Pseudo-gauge ambiguity of the mechanical properties inside a nucleon

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1. Introduction ---- Internal structure of nucleon

We measure form factors to reveal the internal structure.

e.g. EM form factor $\xrightarrow[\text{Fourier transformation}]{}$ Charge distribution $q(r)$

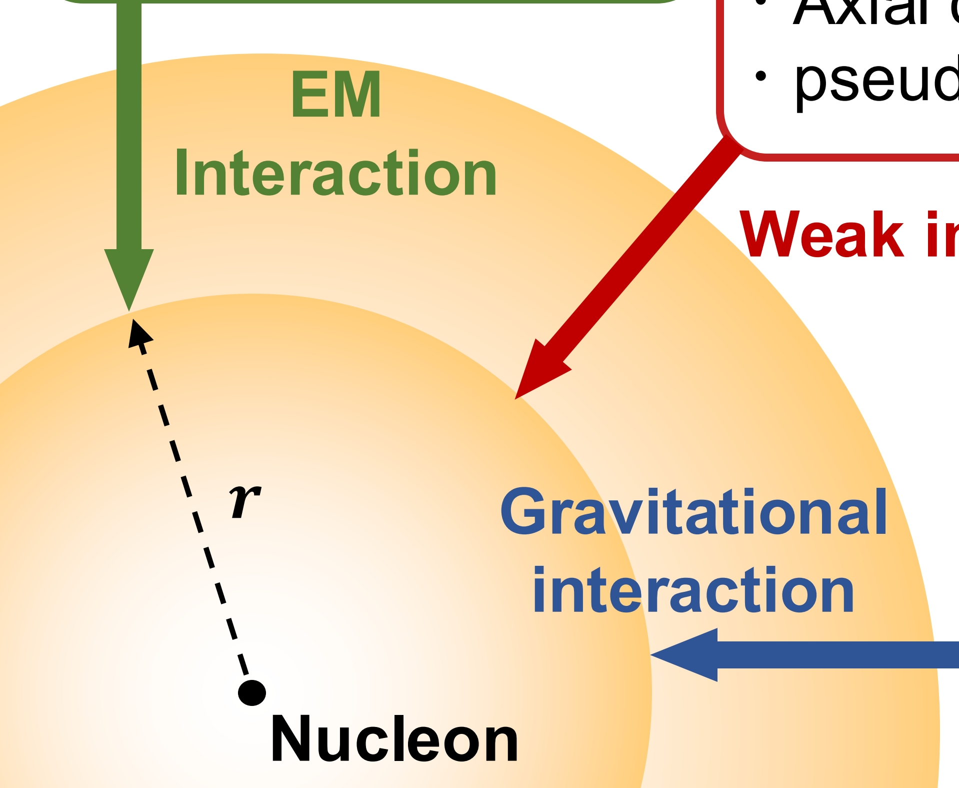
EM form factor

- Charge Q
- Magnetic moment μ

Axial form factor

- Axial coupling g_A
- pseudoscalar coupling g_P

Weak interaction



EMT form factor

- Mass M
- Spin J
- D-term D

Charge
$Q = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$ $\mu = 2.792847356(23)\mu_N$
$g_A = 1.2694(28)$ $g_P = 8.06(55)$
$M = 938.272013(23)\ \mathrm{MeV/c^2}$ $J = 1/2$ $D = ???$

the "last unknown global property"

[Polyakov, Schweitzer (2018)]

1. Introduction ---- What is *D*-term?

EMT form factor

$$P = (p + p')/2\,, \quad \Delta = p' - p\,, \quad t = \Delta^2$$

$$\langle p' | T^{\mu\nu} | p \rangle = \bar{u} \left[\underline{A(t)} \frac{P^\mu P^\nu}{M} + \underline{J(t)} \frac{iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + \underline{D(t)} \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{4M} \right] u$$

Mass

$T^{00} = \varepsilon$

Energy density

Angular Momentum

$x^i T^{0j} - x^j T^{0i} = J^{0ij}$

Angular momentum density

***D*-term**

T^{ij} Pressure

Shear force

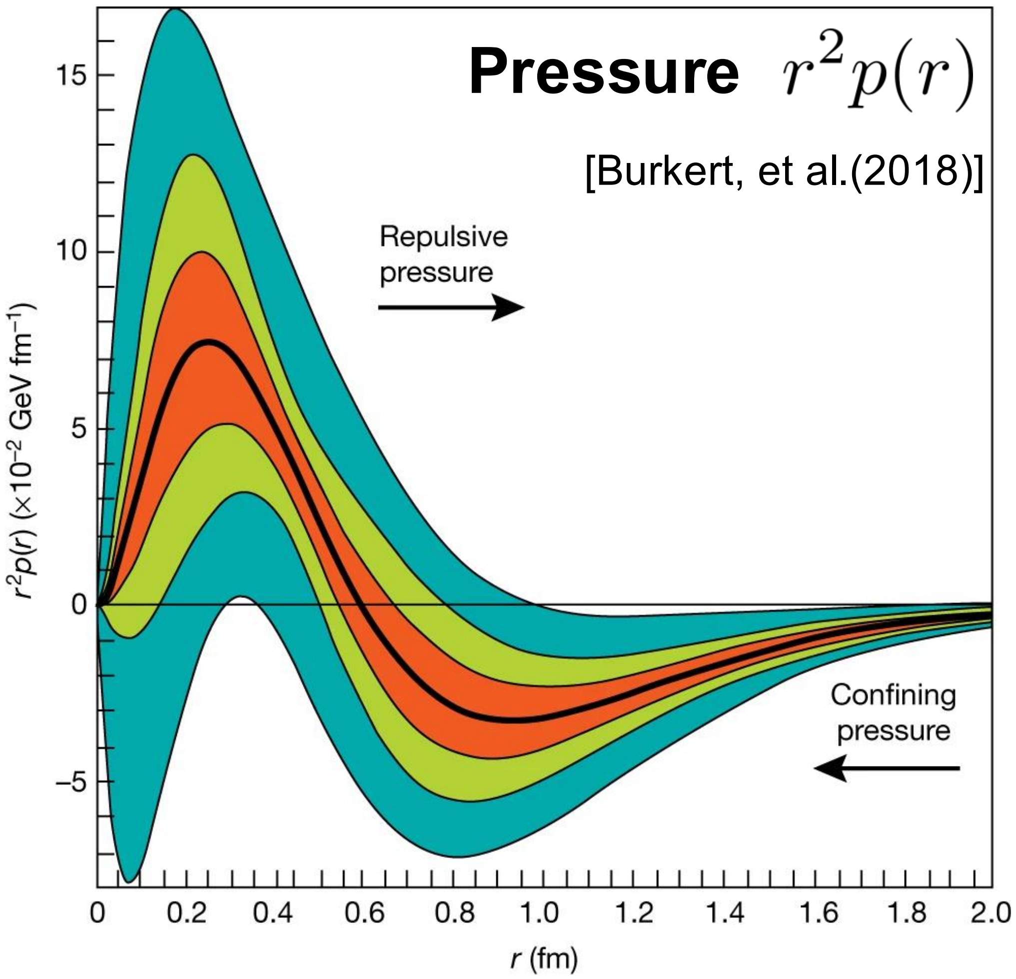
@ Breit frame

$$P^\mu = (E, \mathbf{0})$$
$$\Delta^\mu = (0, \boldsymbol{\Delta})$$

$$T^{ij}(r) = \delta^{ij} p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) s(r)$$

By measuring *D*-form factor, we can know the distribution of the pressure and the shear force inside a nucleon. [Polyakov, Schweitzer (2018)]

1. Introduction ---- Confining force



von Laue condition

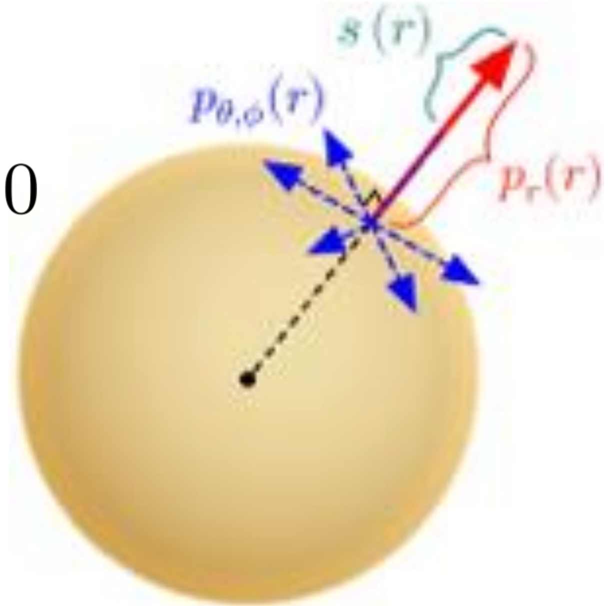
$$\int \mathrm{d}r\, r^2 p(r) = 0$$

Radial pressure

$$p_r(r) = p(r) + \frac{2}{3}s(r) > 0$$

Tangential Pressure

$$p_{\theta,\phi}(r) = p(r) - \frac{1}{3}s(r)$$



Nucleon

Hydrostatic eq.

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} = -\frac{2s}{r}$$

Neutron Star

TOV eq.

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G(\varepsilon + p)(M + 4\pi r^3 p)}{r^2\,(1 - 2GM/r)}$$

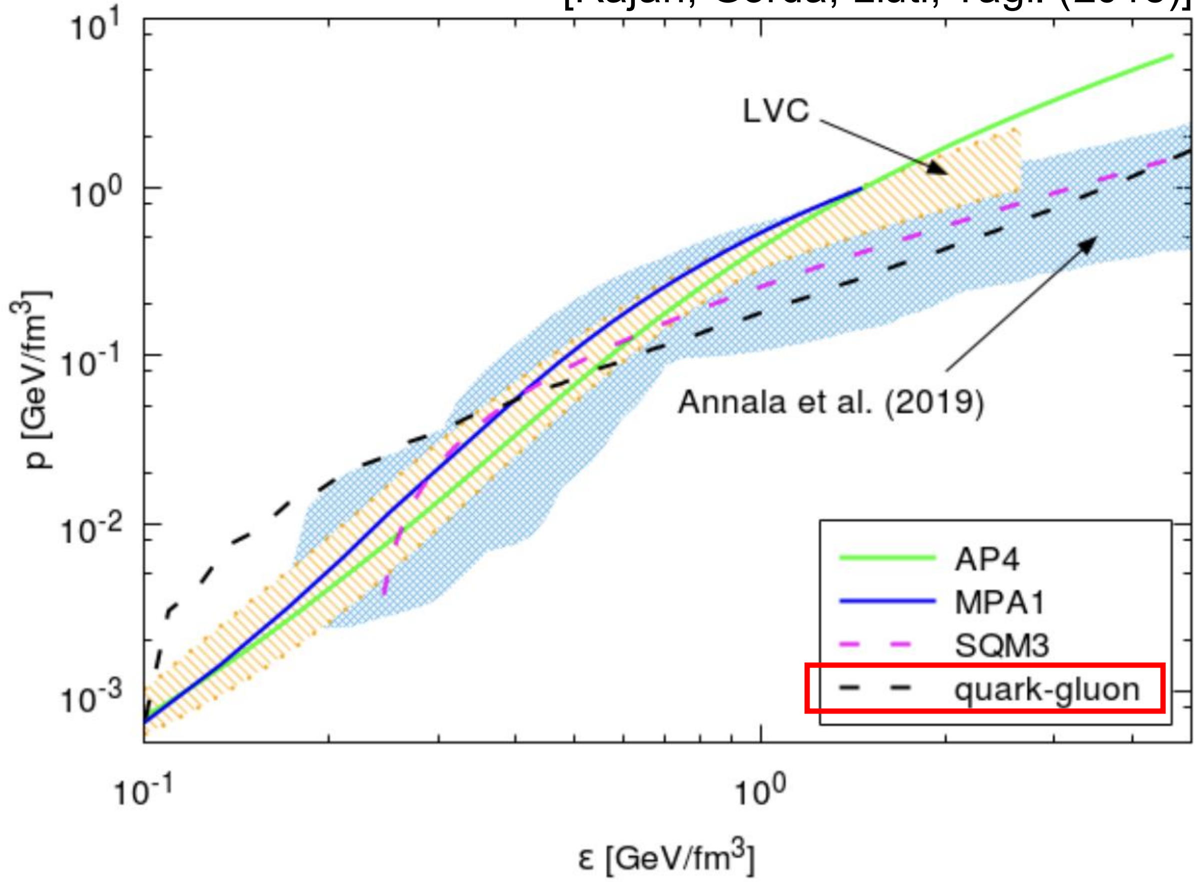
Shear force takes a role of confinement.

[Lorce, et al.(2018)]

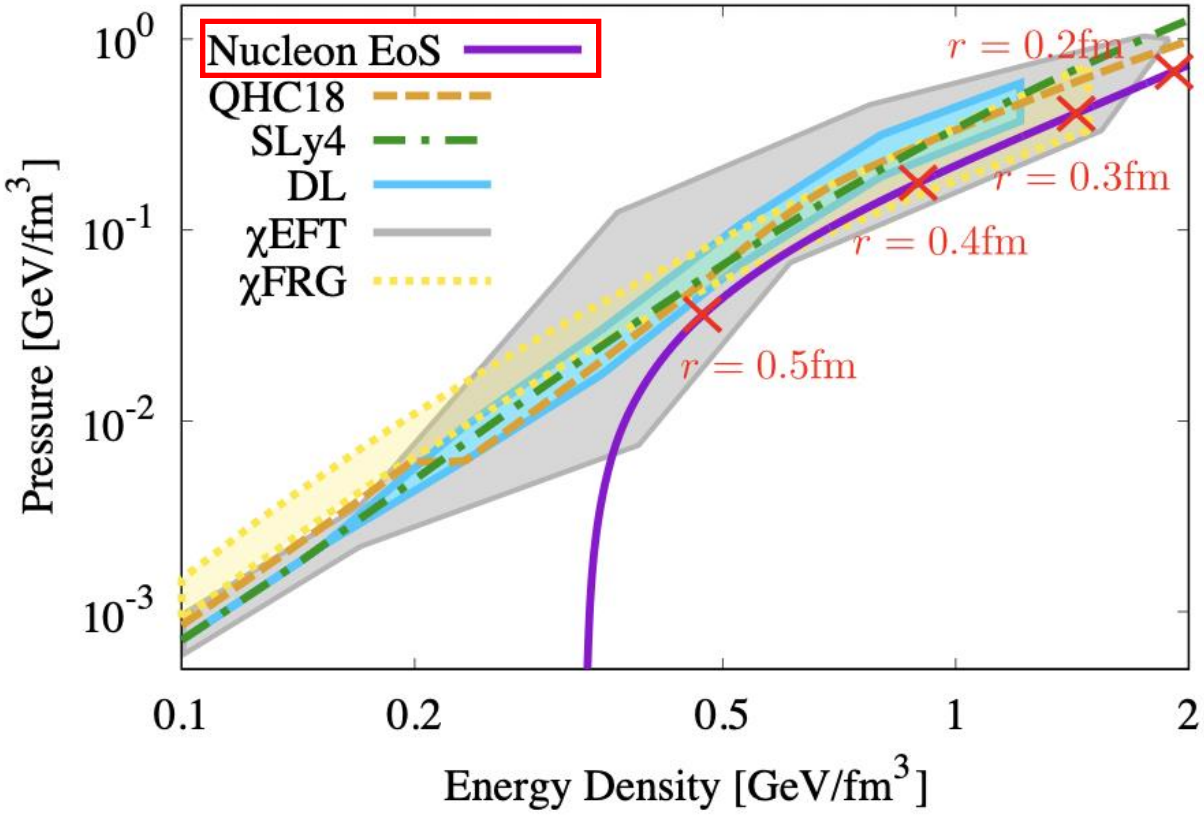
1. Introduction ---- Equation of State

By analogy with NS, a single nucleon EoS was calculated.

[Rajan, Gorda, Liuti, Yagi. (2018)]



[Fukushima, Kojo, Weise. (2020)]



Single nucleon EoS and NS EoS are well matched.

1. Introduction ---- Strategy for $p(r)$, $s(r)$

Experiment

- Deeply Virtual Compton Scattering
- J/ψ photoproduction

factorization →

GPD

- Quark GPD H^q, E^q
- Gluon GPD H^g, E^g

Melin moment

EMT Form factor

$$\int_{-1}^{1} dx\, x H^a(x,\xi,t) = A^a(t) + \xi^2 D^a(t)$$

$$\int_{-1}^{1} dx\, x E^a(x,\xi,t) = B^a(t) - \xi^2 D^a(t)$$

→

Pressure / Shear

$$p(r) = \frac{1}{6Mr^2}\frac{d}{dr}r^2\frac{d}{dr}\tilde{D}(r)$$

$$s(r) = -\frac{r}{4M}\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\tilde{D}(r)$$

This seems a straightforward formulation, but…

1. Introduction ---- Pseudo-gauge ambiguity

But the energy-momentum tensor (EMT) is not unique.
Are the energy density, pressure, and shear force uniquely defined? [Ji, Yang. (2025)]

$$T^{00}(r) = \varepsilon(r)\,, \qquad T^{ij}(r) = \delta^{ij} p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) s(r).$$

In general, we can transform the EMT w/o violating the conservation law.

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu\nu}. \qquad \text{(Pseudo-gauge transformation)}$$

$\mathcal{K}^{\lambda\mu\nu}$ is antisymmetric w.r.t $\lambda,\ \mu,$ and so $\partial_\mu \partial_\lambda \mathcal{K}^{\lambda\mu\nu} = 0.$

Global quantity does not depend on the pseudo-gauge choice.

Forward limit: $\lim_{p'-p\to 0} \langle p'|\partial_\lambda \mathcal{K}^{\lambda\mu\nu}|p\rangle = \lim_{p'-p\to 0} i(p'_\lambda - p_\lambda)\langle p'|\mathcal{K}^{\lambda\mu\nu}|p\rangle = 0.$

Volume integration: $\int d^3\boldsymbol{r}\, \partial_\lambda \mathcal{K}^{\lambda\mu\nu} = 0.$

1. Introduction ---- Pseudo-gauge ambiguity

Canonical EMT

- defined as a conserved current for Noether's theorem: $T^{\mu\nu}_{\rm can} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - g^{\mu\nu}\mathcal{L}.$
- In QCD,

$$T^{\mu\nu}_{\rm can} = \underbrace{-F^{\mu}{}_{\alpha}\partial^\nu A^\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}}_{\text{gauge part}} + \underbrace{\bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha D_\alpha - m)\psi}_{\text{fermionic part}}.$$

- It is not gauge invariant and not symmetric. $\rightarrow$ need to be improved

Cf. Canonical angular momentum tensor

$$J^{\lambda\mu\nu}_{\rm can} = \underbrace{x^\mu T^{\lambda\nu}_{\rm can} - x^\nu T^{\lambda\mu}_{\rm can}}_{\text{orbital angular momentum } L^{\lambda\mu\nu}_{\rm can}} \underbrace{- \frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}\gamma_5\gamma_\rho\psi + F^{\lambda\nu}A^\mu - F^{\lambda\mu}A^\nu}_{\text{Spin angular momentum } S^{\lambda\mu\nu}_{\rm can}}.$$

1. Introduction ---- Pseudo-gauge ambiguity

Belinfante improved EMT

・To be gauge invariant and symmetric, the canonical EMT is improved:

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda (S^{\lambda\mu\nu}_{\rm can} - S^{\mu\lambda\nu}_{\rm can} + S^{\nu\mu\lambda}_{\rm can}).$$

・In QCD,

$$T^{\mu\nu}_{\rm Bel} = \underbrace{-F^{\mu}{}_{\alpha}F^{\nu\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}}_{\text{gauge part}} + \underbrace{\bar{\psi}i\gamma^{\mu}\overleftrightarrow{D}^{\nu}\psi + \frac{1}{4}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(\bar{\psi}\gamma_5\gamma_\rho\psi)}_{\text{fermionic part}},$$

・It is gauge invariant and symmetric.

・It corresponds to the Hilbert EMT defined by metric variation.

$$T^{\mu\nu}_{\rm Bel} = 2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}.$$

1. Introduction ---- Pseudo-gauge ambiguity

Question

Which EMT is used for energy density, pressure, and shear force in a nucleon?

・Observed *D*-form factor corresponds to the spatial component of the Belinfante EMT.

$$D(t) \longleftrightarrow \underline{\frac{1}{2}\bar{\psi}i\gamma^i \overleftrightarrow{D}^j \psi,\ -\frac{1}{4}\delta^{ij}F^2 - F^i{}_\alpha F^{j\alpha}} \longleftrightarrow T^{ij}_{\mathrm{Bel}}.$$

In the experiment,
we measure these operators

○ We can measure *D*-form factor.

○ We can measure T^{ij}_{Bel}.

? We can measure pressure and shear force in a nucleon.

→What happens if we use another EMT for energy density, pressure, and shear force?

2. Method ---- Model choice

To calculate the two types of EMT, we choose the following model.

2-flavor Skyrme model with vector mesons

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\mathrm{WZW}} + \mathcal{L}_m.$$

$$\mathcal{L}_\pi = \frac{1}{4} f_\pi^2 \operatorname{tr}\big(\partial_\mu U \partial^\mu U^\dagger\big).$$ → Non-linear sigma term. Pion: $U = \xi^2$.

Rho and omega mesons

$$\mathcal{L}_1 = -\frac{1}{2} f_\pi^2 \operatorname{tr}\big(D_\mu \xi \cdot \xi^\dagger + D_\mu \xi^\dagger \cdot \xi\big)^2.$$

$$\mathcal{L}_2 = -\frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}).$$

→ Vector meson term. $V^\mu \sim \tau^a \rho^{\mu,a} + \omega^\mu.$

$$F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu - \mathrm{i}[V^\mu, V^\nu].$$

$$D^\mu = \partial^\mu - \mathrm{i}V^\mu.$$

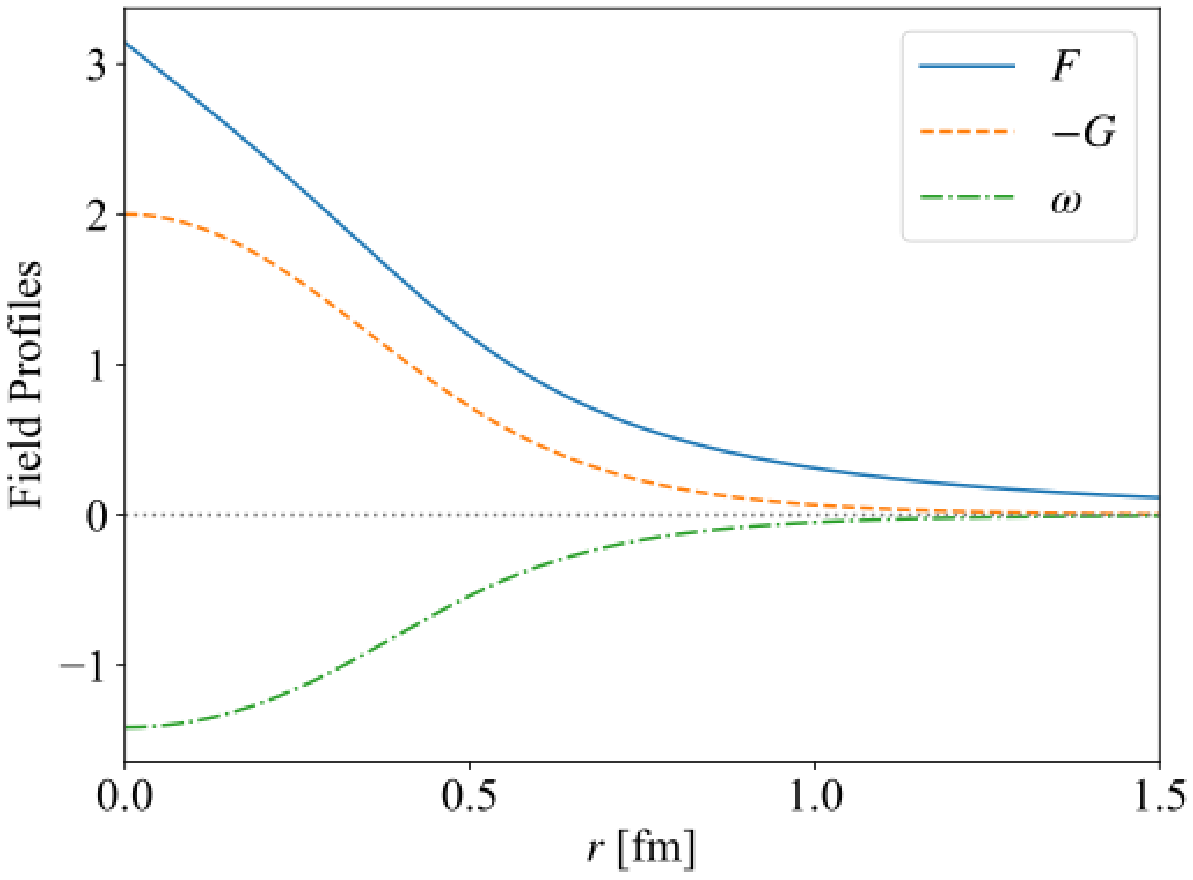
$$\mathcal{L}_{\mathrm{WZW}} = \frac{3}{2} g \omega_\mu B^\mu.$$ → WZW term. Baryon current: $B^\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \operatorname{tr}\big(U^\dagger \partial_\nu U U^\dagger \partial_\alpha U U^\dagger \partial_\beta U\big).$

$$\mathcal{L}_m = \frac{1}{2} m_\pi^2 f_\pi^2 \operatorname{tr}\big(U + U^\dagger - 2\big)$$ → Mass term.

2. Method ---- Model Ansatz

Ansatze

$$U(\boldsymbol{r}) = \exp[\mathrm{i}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)]\,, \quad \rho^0(\boldsymbol{r}) = 0\,, \quad \rho^{i,a}(\boldsymbol{r}) = \varepsilon^{ika}\hat{r}^k\frac{G(r)}{gr}\,, \quad \omega^\mu(\boldsymbol{r}) = \omega(r)\delta^{\mu 0}$$



- The vector mesons stabilize a nucleon as a soliton.
- Substituting the ansatze for the specific EMT forms, we get the energy density, pressure, and shear force.
- This model is not so quantitative, but it can explain the nucleon structure qualitatively.

e.g. Nucleon mass: 1460 GeV

3. Results ---- Analytical expressions

$$\varepsilon_{\mathrm{can}}(r) = \frac{1}{2}f_\pi^2\left(F'^2 + 2\frac{\sin^2 F}{r^2}\right) + f_\pi^2 m_\pi^2(1-\cos F) + \frac{2}{r^2}f_\pi^2(G+1-\cos F)^2 - f_\pi^2 g^2\omega^2$$
$$+ \frac{1}{2g^2r^4}\left[2r^2G'^2 + G^2(G+2)^2\right] - \frac{1}{2}\omega'^2 + \underbrace{\frac{3g\omega}{4\pi^2r^2}F'\sin^2 F}_{\text{WZW}},$$

$$\varepsilon_{\mathrm{Bel}}(r) = \frac{1}{2}f_\pi^2\left(F'^2 + 2\frac{\sin^2 F}{r^2}\right) + f_\pi^2 m_\pi^2(1-\cos F) + \frac{2}{r^2}f_\pi^2(G+1-\cos F)^2 + f_\pi^2 g^2\omega^2$$
$$+ \frac{1}{2g^2r^4}\left[2r^2G'^2 + G^2(G+2)^2\right] + \frac{1}{2}\omega'^2,$$

Absence of WZW

$$p_{\mathrm{can}}(r) = -\frac{1}{6}f_\pi^2\left(F'^2 + 2\frac{\sin^2 F}{r^2}\right) - f_\pi^2 m_\pi^2(1-\cos F) - \frac{2}{3r^2}f_\pi^2(1-\cos F)(G+1-\cos F)$$
$$- \frac{2}{r^2}f_\pi^2 G(G+1-\cos F) + f_\pi^2 g^2\omega^2 - \frac{1}{6g^2r^4}\left[2r^2G'^2 + 3G^3(G+2) + 2G^2(G+2)\right] + \frac{1}{6}\omega'^2,$$

$$p_{\mathrm{Bel}}(r) = -\frac{1}{6}f_\pi^2\left(F'^2 + 2\frac{\sin^2 F}{r^2}\right) - f_\pi^2 m_\pi^2(1-\cos F) - \frac{2}{3r^2}f_\pi^2(G+1-\cos F)^2 + f_\pi^2 g^2\omega^2$$
$$+ \frac{1}{6g^2r^4}\left[2r^2G'^2 + G^2(G+2)^2\right] + \frac{1}{6}\omega'^2,$$

$$s_{\mathrm{can}}(r) = f_\pi^2\left(F'^2 - \frac{\sin^2 F}{r^2}\right) - \frac{2}{r^2}f_\pi^2(1-\cos F)(G+1-\cos F) + \frac{1}{g^2r^4}\left[2r^2G'^2 - G^2(G+2) - 3rGG'\right] - \omega'^2,$$

$$s_{\mathrm{Bel}}(r) = f_\pi^2\left(F'^2 - \frac{\sin^2 F}{r^2}\right) - \frac{2}{r^2}f_\pi^2(G+1-\cos F)^2 + \frac{1}{g^2r^4}\left[r^2G'^2 - G^2(G+2)^2\right] - \omega'^2.$$

$$S_{\mathrm{WZW}} = \frac{g}{16\pi^2}\int \omega \wedge \mathrm{tr}(L\wedge L\wedge L).$$

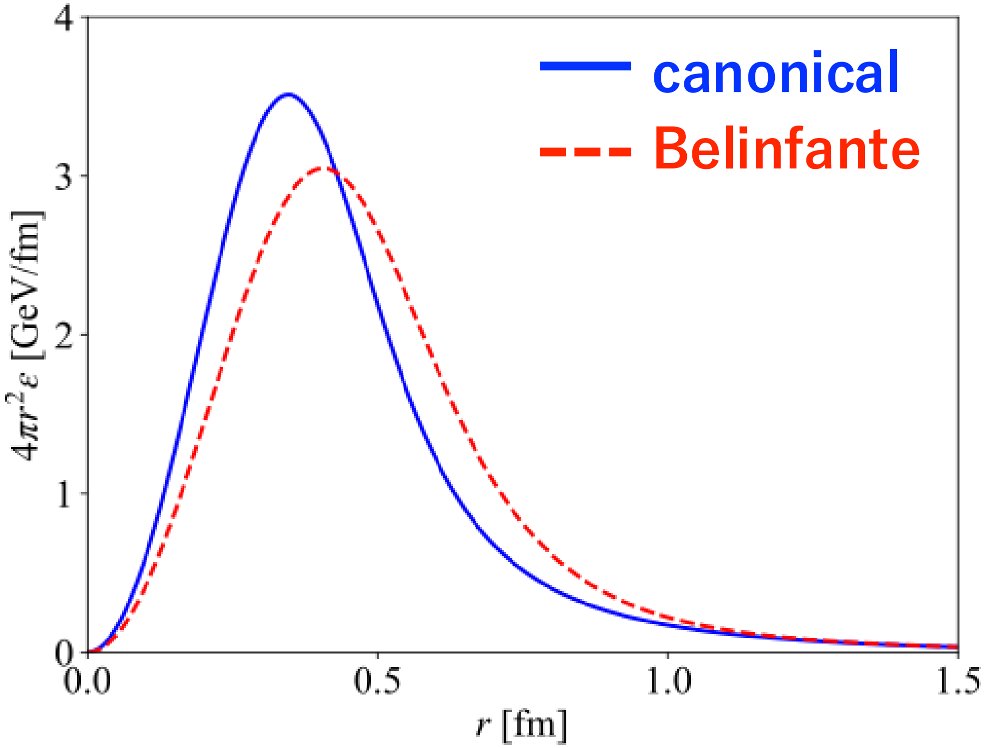
topological term

metric independent

$\rightarrow$ no contribution to Bel

3. Results ---- Energy density

Energy density distribution $T^{00}(r) = \varepsilon(r)$



- The distributions are different for canonical and Belinfante EMT.
- But the total charge, i.e., mass, is the same.

$$\int \mathrm{d}r\, 4\pi r^2 \varepsilon = M,$$

$$\because T^{\mu\nu}_{\mathrm{Bel}} - T^{\mu\nu}_{\mathrm{can}} = \partial_\lambda \mathcal{K}^{\lambda\mu\nu}.$$

- Does the radius depend on the pseudo-gauge choice? → No.

  It looks so, but the (charge) radius is not related to the EMT but the EM current.

3. Results ---- Energy density

Discussion for the radius

- The charge radius is defined as:

$$\langle p' | J^{\mu}_{\mathrm{EM}} | p \rangle \longrightarrow \underbrace{G_E(Q^2)}_{\text{Sachs electric FF}}, \qquad \langle r_E^2 \rangle = -6 \frac{dG_E(Q^2)}{dQ^2}\bigg|_{Q^2=0} = \frac{\int d^3\boldsymbol{r}\, r^2 q(\boldsymbol{r})}{\int d^3\boldsymbol{r}\, q(\boldsymbol{r})}.$$

This is well-defined. J^{μ}_{EM} also has a similar transformation, i.e., $J^{\mu}_{\mathrm{EM}} \to J^{\mu}_{\mathrm{EM}} + \partial_\lambda K^{\lambda\mu}$, but this transformation modifies the action, or theory: $-\int J^\mu A_\mu \to -\int J^\mu A_\mu + \int K^{\lambda\mu} F_{\lambda\mu}.$

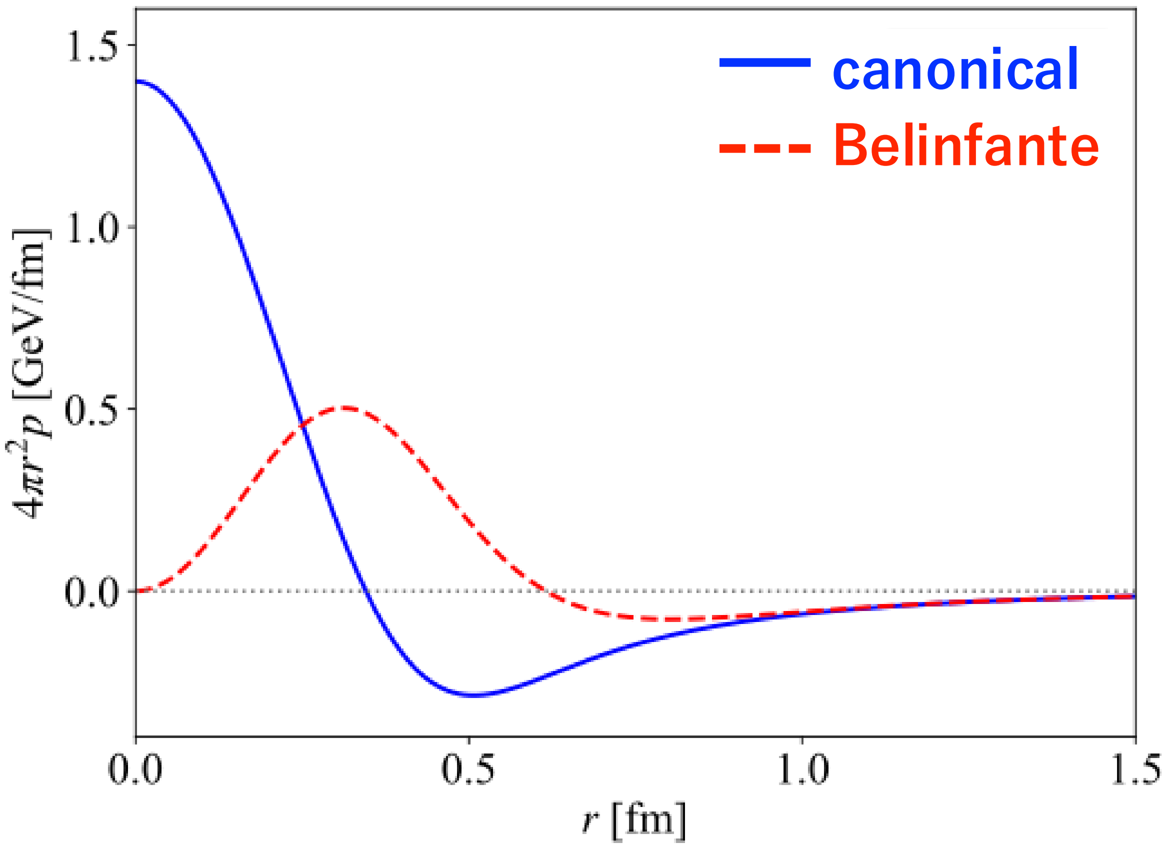
- How about the mass radius and the mechanical radius?

$$\langle r^2_{\mathrm{mass}} \rangle = \frac{\int d^3\boldsymbol{r}\, r^2 T^{00}(\boldsymbol{r})}{\int d^3\boldsymbol{r}\, T^{00}(\boldsymbol{r})}, \qquad \langle r^2_{\mathrm{mech}} \rangle = \frac{\int d^3\boldsymbol{r}\, r^2 \hat{r}_i \hat{r}_j T^{ij}(\boldsymbol{r})}{\int d^3\boldsymbol{r}\, \hat{r}_i \hat{r}_j T^{ij}(\boldsymbol{r})}.$$

Without external gravitational fields, we cannot fix the choice of EMT in the effective field theory.

3. Results ---- Pressure

Pressure distribution $T^{ii}(r) = 3p(r)$



- The canonical pressure diverges at the center of a nucleon.
- The stability condition, i.e., the von Laue condition,

$$\int \mathrm{d}r\, r^2 p = 0,$$

is robust.

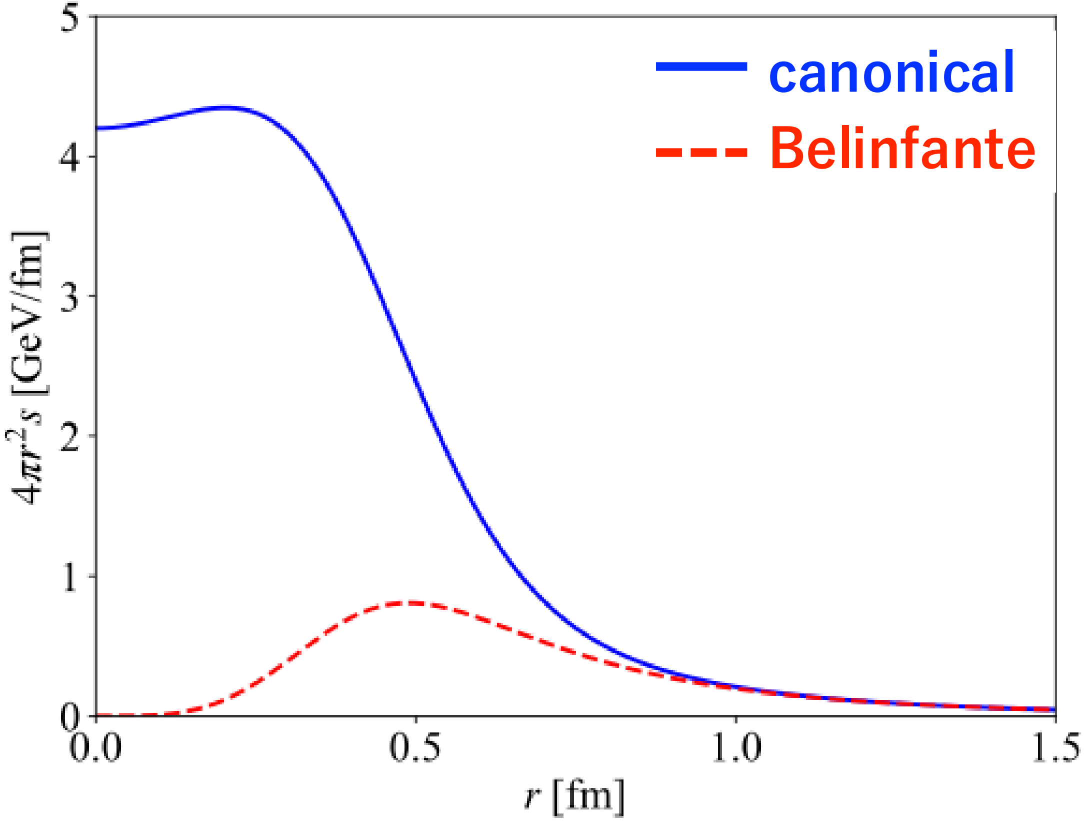
$$\because\ T^{\mu\nu}_{\mathrm{Bel}} - T^{\mu\nu}_{\mathrm{can}} = \partial_\lambda \mathcal{K}^{\lambda\mu\nu}.$$

- The divergence might be a model artifact. In our model ansatz, the rho meson fields diverge at the center.

3. Results ---- Shear force

Shear force distribution

$$T^{ij}(r) = \delta^{ij} p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) s(r).$$



- Shear force relates to surface tension:

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} = -\frac{2s}{r}.$$

- The surface tension energy, [Polyakov, Schweitzer (2018)]

$$\int \mathrm{d}r\, 4\pi r^2 s,$$

depends on the pseudo-gauge choice.

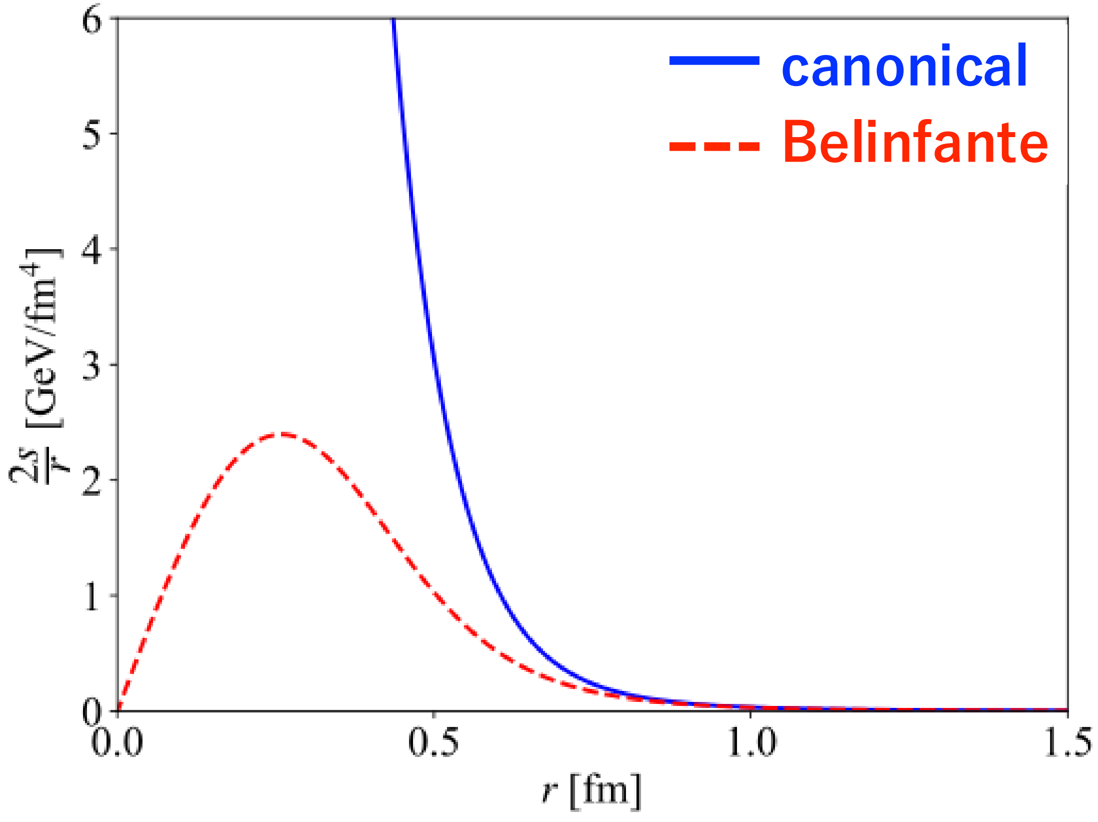
$$\int \mathrm{d}r\, 4\pi r^2 s_{\mathrm{can}} = 2.5\,\mathrm{GeV}, \quad \int \mathrm{d}r\, 4\pi r^2 s_{\mathrm{Bel}} = 0.47\,\mathrm{GeV}.$$

- Yet the volume integration of the EMT does not depend on pseudo-gauge choice.

$$\int \mathrm{d}^3\boldsymbol{r} \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) s = 0.$$

4. Discussions ---- Confining force

Confining force $f_{\text{confining}} = \dfrac{2s}{r}$



- Although the distribution of the confining force differs greatly, the hydrostatic equation holds everywhere.

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} = -\frac{2s}{r}.$$

- Since the hydrostatic equation is a direct consequence of the conservation law:

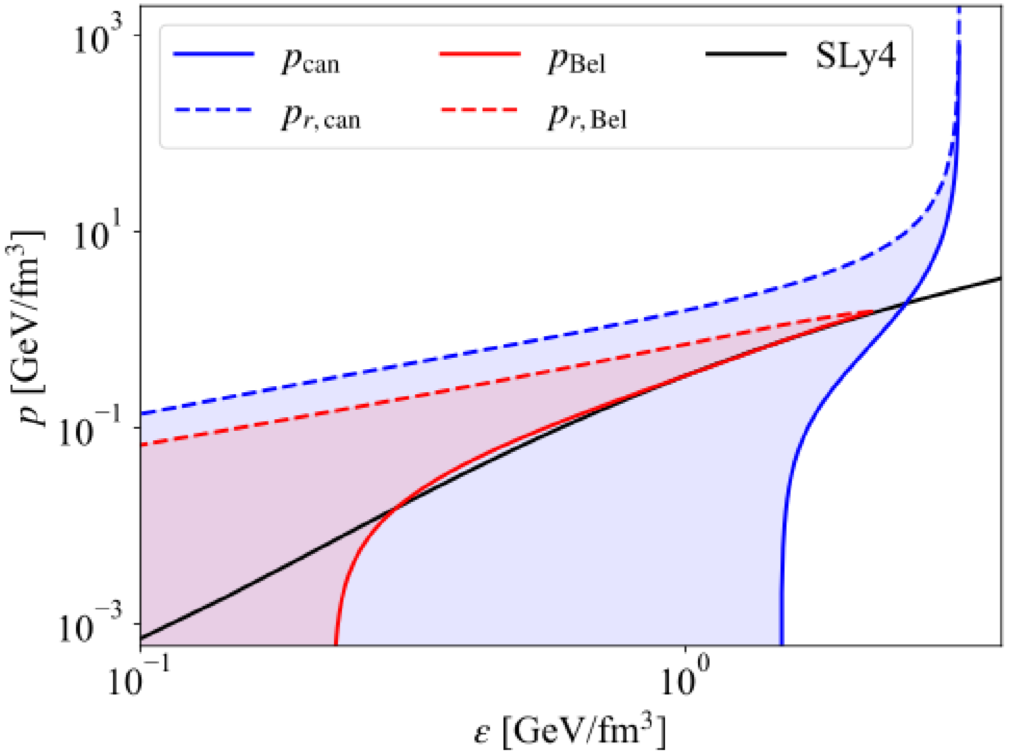
$$\partial_i T^{ij} = 0,$$

it is insensitive to the pseudo-gauge.

4. Discussions ---- Equation of State

Single nucleon EoS

$$\varepsilon(r) = T^{00}(r), \qquad p(r) = \frac{1}{3}T^{ii}(r), \qquad p_r(r) = T^{rr}(r) = p(r) + \frac{2}{3}s(r)$$



- The Belinfante EoS is pretty well matched to the NS EoS.
- But there is pressure anisotropy in a nucleon.
- To compare to NS EoS, we should use $p(\varepsilon)$ or $p_r(\varepsilon)$?

- There is no a priori reason for NS matter to coincide with a single nucleon matter because systems are very different, i.e., size, homogeneity, isospin density, …

4. Discussions

- We cannot determine which EMT is suitable for the pressure inside a nucleon, while the Belinfante one looks natural.
- We note that the Belinfante EMT in the effective field theory does not need to correspond to the Belinfante EMT in QCD.
- In the EIC experiment, we can constrain the expectation values of operators that appear in the Belinfante EMT, such as $\frac{1}{2}\bar{\psi}i\gamma^i\overleftrightarrow{D}^j\psi,\ -\frac{1}{4}\delta^{ij}F^2 - F^i{}_\alpha F^{j\alpha},$ but it does not necessarily mean we can measure the pressure inside a nucleon.
- We realized that the canonical EMT is not symmetric if we quantize Skyrmion.
- The pressure interpretation of *D*-form factor may well need refinement.

  $\rightarrow$ T^{ij} is momentum flux.

4. Discussions ---- Spin decomposition

Cf. There are two different way to decompose the proton spin

Jaffe-Manohar decomposition $J^{\lambda\mu\nu}_{\rm can} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}\gamma_5\gamma_\rho\psi + F^{\lambda\nu}A^\mu - F^{\lambda\mu}A^\nu + x^\mu T^{\lambda\nu}_{\rm can} - x^\nu T^{\lambda\mu}_{\rm can}.$

$$\boldsymbol{J}_{\rm can} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} \underbrace{- i\psi^\dagger(\boldsymbol{x}\times\boldsymbol{\nabla})\psi}_{L^q_{\rm can}} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\boldsymbol{\nabla})\boldsymbol{A}}_{L^g_{\rm can}}.$$

quark helicity gluon helicity orbital angular momentum

Ji decomposition $J^{\lambda\mu\nu}_{\rm can} = x^\mu T^{\lambda\nu}_{\rm Bel} - x^\nu T^{\lambda\mu}_{\rm Bel}.$

$$\boldsymbol{J}_{\rm Bel} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} \underbrace{- i\psi^\dagger(\boldsymbol{x}\times\boldsymbol{D})\psi}_{L^q_{\rm Bel}} + \underbrace{\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})}_{L^g_{\rm Bel}}.$$

quark helicity orbital angular momentum

4. Discussions ---- Momentum flux decomposition?

We can decompose momentum flux in two different ways.

Canonical decomposition

$$T^{ij}_{\rm can} = \underbrace{-F^{i}{}_{\alpha}\partial^{j}A^{\alpha} - \frac{1}{4}\delta^{ij}F^{\alpha\beta}F_{\alpha\beta}}_{\text{gauge field part}} + \underbrace{\bar{\psi}i\gamma^{i}\partial^{j}\psi}_{\text{fermion part}}.$$

Belinfante decomposition

$$T^{ij}_{\rm Bel} = \underbrace{-F^{i}{}_{\alpha}F^{j\alpha} - \frac{1}{4}\delta^{ij}F^{\alpha\beta}F_{\alpha\beta}}_{\text{Maxwell stress tensor}} + \underbrace{\overbrace{\bar{\psi}i\gamma^{i}\overleftrightarrow{D}^{j}\psi + \frac{1}{4}\epsilon^{ij\lambda\rho}\partial_{\lambda}(\bar{\psi}\gamma_5\gamma_\rho\psi)}^{\bar{\psi}i\gamma^{(i}\overleftrightarrow{D}^{j)}\psi}}_{\text{fermion part}},$$

Does the canonical decomposition contain interesting physics?

Is this the same level topic as the spin decomposition?