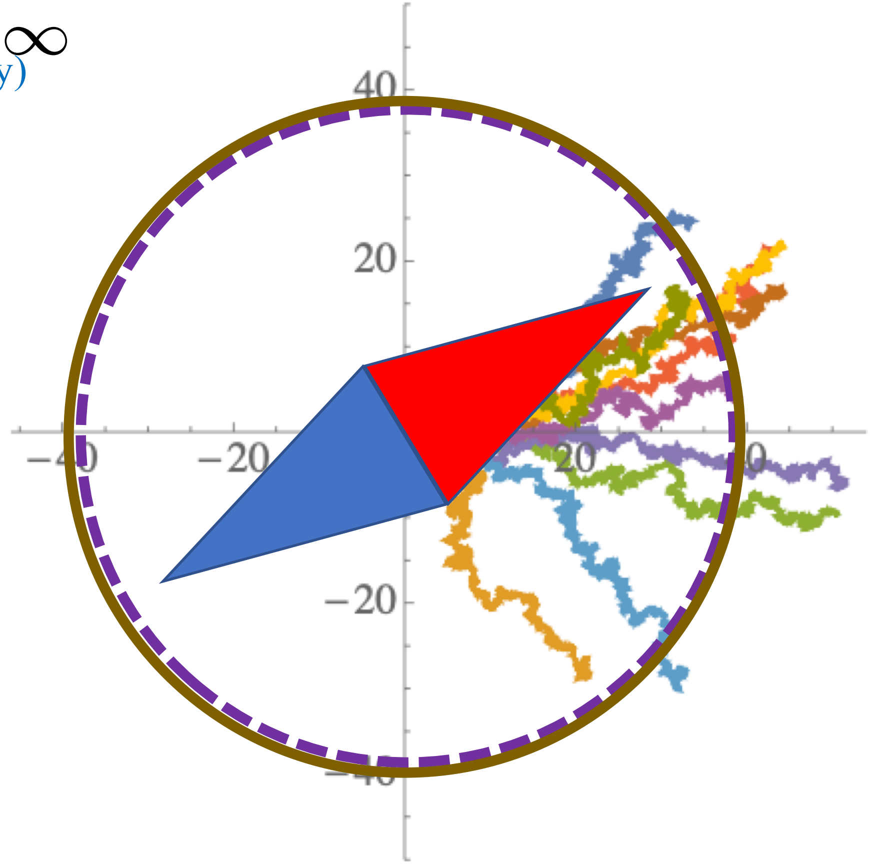
"How can the spin appear ?" $\Rightarrow$ *"Spin appears as macro-orientation"*

Conclusion: Martingale can *somehow* transmit canonical characteristics

$$\hat{S} \simeq \vec{a}^{\,*}_{\infty}$$

(statistically)





Open questions :

Why does the Langevin function appear ?

Why does the canonical "spin" $\boldsymbol{a}_\infty^*$ *appear ?*

What stocks the information of $\boldsymbol{x}_0$ through $0<t<\infty$ *?*

Is there generalization to the state spaces other than $\mathbb{R}^d$ *?* (e.g., *compact, non-commutative, ...*)

Is there a discrete version, with jumps, or quantum version ?

...

Acknowledgements

Charles Moslonka (ESPCI, *PhD defence* on 20th Oct., 2023)

Matteo Polettini (Luxembourg Univ.)

Felix Ritort (Barcelona Univ.)

Thank you for your attention !

ref. K.S.: *to be posted soon on arXiv.*