THE EXOTIC EXCHANGE OF SMOKE RINGS

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Smoke rings are fascinating, to humans and animals alike. Experienced cigarette smokers blow them for entertainment while dolphins play with air-filled underwater rings that they know how to puff. Smoke ring machines can be bought from science gadget shops and Lord Kelvin explains in[1] how one can be constructed from a cardboard box. Even Mount Etna[2] and our Sun[3] are known to be sources of huge smoke rings. But a smoke ring is not only fun to watch. It is also an organized structure with the ability to engage in complex acts, best exemplified by the leapfrogging motion of two smoke rings. Here we propose that the leapfrogging actually encodes very important Physics: It is a direct three dimensional generalization of the motion that in the two dimensional context is responsible for exotic exchange statistics which rules the properties of structures and materials such as quantum Hall systems and high-temperature superconductors. By employing very simple and universal concepts with roots in the hydrodynamical Euler equation, the universal law that describes the properties of fluids and gases, we argue that three dimensional exotic exchange statistics is commonplace. Our observations could have far reaching consequences in fluids and gases which are subject to the laws of quantum mechanics, from helium supefluids to Bose-Einstein condensed alkali gases and even metallic hydrogen in its liquid phases.

Closed vortex loops such as an ordinary smoke ring, are common in inviscid and low viscosity fluids. They are among the simplest examples of organized structure formation, and thus subject to vigorous scientific interest. In the past vortex loops have even been proposed to model atoms,[1] and today they are becoming increasingly important in a wide variety of scenarios: In cosmology cosmic string loops might explain a myriad of phenomena, from baryon number asymmetry to galactic structure formation.[4] In particle physics the string that confines quarks into protons, neutrons and other hadrons might also exist in solitude as a closed loop. The physical principle that explains its properties relates to the origin of most observed mass in our Universum, and its theoretical understanding of this principle is among the Millenium Problems.[5] Properties of vortex loops are vigorously investigated in a variety of fluids and gases, including quantum superfluids,[6][7] atomic Bose-Einstein condensates,[8] nematic liquid crystals,[9] and in the future maybe even the liquid phase of metallic hydrogen.[10] The dynamics of vortex loops is also studied extensively in connection of hydrodynamical turbulence, widely considered as *the* unexplained phenomenon in classical physics. Consequently the theoretical study of hydrodynamical vortex loops could also yield a solution to another Millenium Problem.[5]

The interactions between two or more vortex loops can lead to stunning phenomena. One of the most impressive sights occurs when two identical smoke rings move in unison in the same direction by leapfrogging through each other. The leapfrogging begins with the rear ring contracting and accelerating, while at the same time the front ring slows down and expands its diameter. Eventually the rear ring catches up with the front ring, slides through its center, and emerges ahead of it. This reverses the role of the smoke rings and under ideal conditions the process can repeat itself almost perpetually.

Here we propose that the leapfrogging motion of two smoke rings engages very deep and far reaching Physics. It is the three dimensional generalization of the motion that in a two dimensional context leads to exotic exchange statistics:[11]

In certain two dimensional physical systems such as very thin (^{4}He) superfluid films, fractional quantum Hall systems and high temperature superconductors, identical point particles which are subject to the laws of quantum mechanics do not need to obey the conventional bosonic and fermionic exchange statistics. Instead, under an exchange their quantum mechanical wave function can acquire an arbitrary phase of the form $(-1)^{\Phi}$, where the statistical parameter Φ can have *any* value. This parameter can be made visible by interference experiments, when a pointlike particle such as a point vortex in a thin ^{4}He superfluid film traverses around a closed loop Γ: With A the area encircled by Γ, the quantum mechanical phase Φ acquires the following two (universal) contributions[12]

$$\Phi = 2\pi Q(\rho_0 A - n \cdot \pi\xi^2 \ \rho_V) \tag{0·1}$$

Here Q is the circulation of the vortex which is being traversed as a multiplet of h/m with m the mass of the background fluid particles, and ρ_0 is the constant London limit density of the background fluid.

The first term in (0·1) is a Bohm-Aharonov phase factor for transporting an isolated pointlike vortex around Γ. It is insensitive to the presence of additional vortices and does not contribute to any exotic statistics. But the second term in (0·1) relates directly to the presence of vortices that reside in the area A, and this is the contribution that leads to exotic statistics: The integer n counts the number of vortices with circulation Q that are being encircled and ξ is a length scale that characterizes the radius (thickness) of a vortex core, and ρ_V is the deficit in the density of background fluid particles inside the core of a vortex; For simplicity we have idealized the vortices as pointlike circular regions of radius ξ, with the thickness of the vortex boundary being negligible. The equation (0·1) then states that the exotic contribution to exchange statistics measures how the presence of vortices influences the number of background fluid particles within the area encircled by Γ.

In a three dimensional physical world the exchange of pointlike particles becomes topologically trivial and the pertinent quantum mechanical wave function can only exhibit either the Bose-Einstein ($\Phi = 0$) or the Fermi-Dirac ($\Phi = 1$) statistics, even though this point of view has been occasionally challenged.[12] But if instead of pointlike particles we consider the exchange of closed three dimensional loops and allow for motions that generalize the leapfrogging motion of two smoke rings, the exchange topology becomes nontrivial. In fact, the topology of the three dimensional leapfrogging motion of loops coincides with the topology of two dimensional point particle motion[13] which leads to calculable exotic exchange statistics in that context. This suggests that in the presence of vortex loops a fluid or gas which is governed by the laws of quantum mechanics could exhibit unexpected behaviour which may have hitherto escaped attention since the statistical properties of leapfrogging vortex loops have not yet been adequately scrutinized.

We propose that exotic exchange statistics of vortex loops is in fact commonplace, it follows directly from the properties of hydrodynamical Euler equation which is widely considered as *the* universal equation for describing homogeneous and inviscid fluids and gases. With $\mathbf{u}(\mathbf{x})$ the fluid velocity the Euler equation reads

$$\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u} \ = \ \mathbf{F} \tag{0·2}$$

where the density ρ is subject to the continuity equation

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{0·3}$$

For us the details of the (conservative) external force $\mathbf{F}$ in (0·2) are quite inessential. For simplicity we could for example assume that the flow is isentropic so that any external force can be combined with the pressure and derived from an enthalpy $V(\rho)$ which only depends on the density ρ. If $V(\rho) \sim \lambda(\rho^2 - \rho_0^2)^2$ with λ large, the fluid density is then practically a constant $\rho(\mathbf{x}) \approx \rho_0$, which defines the London limit.

The version of (0·2) and (0·3) which is relevant for describing the motion of vortex loops derives from the classical action

$$S = \int d^3x dt \left\{ \rho\dot{\theta} - \theta\nabla\cdot(\rho\mathbf{u}) - \frac{1}{2}\rho\mathbf{u}^2 - V[\rho] \right\} \tag{0·4}$$

Indeed, a variation of θ in (0·4) yields (0·3), a variation of $\mathbf{u}$ gives

$$\mathbf{u} \ = \ \nabla\theta \qquad \text{when} \quad \rho \neq 0 \tag{0·5}$$

and a variation of ρ leads to the Euler equation (0·2) when we use (0·5). The action (0·4) has the standard Hamiltonian form with θ and ρ as canonical conjugates. Consequently it allows, in principle, for a quantum mechanical treatment which means in particular that we can employ it to inspect the exotic exchange statistics of vortex loops in fluids and gases that are subject to quantum laws.

Instead of the universal Euler equation we could certainly employ some more sophisticated model. In experimental scenarios this may even become a practical concern since as it stands, (0·4) yields a gapless energy spectrum. However, that would not lead to any modification in our main conclusions and for this reason we prefer the Euler equation. The very fact that exotic statistics is encoded already at this level is a clear indication that our observations also are universal, and with a wide applicability.

We now proceed to employ (0·4) to describe the motion of (very thin) vortex loops in a background fluid (or gas). Without loss of generality we idealize the vortex loops as very thin but not necessarily very short tubes, which are closed and possibly knotted. The equation (0·5) states that in a region where $\rho \neq 0$ the vorticity $\omega = \nabla\times\mathbf{u}$ necessarily vanishes. Since a vortex tube carries a nontrivial vorticity, for consistency we must then require that the fluid density $\rho(\mathbf{x})$ vanishes at the core of a vortex in the fluid. Consequently in the presence of closed vortex tubes the region where $\rho \neq 0$ becomes multiply connected, the circulation of $\mathbf{u}$ along irreducible curves that encircle vortices does not need to vanish, and the variable θ becomes multivalued. As customary in fluid mechanics[14] we realize this scenario by approximating a thin vortex tube by a closed curve $\mathbf{R}(s)$ where s measures distance along the curve. Our consistency requirement for the density means that on these curves $\rho[\mathbf{R}(s)] = 0$, while outside of the vortices the density very rapidly approaches its London limit value $\rho(\mathbf{x}) = \rho_0$ with a constant ρ_0.

In the vicinity of a vortex loop the velocity $\mathbf{u}(\mathbf{x})$ is computed from the Biot-Savart law

$$\mathbf{u}(\mathbf{x}) = \frac{Q}{4\pi} \nabla \times \oint \frac{d\mathbf{s}}{|\mathbf{x} - \mathbf{R}(s)|} + \nabla\psi \tag{0·6}$$

where Q is the circulation of the vortex filament, and ψ is some (single valued) function in the fluid that we include for completeness as the fluid does not need to be incompressible. The vorticity has indeed support only on the curve $\mathbf{R}(s)$

$$\omega(\mathbf{x}) \ = \ \nabla \times \mathbf{u} \ = \ \frac{\Gamma}{4\pi} \oint d\mathbf{R} \ \delta(\mathbf{x} - \mathbf{R}) \tag{0·7}$$

and outside of the vortex filaments we have indeed a potential flow, with velocity field $\mathbf{u}$ given by

$$\mathbf{u}(\mathbf{x}) \ = \ -\frac{Q}{4\pi} \nabla \Omega(\mathbf{x}) + \nabla\psi \tag{0·8}$$

where $\Omega(\mathbf{x})$ is the (signed) solid angle at the point $\mathbf{x}$ which is subtended by some two dimensional surface Σ which is bounded by $\mathbf{R}(s)$. The solid angle has a number of valuable properties which are obtained by inspecting the two-form[15]

$$\hat{\omega}(\mathbf{x}_0) \ = \ \frac{1}{2} \epsilon_{ijk} \frac{(x^i - x_0^i) dx^j \wedge dx^k}{|\mathbf{x} - \mathbf{x}_0|^3} \tag{0·9}$$

Its integral

$$\Omega(\mathbf{x}_0) = \int_{\Sigma} \hat{\omega}(\mathbf{x}_0) = \int_{\Sigma} d^2\sigma \epsilon^{ab} \epsilon_{ijk} \frac{\partial z^i}{\partial \sigma^a} \frac{\partial z^j}{\partial \sigma^b} \frac{x^k - z^k}{|\mathbf{x} - \mathbf{z}|^3} \tag{0·10}$$

over the surface Σ coincides with the (signed) solid angle which is subtended by Σ at $\mathbf{x}_0$. Since $d\hat{\omega} = 0$ the solid angle remains intact under such local deformations of the surface Σ that leave its boundary curve $\mathbf{R}(s)$ intact: The solid angle depends only on the boundary curve, and when we move around the boundary curve by linking it once the solid angle jumps by $\pm 4\pi$. Consequently in the presence of closed vortex filaments the variable θ indeed becomes multivalued.

We now consider an (imaginary) periodic adiabatic motion of a closed vortex loop $\mathbf{R}(s)$ in the fluid. We parametrize the motion by $\mathbf{R}(s, t)$ where t is the adiabatic time, and since the motion is periodic we have $\mathbf{R}(s, 0) = \mathbf{R}(s, T)$ for some T. The surface $\mathbf{R}(s, t)$ encloses a toroidal volume V_T in the fluid. We first assume that V_T does not intersect or enclose any other vortex. In the London limit we then get from (0·4), (0·5), (0·8) for the quantum mechanical adiabatic phase

$$\Phi = \int d^3x dt \ \rho\dot{\theta} = - \lim_{\Delta t \to 0} \frac{\rho_0 Q}{4\pi} \int d^3x \int_0^T dt \ \frac{\Omega(\mathbf{x}, t + \Delta t) - \Omega(\mathbf{x}, t)}{\Delta t} \tag{0·11}$$

The difference $\Omega(t+\Delta t)-\Omega(t)$ in (0·11) emanates an evolution of the surfaces $\Sigma(t)$ which is characterized by an evolution of the coordinates $z^i(\sigma^a;t)$ that embed $\Sigma(t)$ in the fluid. The general evolution of the z^i admits two contributions, there is a flow which is normal to the surface $\Sigma(t)$ and there is a flow which is tangential to this surface. The latter corresponds to a reparametrization of $\Sigma(t)$ which we exclude. When we substitute the evolution of z^i normal to the surface in (0·11) we get for the phase in the ensuing quantum mechanical wave function

$$\Phi \;=\; \frac{1}{3}\rho_0 Q \int\limits_{V_T} d^3\sigma \; \epsilon^{\alpha\beta\gamma}\epsilon_{ijk}\frac{\partial z^i}{\partial\sigma^\alpha}\frac{\partial z^j}{\partial\sigma^\beta}\frac{\partial z^k}{\partial\sigma^\gamma} \;=\; \frac{1}{3}Q\rho_0 V_T \tag{0·12}$$

where $\sigma^3 = t$ and we have recognized in the integrand the volume three-form. The result should be compared to the first term in (0·1), it is clear that (0·12) is a three dimensional version of that term.

We now proceed to the general case where the adiabatic transport $\mathbf{R}(s,t)$ of the vortex loop surrounds, without touching, a number of other vortex loops in the fluid. This motion could be for example the leapfrogging motion of two identical vortex loops in a scenario which is subject to the laws of quantum mechanics. The integral (0·11) now acquires two distinct contributions. One of these contributions is an integral that extends over that subregion of the volume V_T which does not contain any vortices. In this subregion we are at the London limit with $\rho(\mathbf{x})=\rho_0$ a constant, and the ensuing integral in (0·12) leads to a contribution where V_T becomes replaced by the volume of the region where there are no vortices. The second contribution is an integral over the volume occupied by the vortices. In this case, we recall that the consistency of our equations implies that at the location of vortices we have $\rho(\mathbf{x})=0$. Consequently the ensuing contribution to the integral (0·11) vanishes. In this way we find that in the presence of N vortices with circulation Q_i, length L_i and (average) tube radius ξ_i we get for the quantum mechanical phase

$$\Phi \;=\; \frac{1}{3}\rho_0(V_T Q - \pi\sum_i^N Q_i L_i \xi_i^2) \tag{0·13}$$

The second term adds up to the total volume occupied by the vortices that are being encircled, weighted by the strength of their circulations. This result is clearly a direct three dimensional generalization of (0·1). In particular, when two identical vortex loops slide through each other in a leapfrogging motion, in addition of the conventional Bohm-Aharonov contribution to the quantum mechanical phase we also have a contribution which is proportional to the (average) volume of the vortices under the leapfrogging period. This additional contribution leads to an exotic exchange statistics, in full parallel to the two dimensional case.

In conclusion, we have shown that if the motion of vortices is governed by the hydrodynamical Euler equation, the ensuing statistics becomes exotic in full parallel with the two dimensional case of anyon statistics. The simplicity of the derivation and the wide applicability of Eulers equations to describe inviscid and very low viscosity fluids and gases is an indication that exotic statistics is a common property of all single component fluids and gases of quantum mechanical relevance. This includes in particular He-II quantum superfluid and one component gases that exhibit Bose-Einstein condensation, and maybe even liquid metallic hydrogen. In these cases we expect that our observation could lead to profound consequences in their description, in particular since closed vortex loops are supposed to have a pivotal role in their turbulent behaviour and phase transition structures.

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