Semiclassical description of confinement via center vortices and anomaly-preserving T^2 compactifications

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based on 2201.06166

Plan

- Introduction
  - Confinement & adiabatic continuity
  - Our conjecture
- T^2 compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua
- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

Introduction

Goal of this talk is to DERIVE

- Confinement of 4d $SU(N)$ Yang-Mills theory, and
- Chiral effective Lagrangian for QCD,

in a CALCULABLE setup with SEMICLASSICAL methods.

However, confinement seems to be a strong-coupling physics.

Why can we expect that it's possible?

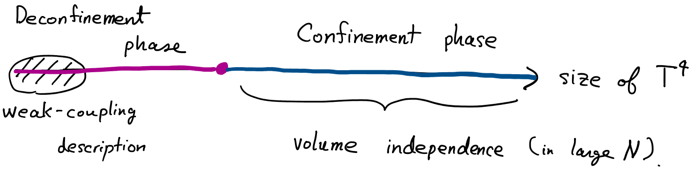
Volume independence

Eguchi, Kawai ('82): In the large-N limit,

$SU(N)$ YM theory on $\mathbb{R}^4$

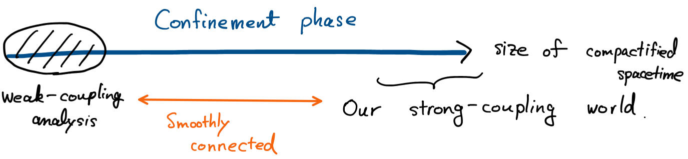
$= SU(N)$ YM theory on T^4

as long as the center symmetry is unbroken.


Deconfinement phase
Confinement phase
size of T^4
weak-coupling description
volume independence (in large N).

Adiabatic Continuity to Weak-Couplings.

We'd like to achieve:



cf. · Twisted Eguchi-Kawai model (Gonzalez-Arroyo, Okawa)

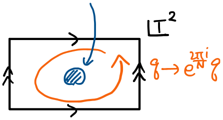
· Double-trace deformed YM
or YM theory w/ adjoint fermions } on $\mathbb{R}^3\times S^1$ (Ünsal, ... 2007–)

Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2\times T^2$

w/ $\boxed{\text{'t Hooft flux}}$ $\Longleftrightarrow$ YM, QCD on $\mathbb{R}^4$

Adiabatic Continuity



Weak-coupling description

Strong-couplings

via $\boxed{\text{Center Vortices}}$

Concrete results that motivate the conjecture

For small $\mathbb{R}^2\times T^2$ w/ 't Hooft flux,

we can use the Dilute Gas Approximation w/ center vortices.

It predicts

- $\boxed{\text{(YM theory)}}$ $\quad E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- $(\mathcal{N}=1 \text{ SYM})$ $\quad \langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$
- $\begin{pmatrix} \text{QCD w/ non-commuting} \\ \text{flavor twist } (N_c = N_f = N) \end{pmatrix}$ $\quad \langle \mathrm{tr}_{cf}(\bar\Psi)\, \mathrm{tr}_{cf}(\Psi) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

(Discrete chiral SSB)

- $\boxed{\begin{pmatrix} \text{QCD w/} \\ U(1)_B \text{ monopole flux} \end{pmatrix}}$ $\quad S_{\mathrm{eff}} \sim \int \left(|dU|^2 + \frac{1}{12\pi}\mathrm{tr}((U^\dagger dU)^3) + \underbrace{\chi_{\mathrm{top}} (i \ln\det U - \theta)^2}_{\eta' \text{ mass from Witten-Veneziano formula}} \right)$

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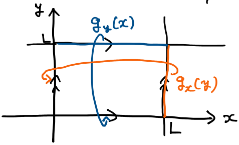
YM theory

- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

't Hooft twisted boundary condition on $T^2 \ni (x, y)$



$x \sim x + L$

$y \sim y + L$

$\phi(x, y)$: adjoint matter field $\quad (\mathrm{Ad}(U^\dagger)\phi = U^\dagger \phi U)$

$$\phi(L,y) = \mathrm{Ad}(\underline{g_x^\dagger(y)})\,\phi(0,y)$$

$$\phi(x,L) = \mathrm{Ad}(\underline{g_y^\dagger(x)})\,\phi(x,0)$$

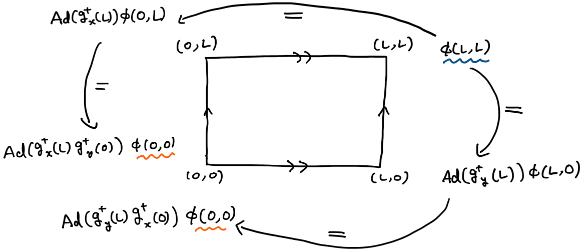
Uniqueness of the matter wavefunction requires

$$g_x^\dagger(L)\,g_y^\dagger(0) = g_y^\dagger(L)\,g_x^\dagger(0)\,e^{\frac{2\pi i}{N}\underset{\substack{\uparrow\\ \text{'t Hooft flux.}}}{n_{xy}}}.$$

(cf When fundamental matters exist, the condition becomes

$$g_x^\dagger(L)\,g_y^\dagger(0) = g_y^\dagger(L)\,g_x^\dagger(0).$$)

Derivation of cocycle condition



$\mathrm{Ad}(g_x^\dagger(L))\phi(0,L)$
=
$(0,L)$
(L,L)
$\underline{\phi(L,L)}$
=
=
$\mathrm{Ad}(g_x^\dagger(L)\,g_y^\dagger(0))\,\underline{\phi(0,0)}$
$(0,0)$
$(L,0)$
$\mathrm{Ad}(g_y^\dagger(L))\,\phi(L,0)$
$\mathrm{Ad}(g_y^\dagger(L)\,g_x^\dagger(0))\,\underline{\phi(0,0)}$
=

Thus, the consistency says

$$\mathrm{Ad}(g_x^\dagger(L)\,g_y^\dagger(0)) = \mathrm{Ad}(g_y^\dagger(L)\,g_x^\dagger(0))$$

$\Longleftrightarrow$ For some n, $\quad g_x^\dagger(L)\,g_y^\dagger(0) = g_y^\dagger(L)\,g_x^\dagger(0)\,e^{\frac{2\pi i}{N}n}.$

In the following, we take the simplest nontrivial example $n = 1 \pmod N$.

Classical configuration and unbroken center symmetry

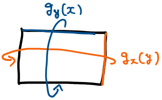
$F = 0$ gives the classical vacuum on T^2.

$\Rightarrow$ We can take a gauge so that $\boxed{a_\mu = 0.}$

However, Polyakov loops are forced to be nontrivial for the 't Hooft twist :

$$g_x(y) \equiv S \propto \begin{pmatrix} 0 & 1 & & \mathbb{0} \\ & 0 & 1 \ddots & \\ & \mathbb{0} & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

$$g_y(x) \equiv C \propto \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \quad (\omega = e^{\frac{2\pi i}{N}})$$



$(S, C \in SU(N)$ and $SC = CS\, e^{\frac{2\pi i}{N}})$.

$\Rightarrow P_x = S, \quad P_y = C.$

This is the $\boxed{\text{center-symmetric}}$ configuration. $(\mathrm{tr}(P_x) = 0$ etc.$)$

Perturbative analysis of $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.
- 2d gluons are gapped.

  $\Leftarrow$ Polyakov loops along T^2 are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S, \ P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N.$$

  Weak-coupling analysis is free from IR divergences.
- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

  $\Downarrow$

  We have to resolve this problem to achieve adiabatic continuity. $\Rightarrow$ Center vortex

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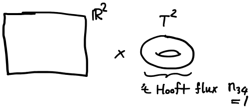
Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$



In this setup, the minimal topological charge is given by

$$\boxed{Q_{\mathrm{top}} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}}$$

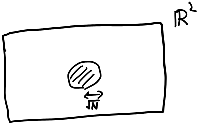
(More precisely, $Q_{\mathrm{top}} \in \frac{1}{N}\left(\frac{-\varepsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$\boxed{S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2\cdot N}}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex
or fractional instanton.



$$\left(\begin{array}{l} Q_{top}=\frac{1}{N} \\ S_{YM}=\frac{8\pi^2}{g^2N}\end{array}\right)$$

(cf. Garcia Perez, Gonzalez-Arroyo, '92, Itou '18)

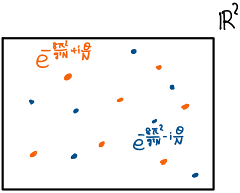
Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Rightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Rightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)


$\mathbb{R}^2$
$e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}$
$e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}$

n : # of vortices

$\bar{n}$: # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

Partition funcion on $M_2 \times T^2$ & θ-dependence

$\to \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

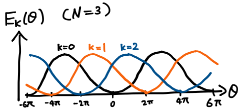
$$K\cdot e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}},$$

we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!}\Big(\overset{\swarrow\text{vortex}}{\underline{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}}\Big)^{n}\Big(\overset{\text{anti-vortex}\swarrow}{\underline{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(\underbrace{-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta-2\pi k}{N}\right)}_{E_k(\theta)\,:\ \text{Ground-state energy densities}}\right)\right]$$

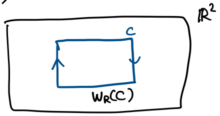
$\Rightarrow \begin{cases} \bullet\ \text{N-branch structure of ground states.} \\ \bullet\ \text{Each branch has a fractional } \theta\text{-dependence.} \end{cases}$



Confinement of Wilson loops inside $\mathbb{R}^2$

Using the dilute gas approximation, we can also compute

$$\langle W_R(C) \rangle =$$



$$= \exp\left(-\text{Area} \times T_R(\theta)\right).$$

For $-\pi < \theta < \pi$, the string tension is given by

$$T_R(\theta) = E_{|R|}(\theta) - E_0(\theta),$$

where $|R|$ is the N-ality of the representation.

In particular, at $\theta = 0$,

$$T_R(\theta=0) \sim \Lambda^2 (\Lambda L)^{\frac{5}{3}} \sin^2\left(\frac{\pi |R|}{N}\right).$$

$\begin{cases} \cdot\ \text{N-ality rule } ✓ \\ \cdot\ \text{Casimir scaling } ✗ \end{cases}$

Hamiltonian picture

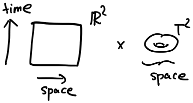
Using the dilute gas approximation of the YM path integral, we obtain

- N-branch structure of vacua
- Fractional θ-dependence $E_k(\theta) \sim \Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta-2\pi k}{N}\right)$
- Confinement of Wilson loops for non-trivial N-alities.

To get a better understandings,

let us construct the wave-function of these confining vacua.

Setup



$$\Rightarrow \quad \underset{\mathbb{R}_{\text{time}}}{\uparrow} \times \underbrace{S^1_A}_{\text{large}} \times \underbrace{S^1_B \times S^1_C}_{\text{small},\ n_{BC}=1}$$

(cf. Yamazaki, Yonekura '17 : $L_A, L_B \ll L_C$ $(\ll \Lambda^{-1})$ w/ $n_{BC}=1$)

Perturbative vacua

We now have three holonomies (Polyakov loops along S^1_A, S^1_B, S^1_C) : P_A, P_B, P_C.

't Hooft flux $n_{BC}=1$: Classical vacuum $F_{BC}=0$ gives

$$P_B = S, \qquad P_C = C.$$

We note that $n_{AB} = n_{AC} = 0$: Classical vacuum $F_{AB} = F_{AC} = 0$ gives

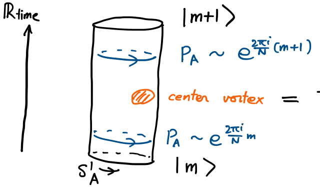
$$P_A P_B = P_B P_A, \qquad P_A P_C = P_C P_A.$$

$$\Rightarrow \quad P_A = e^{\frac{2\pi i}{N} m} \mathbb{1}_N \qquad (m = 0, 1, \cdots, N-1).$$

Classically, $\mathbb{Z}_N$ center symmetry for P_A is spontaneously broken, and we get N degenerate vacua.

$$|m\rangle : \quad \frac{1}{N} \mathrm{tr}(P_A) \, |m\rangle = e^{\frac{2\pi i}{N} m} \, |m\rangle.$$

Tunneling amplitude via center vortex


$\mathbb{R}_{\text{time}}$
$|m+1\rangle$
$P_A \sim e^{\frac{2\pi i}{N}(m+1)}$
center vortex
$P_A \sim e^{\frac{2\pi i}{N} m}$
$|m\rangle$
S^1_A

= Tunneling from $|m\rangle$ to $|m+1\rangle$.

Within the one-center-vortex approximation,

$$\langle n | \exp(-T \hat{H}_{YM}) | m \rangle$$

$$\sim \underbrace{\delta_{nm}}_{\text{No tunneling}} + T\cdot L_A \cdot K e^{-\frac{8\pi^2}{g^2 N}} \left(\underbrace{e^{i\frac{\theta}{N}} \delta_{n,m+1}}_{\substack{\text{tunneling via} \\ \text{center vortex}}} + \underbrace{e^{-i\frac{\theta}{N}} \delta_{n,m-1}}_{\substack{\text{tunneling via} \\ \text{anti center vortex}}} \right)$$

Especially, $|m\rangle$ is not an eigenstate of $\hat{H}_{YM}$.

Eigenstate of $\exp(-T\hat{H}_{YM})$ and confining vacua

Let us consider the following wavefunction

$$|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} km} |m\rangle .$$

This is an eigenstate of $\exp(-T\hat{H}_{YM})$:

$$\exp(-T\hat{H}_{YM})|\tilde{k}\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{n,m} e^{\frac{2\pi i}{N} km} |n\rangle \underbrace{\langle n| e^{-T\hat{H}_{YM}} |m\rangle}_{\sim\ \delta_{nm} + TL_A K e^{-\frac{8\pi^2}{g^2 N}} \left(e^{i\frac{\theta}{N}} \delta_{n,m+1} + e^{-i\frac{\theta}{N}} \delta_{n,m-1}\right)}$$

$$\sim \exp\Big[-T\cdot L_A \cdot \underbrace{\Big(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)\Big)}_{=\ E_k(\theta)}\Big] \cdot |\tilde{k}\rangle .$$

$\Rightarrow$ $|\tilde{k}\rangle$ give the wavefunction of confining vacua for $\mathbb{R}^2 \times \underset{\text{'t Hooft flux}}{\underset{\sim}{T^2}}$.

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QCD

't Hooft flux with fundamental quarks

We have seen that

Semiclassical theory w/ center vortices

$\Rightarrow$ Confinement vacua of YM theory.

Next, we would like to apply this idea for

Chiral effective Lagrangian of QCD.

Obstacle : We cannot introduce the 't Hooft flux when fundamental matters exist.

$U(1)_B$ vs $U(1)_q$

$$\text{Baryon numbers} = \frac{\text{quark numbers}}{N}$$

i.e.
$$U(1)_B = U(1)_q / \mathbb{Z}_N .$$

Let us denote $U(1)_B$, $U(1)_q$ - gauge fields as A_B, A_q :

$$\frac{1}{N} A_B = A_q .$$

$$\left(\Leftarrow \quad A_{B\mu} J_B^{\mu} = A_{q\mu} J_q^{\mu} \quad \text{and} \quad J_B^{\mu} = \frac{1}{N} J_q^{\mu} \right)$$

However, this naive relation looks to be problematic when we consider monopoles:

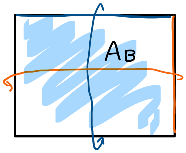
$U(1)_B$ monopole : $\int_{S^2} dA_B = 2\pi.$ $\left(\Leftrightarrow \int_{S^2} dA_q = \frac{2\pi}{N}\right)$

$U(1)_q$ monopole : $\int_{S^2} dA_q = 2\pi$ ← normalization doesn't match.

$U(1)_B$ monopole violates Dirac quantization.

monopole.
S^2

$U(1)_B$ monopole flux & 't Hooft flux on T^2



$$\begin{cases}\psi(L,y) = g_x^\dagger(y)\, e^{-i\frac{\phi_x(y)}{N}}\,\psi(0,y)\\ \psi(x,L) = \underbrace{g_y^\dagger(x)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_y(x)}{N}}}_{U(1)_q\text{-transition functions}}\,\psi(x,0)\end{cases}$$

Cocycle condition

$$g_x^\dagger(L)\, g_y^\dagger(0)\, e^{-i\frac{1}{N}(\phi_x(L)+\phi_y(0))} = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{-i\frac{1}{N}(\phi_y(L)+\phi_x(0))}.$$

$U(1)_B$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_x(L)-\phi_x(0)) - (\phi_y(L)-\phi_y(0))$$

$$\Longrightarrow \quad g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

Perturbative spectrum

Gauge fields : Same with $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

$\Rightarrow$ $\mathbb{Z}_N$ gauge field.

Quark fields : Solve the Dirac zero-mode eq.

$$\left[\gamma_3\left(\partial_3 + i\frac{1}{N}A_{B,3}\right) + \gamma_4\left(\partial_4 + i\frac{1}{N}A_{B,4}\right)\right]\psi = 0.$$

$\Rightarrow$ For each 4d fundamental Dirac fermion, there is a 2d massless Dirac fermion.

For N_f-flavor massless QCD, we have 2d N_f Dirac fermions.

$\xleftrightarrow[\text{bosonization}]{\text{non-Abelian}}$ $U(N_f)_1$ WZW model $\frac{1}{8\pi}\int_{M_2} \mathrm{tr}(dU^\dagger \wedge \star dU) + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger dU)^3]$

w/ $U(N_f)$-valued field $U: M_2 \to U(N_f)$.

Center vortex

Let us construct the center-vortex vertex $\sim e^{-\frac{8\pi^2}{g^2N} + i\frac{\theta}{N}}$.

$U(1)$ axial anomaly requires the spurious symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar\psi \to \bar\psi e^{i\alpha\gamma_5}, \quad \theta \to \theta + 2N_f\alpha.$$

In the bosonized description

$$U \to e^{-2i\alpha} U.$$

$$\Rightarrow \quad e^{-\frac{8\pi^2}{g^2N} + i\frac{\theta}{N}} \cdot (\det U)^{\frac{1}{N}} \quad \text{satisfies}$$

- $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry,
- $U(1)$ axial anomaly relation.

$$\Rightarrow \quad \Delta S_{\text{center-vortex}} \sim \underbrace{\Lambda^2(\Lambda L)^{\frac{5N-2N_f}{3N}}}_{\eta' \ \text{mass}} \cdot \cos\left(\frac{i\ln(\det U) - \theta}{N}\right).$$

QCD for large $\mathbb{R}^2 \times T^2$ w/ $U(1)_B$ monopole flux

So far, we explicitly solved QCD for small $\mathbb{R}^2 \times T^2$ w/ $U(1)_B$ flux.

What happens for large $\mathbb{R}^2 \times T^2$?

Chiral effective Lagrangian is now available :

$$\int_{M_4} f_\pi^2 |dU|^2 + \int_{M_4} A_{B\mu} J_B^\mu .$$

$$\star J_B = \star J_{\text{skyrmion}} = \frac{1}{24\pi^2}\mathrm{tr}[(U^\dagger dU)^3].$$

$$\Downarrow \quad T^2\text{-compactification w/ } \int_{T^2} dA_B = 2\pi$$

$$\int_{M_2} (L^2 f_\pi^2)|dU|^2 + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger dU)^3].$$

$$\Rightarrow \quad SU(N_f)_1 \ \text{WZW model when } L^2 f_\pi^2 = \frac{1}{8\pi}.$$

Consistent result with the explicit QCD computation.

SUMMARY

YM, QCD on $\mathbb{R}^2 \times T^2$ $\Longleftrightarrow$ YM, QCD on $\mathbb{R}^4$

w/ 't Hooft flux

Semiclassics w/ center vortices

Adiabatic Continuity

Strong-couplings

- **(YM theory)** $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\langle \mathrm{tr}_{cf}(\bar{\psi})\, \mathrm{tr}_{cf}(\psi) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

(Discrete chiral SSB)

- **(QCD w/ $U(1)_B$ monopole flux)** $S_{\mathrm{eff}} \sim \int \left(|dU|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \underbrace{\chi_{\mathrm{top}} (i \ln \det U - \theta)^2}_{} \right)$

η' mass from Witten-Veneziano formula