Supersymmetric $U(N)$ Gauge Model and Partial Breaking of $\mathcal{N}=2$ Supersymmetry *)

K. Fujiwara[1] , H. Itoyama[1] & M. Sakaguchi[2]

[1] *Department of Mathematics and Physics, Graduate School of Science, Osaka City University*
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

[2] *Okayama Institute for Quantum Physics,*
1-9-1 Kyoyama, Okayama 700-0015, Japan

We review the construction of the $\mathcal{N}=2$ $U(N)$ gauge model and the analysis of vacua of the model. On the vacua, $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=1$, and the gauge symmetry is broken to a product gauge group $\prod_{i=1}^{n} U(N_i)$. The masses of the supermultiplets appearing on the $\mathcal{N}=1$ vacua are given. We provide a manifestly $\mathcal{N}=2$ supersymmetric formulation of the $U(N)$ gauge model coupled with $\mathcal{N}=2$ hypermultiplets, and show that $\mathcal{N}=2$ supersymmetry is partially broken down to $\mathcal{N}=1$ spontaneously.

§1. Introduction

Until mid nineties, partial breaking of global extended supersymmetries was thought not to be possible. The statement is as follows:

Start with the $\mathcal{N}$-extended supersymmetry algebra

$$\{\bar{Q}^I_\alpha, Q_{J\dot{\alpha}}\} = 2\delta_{\alpha\dot{\alpha}}\delta^I_J H \ , \qquad I, J = 1, ..., \mathcal{N} \ .$$

This implies that

$$2H = \sum_{\dot{\alpha}} ||Q_{I\dot{\alpha}}|0\rangle||^2 \qquad \forall I \ .$$

If $Q_I|0\rangle = 0$ for some I, then $H = 0$. This implies that $Q_I|0\rangle = 0$ for all I because the right hand side is positive definite. On the other hand, if $Q_I|0\rangle \neq 0$ for some I, then $H > 0$. This implies that $Q_I|0\rangle \neq 0$ for all I. Thus, in an $\mathcal{N}$-extended global supersymmetric theory, either all or no supersymmetry is spontaneously broken.

Obviously, this does not apply to *local* supersymmetry, because the Hilbert space metric is not positive definite. For the *rigid* case a loophole for this statement is to use the supercurrent algebra, and the most general form is

$$\{\bar{Q}^J_{\dot{\alpha}}, \mathcal{S}^m_{\alpha I}(x)\} = 2(\sigma^n)_{\alpha\dot{\alpha}}\delta^J_I T^m_n(x) + (\sigma^m)_{\alpha\dot{\alpha}} C_I{}^J \tag{1·1}$$

where $\mathcal{S}^m_{\alpha I}$ are extended supercurrents, T^m_n is the stress-energy tensor and $C_I{}^J$ is a field independent constant matrix, which is permitted by the constraint for the Jacobi

*) Talk given by H.I..

identities.[1] The new term does not modify the supersymmetry algebra on the fields. Partial supersymmetry breaking discussed in the present paper corresponds to this case.

Besides active researches on the *non-linear* realization of extended supersymmetry in the partially broken phase, a model in which *linearly* realized $\mathcal{N}=2$ supersymmetry is partially broken to $\mathcal{N}=1$ spontaneously was given by Antoniadis, Partouche and Taylor (APT)[2] (see also 3)4)5)). APT model is $\mathcal{N}=2$ supersymmetric, self-interacting $U(1)$ model with one (or several) abelian $\mathcal{N}=2$ vector multiplet(s) $\mathcal{A}$[6] with electric & magnetic Fayet-Iliopoulos (FI) terms. In 7)8), we have generalized this model to the $U(N)$ gauge model and shown that the $\mathcal{N}=2$ supersymmetry is partially broken to $\mathcal{N}=1$ spontaneously. Further in 9), we have analyzed the vacua with broken gauge symmetry and revealed the $\mathcal{N}=1$ supermultiplets on the vacua. In addition, a manifestly $\mathcal{N}=2$ supersymmetric formulation of the $U(N)$ gauge model coupled with/without $\mathcal{N}=2$ hypermultiplets was given in 10) by using unconstrained $\mathcal{N}=2$ superfields on harmonic superspace.[11] We introduce the magnetic Fayet-Iliopoulos term so as to shift the auxiliary field in $\mathcal{N}=2$ vector multiplet by an imaginary constant. We find that in presence of hypermultiplets in the fundamental representation of $U(N)$, the magnetic FI term develops an additional term which overcomes the difficulty[4][5] in coupling fundamental hypermultiplets with the APT model. In these models, the renormalizability is not imposed and the prepotential $\mathcal{F}$ appears from the beginning. Thus, our model should be regarded as a low-energy effective action for systems given by $\mathcal{N}=2$ bare actions spontaneously broken to $\mathcal{N}=1$. See 12)13)14)15) for related references.

This paper is organized as follows. In the next section, $\mathcal{N}=2$ $U(N)$ gauge model is constructed by requiring $\mathcal{N}=1$ $U(N)$ gauge model to be $\mathfrak{R}$-invariant. The $\mathfrak{R}$-action is composed of the discrete element of the $SU(2)$ R-symmetry, the automorphism of $\mathcal{N}=2$, and a sign flip of the FI D-term. $\mathcal{N}=2$ supersymmetry transformations are given in section 3. In section 4, we analyze the vacua of the model, and find that $\mathcal{N}=2$ supersymmetry and the $U(N)$ gauge symmetry are partially broken to $\mathcal{N}=1$ and $\prod_i U(N_i)$, respectively. We clarify the mass spectrum of the model, and reveal the $\mathcal{N}=1$ supermultiplets on the vacua in section 5. In section 6, we discuss the $\mathcal{N}=2$ supercurrents and the "central charge" in (1·1). The last section is devoted to a manifestly $\mathcal{N}=2$ supersymmetric formulation of the $U(N)$ gauge model coupled with $\mathcal{N}=2$ hypermultiplets, and we show that $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=1$.

§2. $\mathcal{N}=2$ $U(N)$ gauge model

We introduce an $\mathcal{N}=1$ chiral superfield, $\Phi(x^m,\theta)=\sum_{a=0}^{N^2-1}\Phi^a t_a$, where $N\times N$ hermitian matrices t_a $(a=0,...,N^2-1)$ generate $u(N)$ algebra, $[t_a,t_b]=if_{ab}^c t_c$, $\mathrm{tr}(t_a t_b)=\frac{1}{2}\delta_{ab}$, ($t_0$ generates the overall u(1)). The kinetic term for A ($\Phi \ni (A,\psi,F)$) we use is given by the Kähler potential for the special Kähler geometry

$$\mathcal{L}_K = \int d^2\theta d^2\bar{\theta}\ K(\Phi^a, \Phi^{*a}), \qquad K = \frac{i}{2}(\Phi^a \mathcal{F}_a^* - \Phi^{*a}\mathcal{F}_a), \tag{2·1}$$

where $\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial \Phi^a}$ and $\mathcal{F}$ is an analytic function of Φ. The Kähler metric $g_{ab^*} = \partial_a \partial_{b^*} K| = \mathrm{Im}\,\mathcal{F}_{ab}|$ admits $U(N)$ isometry generated by holomorphic Killing vectors $k_a = {k_a}^b \partial_b$ and $k_a^* = k_a^{*b}\partial_b$ with

$${k_a}^b = -ig^{bc^*}\partial_{c^*}\mathcal{P}_a, \quad k_a^{*b} = +ig^{cb^*}\partial_c \mathcal{P}_a, \tag{2·2}$$

where $\mathcal{P}_a$ is called as the Killing potential. In the present case, $\mathcal{P}_a$ is give by

$$\mathcal{P}_a = -\frac{1}{2}(\mathcal{F}_b f_{ac}^b A^{*c} + \mathcal{F}_b^* f_{ac}^b A^c) \ . \tag{2·3}$$

A^a and $\mathcal{F}_b$ transform in the adjoint representation of $U(N)$

$$k_a^c \partial_c A^b = f_{ac}^b A^c, \quad k_a^c \partial_c \mathcal{F}_b = -f_{ab}^c \mathcal{F}_c \ . \tag{2·4}$$

To gauge this isometry, we introduce an $\mathcal{N}=1$ vector superfields, $V(x^m, \theta, \bar{\theta}) = \sum_{a=0}^{N^2-1} V^a t_a$, $V^a \ni (v_m^a, \lambda^a, D^a)$. The $U(N)$ gauging is accomplished by adding[18)19)]

$$\mathcal{L}_\Gamma = \int d^2\theta d^2\bar{\theta}\ \Gamma, \qquad \Gamma = \left[\int_0^1 d\alpha\ e^{\frac{i}{2}\alpha v^a (k_a - k_a^*)} v^c \mathcal{P}_c\right]_{v^a \to V^a} . \tag{2·5}$$

For the gauge kinetic term, we introduce

$$\mathcal{L}_{\mathcal{W}^2} = -\frac{i}{4}\int d^2\theta\ \tau_{ab}\, \mathcal{W}^a \mathcal{W}^b + c.c. \qquad \mathcal{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V = \mathcal{W}_\alpha^a t_a \tag{2·6}$$

where $\tau_{ab}(\Phi)$ is an analytic function of Φ. In addition, we introduce a gauge invariant superpotential term and the FI D-term[20)]

$$\mathcal{L}_W = \int d^2\theta\ W(\Phi) + c.c.\ , \qquad \mathcal{L}_D = \xi \int d^2\theta d^2\bar{\theta} V^0 = \sqrt{2}\xi D^0. \tag{2·7}$$

In summary, the total Lagrangian of the $\mathcal{N}=1$ $U(N)$ gauge model is

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_\Gamma + \mathcal{L}_{\mathcal{W}^2} + \mathcal{L}_W + \mathcal{L}_D\ . \tag{2·8}$$

The auxiliary fields are evaluated as

$$\begin{aligned}
D^a &= \hat{D}^a - (\tau_2^{-1})^{ab}\left(\tfrac{1}{2}\mathcal{P}_b + \sqrt{2}\xi\delta_b^0\right), & \hat{D}^a &= -\tfrac{\sqrt{2}}{4}(\tau_2^{-1})^{ab}\left(\partial_d \tau_{bc}\psi^d\lambda^c + \partial_{d^*}\tau_{bc}^*\bar{\psi}^d\bar{\lambda}^c\right),\\
F^a &= \hat{F}^a - g^{ab^*}\partial_{b^*}W^*, & \hat{F}^a &= -g^{ab^*}\left(-\tfrac{i}{4}\partial_{b^*}\tau_{cd}^*\bar{\lambda}^c\bar{\lambda}^d - \tfrac{1}{2}g_{cb^*,d}\psi^c\psi^d\right),\\
F^{*a} &= \hat{F}^{*a} - g^{ba^*}\partial_b W\ , & \hat{F}^{*a} &= -g^{ba^*}\left(\tfrac{i}{4}\partial_b\tau_{cd}\lambda^c\lambda^d - \tfrac{1}{2}g_{bc^*,d^*}\bar{\psi}^c\bar{\psi}^d\right)\ ,
\end{aligned} \tag{2·9}$$

where $(\tau_2)_{ab} = \mathrm{Im}\,\tau_{ab}$, and $\hat{D}^a$, $\hat{F}^a$ and $\hat{F}^{*a}$ are fermion bilinear terms. Eliminating auxiliary fields by using the above expressions and defining covariant derivatives by

$$\begin{aligned}
&\mathcal{D}_m \Psi^a = \partial_m \Psi^a - \frac{1}{2}f_{bc}^a v_m^b \Psi^c\ , \quad \Psi = \{A, \psi, \lambda\},\\
&\mathcal{D}'_m \psi^a = \mathcal{D}_m \psi^a + \Gamma_{bc}^a \mathcal{D}_m A^b \psi^c\ , \qquad v_{mn}^a = \partial_m v_n^a - \partial_n v_m^a - \frac{1}{2}f_{bc}^a v_m^b v_n^c\ ,
\end{aligned} \tag{2·10}$$

the total action is summarized as $\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm pot} + \mathcal{L}_{\rm Pauli} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm fermi^4}$ with

$$\begin{aligned}
\mathcal{L}_{\rm kin} &= -g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{1}{4}(\tau_2)_{ab} v^a_{mn} v^{bmn} - \frac{1}{8}{\rm Re}\,\tau_{ab}\,\epsilon^{mnpq} v^a_{mn} v^b_{pq} \\
&\quad + [-\frac{1}{2}\tau_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{i}{2}g_{ab^*}\psi^a\sigma^m\mathcal{D}'_m\bar{\psi}^b \quad + \quad c.c.]\ , \qquad (2\cdot11)\\
\mathcal{L}_{\rm pot} &= -\frac{1}{2}\left(\tau_2^{-1}\right)^{ab}\left(\frac{1}{2}\mathcal{P}_a + \sqrt{2}\xi\delta^0_a\right)\left(\frac{1}{2}\mathcal{P}_b + \sqrt{2}\xi\delta^0_b\right) - g^{ab^*}\partial_a W \partial_{b^*} W^*, \qquad (2\cdot12)\\
\mathcal{L}_{\rm Pauli} &= [-i\frac{\sqrt{2}}{8}\partial_c\tau_{ab}\psi^c\sigma^n\bar{\sigma}^m\lambda^a v^b_{mn} \quad + \quad c.c.]\ , \qquad (2\cdot13)\\
\mathcal{L}_{\rm Yukawa} &= [-\frac{1}{2}\left(\partial_a\partial_b W - g^{cd^*}\partial_c W g_{ad^*,b}\right)\psi^a\psi^b - \frac{i}{4}g^{cd^*}\partial_{d^*}W^*\partial_c\tau_{ab}\lambda^a\lambda^b \quad + \quad c.c.] \\
&\quad + [\left(\frac{1}{\sqrt{2}}g_{ac^*}k^{*c}_b - \frac{\sqrt{2}}{4}\left(\tau_2^{-1}\right)^{cd}\left(\frac{1}{2}\mathcal{P}_d + \sqrt{2}\xi\delta^0_d\right)\partial_a\tau_{bc}\right)\psi^a\lambda^b \quad + \quad c.c.]\ , \\
&\qquad (2\cdot14)\\
\mathcal{L}_{\rm fermi^4} &= [-\frac{i}{8}\partial_c\partial_d\tau_{ab}\psi^c\psi^d\lambda^a\lambda^b \quad + \quad c.c.] \qquad (2\cdot15)\\
&\quad - \frac{1}{16}\left(\tau_2^{-1}\right)^{ab}\left(\partial_d\tau_{ac}\psi^d\lambda^c + \partial_{d^*}\tau^*_{ac}\bar{\psi}^d\bar{\lambda}^c\right)\left(\partial_f\tau_{be}\psi^f\lambda^e + \partial_{f^*}\tau^*_{be}\bar{\psi}^f\bar{\lambda}^e\right) \\
&\quad - g^{ab^*}\left(\frac{i}{4}\partial_a\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{ac^*,d^*}\bar{\psi}^c\bar{\psi}^d\right)\left(-\frac{i}{4}\partial_{b^*}\tau^*_{ef}\bar{\lambda}^e\bar{\lambda}^f - \frac{1}{2}g_{eb^*,f}\psi^e\psi^f\right).
\end{aligned}$$

Now, we require that the action is invariant under $\mathfrak{R}$-action, $\mathfrak{R} : S \to S$. The $\mathfrak{R}$-action is composed of a discrete element of the $SU(2)$ R-symmetry, automorphism of $\mathcal{N} = 2$, and a sign flip of the FI parameter

$$R : \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix} \longrightarrow \begin{pmatrix} \psi^a \\ -\lambda^a \end{pmatrix} \qquad \& \qquad R_\xi : \xi \to -\xi\ , \qquad (2\cdot16)$$

so that $S^{(+\xi)} \xrightarrow{R} S^{(-\xi)} \xrightarrow{R_\xi} S^{(+\xi)}$ where we have made the sign of the FI parameter manifest. This ensures the $\mathcal{N} = 2$ supersymmetry of our action as follows (see also Appendix A in 7)). By construction, the action is invariant under the first supersymmetry $\delta_{\eta_1} S^{(+\xi)} = 0$. Acting $\mathfrak{R}$ on it, we have

$$\delta_{\eta_1} S^{(+\xi)} = 0 \quad \xrightarrow{R} \quad R(\delta_{\eta_1}) S^{(-\xi)} = 0 \quad \xrightarrow{R_\xi} \quad \mathfrak{R}(\delta_{\eta_1}) S^{(+\xi)} = 0\ , \qquad (2\cdot17)$$

which implies that the resulting $\mathfrak{R}$-invariant action is invariant under the second supersymmetry $\delta_{\eta_2} \equiv \mathfrak{R}(\delta_{\eta_1})$ as well.

In 7), we find that the action is invariant under $\mathfrak{R}$-action, and thus $\mathcal{N} = 2$ supersymmetric, if

$$\tau_{ab} = \mathcal{F}_{ab}\ , \quad W = eA^0 + m\mathcal{F}_0\ . \qquad (2\cdot18)$$

The $\mathfrak{R}$-action on the auxiliary fields are

$$F^a + g^{ac^*}\partial_{c^*}W^* \ \to\ F^{*b} + g^{db^*}\partial_d W\ ,\quad D^c + \frac{1}{2}g^{cd}\mathcal{P}_d \ \to\ -(D^c + \frac{1}{2}g^{cd}\mathcal{P}_d) \qquad (2\cdot19)$$

or equivalently, $\hat{F}^a \to \hat{F}^{*a}$, $\hat{D}^a \to -\hat{D}^a$, which are consistent with the $\mathfrak{R}$-action on the fermions.

§3. $\mathcal{N}=2$ supersymmetry transformation

We construct the second supersymmetry transformation by applying the $\mathfrak{R}$-action on the first supersymmetry transformation

$$\begin{aligned}&\delta_{\eta_1} A^a = \sqrt{2}\eta_1\psi^a \ , \quad \delta_{\eta_1}\psi^a = i\sqrt{2}\sigma^m\bar{\eta}_1\mathcal{D}_m A^a + \sqrt{2}\eta_1 F^a \ ,\\ &\delta_{\eta_1} v_m^a = i\eta_1\sigma_m\bar{\lambda}^a - i\lambda^a\sigma_m\bar{\eta}_1 \ , \quad \delta_{\eta_1}\lambda^a = \sigma^{mn}\eta_1 v_{mn}^a + i\eta_1 D^a \ .\end{aligned} \tag{3·1}$$

The $\mathfrak{R}$-action on the fields is summarized as

$$\boldsymbol{\lambda}_I{}^a \equiv \begin{pmatrix}\lambda^a\\ \psi^a\end{pmatrix} \longrightarrow \begin{pmatrix}\psi^a\\ -\lambda^a\end{pmatrix} \equiv \boldsymbol{\lambda}^{Ia} = \epsilon^{IJ}\boldsymbol{\lambda}_J{}^a \ , \qquad \epsilon^{12} = \epsilon_{21} = 1 \ , \tag{3·2}$$

$$F^a = \hat{F}^a - g^{ab^*}\partial_{b^*}W^* \longrightarrow \hat{F}^{*a} - g^{ab^*}\partial_{b^*}W^* \ , \tag{3·3}$$

$$D^a = \hat{D}^a - (\tau_2^{-1})^{ab}(\frac{1}{2}\mathcal{P}_b + \sqrt{2}\xi\delta_b^0) \longrightarrow -\hat{D}^a - (\tau_2^{-1})^{ab}(\frac{1}{2}\mathcal{P}_b - \sqrt{2}\xi\delta_b^0) \tag{3·4}$$

while A^a and v_m^a are $\mathfrak{R}$-invariant. For $\boldsymbol{\delta} A^a$ and $\boldsymbol{\delta} v_m^a$ ($\boldsymbol{\delta} = \delta_{\eta_1} + \delta_{\eta_2}$) to be $\mathfrak{R}$-invariant, the supersymmetry parameters η_1 and η_2 must form a doublet $\boldsymbol{\eta}_I \equiv \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \to \begin{pmatrix}\eta_2\\ -\eta_1\end{pmatrix} \equiv \boldsymbol{\eta}^I = \epsilon^{IJ}\boldsymbol{\eta}_J$. As a result, we obtain the $\mathcal{N}=2$ supersymmetry transformation

$$\boldsymbol{\delta} A^a = \sqrt{2}\boldsymbol{\eta}_J\boldsymbol{\lambda}^{Ja} \ , \tag{3·5}$$

$$\boldsymbol{\delta} v_m^a = i\boldsymbol{\eta}_J\sigma_m\bar{\boldsymbol{\lambda}}^{Ja} - i\boldsymbol{\lambda}_J^a\sigma_m\bar{\boldsymbol{\eta}}^J \ , \tag{3·6}$$

$$\boldsymbol{\delta}\boldsymbol{\lambda}_J^a = (\sigma^{mn}\boldsymbol{\eta}_J)v_{mn}^a + \sqrt{2}i(\sigma^m\bar{\boldsymbol{\eta}}_J)\mathcal{D}_m A^a + i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_J{}^K\boldsymbol{\eta}_K - \frac{1}{2}\boldsymbol{\eta}_J f^a_{\ bc}A^{*b}A^c, \tag{3·7}$$

where $\boldsymbol{\tau}$ are Pauli matrices, and 3-dimensional vector $\boldsymbol{D}^a$ is given by

$$\boldsymbol{D}^a = \hat{\boldsymbol{D}}^a - \sqrt{2}g^{ab^*}\partial_{b^*}\left(\boldsymbol{\mathcal{E}}A^{*0} + \boldsymbol{\mathcal{M}}\mathcal{F}_0^*\right), \tag{3·8}$$

$$\hat{\boldsymbol{D}}^a = (\hat{D}_1^a,\ \hat{D}_2^a,\ \hat{D}_3^a) = (\sqrt{2}\mathrm{Im}\,\hat{F}^a, -\sqrt{2}\mathrm{Re}\,\hat{F}^a, \hat{D}^a), \tag{3·9}$$

$$\boldsymbol{\mathcal{E}} = (0,\ -e,\ \xi) \ , \quad \boldsymbol{\mathcal{M}} = (0,\ -m,\ 0) \ . \tag{3·10}$$

This would be $SU(2)$ covariant if $\boldsymbol{D}^a$ transformed as a triplet. In reality, the rigid $SU(2)$ has been gauge fixed by making $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{M}}$ point to specific directions.

Under the symplectic transformation, $\Omega = \begin{pmatrix}A^0\\ \mathcal{F}_0\end{pmatrix} \to \Lambda\Omega$, $\Lambda \in Sp(2,\mathbb{R})$, $\begin{pmatrix}-\mathcal{M}\\ \mathcal{E}\end{pmatrix}$ changes to $\begin{pmatrix}-\mathcal{M}'\\ \mathcal{E}'\end{pmatrix} = \Lambda^{-1}\begin{pmatrix}-\mathcal{M}\\ \mathcal{E}\end{pmatrix}$. So the electric and magnetic charges are interchanged $(\mathcal{E}',\mathcal{M}') = (\mathcal{M},-\mathcal{E})$ when $\Lambda = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. This explains the name of the electric and magnetic FI terms (see §7).

The $\boldsymbol{D}^a$ is not real but complex, Im $\boldsymbol{D}^a = \delta^a{}_0(-\sqrt{2}m)\,(0,1,0)$. As is seen in subsection 4.2, this is necessary for the partial supersymmetry breaking.

§4. Analysis of vacua

We examine vacua of the model and find that the $\mathcal{N}=2$ supersymmetry and the $U(N)$ gauge symmetry are partially broken to $\mathcal{N}=1$ and $\prod_i U(N_i)$, respectively.

The scalar potential of our model is given by $\mathcal{V}\equiv -\mathcal{L}_{\rm pot}$

$$\mathcal{V}=\frac{1}{8}g_{ab}\mathcal{P}^a\mathcal{P}^b+\frac{1}{2}g_{ab}\widetilde{\boldsymbol{D}}^a\cdot\widetilde{\boldsymbol{D}}^{*b}\ ,\quad \left\{\begin{array}{l}\mathcal{P}^a\equiv g^{ab}\mathcal{P}_b=-if^a_{bc}A^{*b}A^c\ ,\\ \widetilde{\boldsymbol{D}}^a\equiv-\sqrt{2}g^{ab^*}\partial_{b^*}(\boldsymbol{\mathcal{E}}A^{*0}+\boldsymbol{\mathcal{M}}\mathcal{F}^*_0)\\ \qquad=\sqrt{2}g^{ab^*}(0,\partial_{b^*}W^*,-\xi\delta^0_b)\ .\end{array}\right. \quad (4\cdot1)$$

We require the positivity of the metric. The first term vanish when $\langle A^r\rangle=0$, where $A=A^at_a\equiv A^it_i+A^rt_r$ with $t_i(t_r)\in$ (non-)Cartan. The vacua are specified by[8)]

$$\langle\frac{\partial\mathcal{V}}{\partial A^a}\rangle=\frac{i}{4}\langle\mathcal{F}_{abc}\boldsymbol{D}^b\cdot\boldsymbol{D}^c\rangle=0\ . \quad (4\cdot2)$$

This determines the vacuum expectation value $\langle\!\langle A^i\rangle\!\rangle$.

Let $E_{\underline{ij}}$ ($\underline{i},\underline{j}=1,\cdots,N$) be the fundamental matrix, which has 1 at the $(\underline{i},\underline{j})$-component and 0 otherwise. u(N) generators are given by

$$\text{Cartan}: H_{\underline{i}}=E_{\underline{ii}}\ ,\qquad \mathrm{tr}(H_{\underline{i}})^2=1 \quad (4\cdot3)$$

$$\text{non-Cartan}:\left\{\begin{array}{l}E^+_{\underline{ij}}=\frac{1}{2}(E_{\underline{ij}}+E_{\underline{ji}})\\ E^-_{\underline{ij}}=-\frac{i}{2}(E_{\underline{ij}}-E_{\underline{ji}})\end{array}\right.\quad(\underline{i}\neq\underline{j})\ ,\quad E^\pm_{\underline{ij}}=\pm E^\pm_{\underline{ij}}\ ,\ \mathrm{tr}(E^\pm_{\underline{ij}})^2=\frac{1}{2} \quad (4\cdot4)$$

and A is expanded as $A=A^it_i+A^rt_r\ =A^{\underline{i}}H_{\underline{i}}+\frac{1}{2}(A^{\underline{ij}}_+E^+_{\underline{ij}}+A^{\underline{ij}}_-E^-_{\underline{ij}})$ with $A^{\underline{ij}}_\pm=\pm A^{\underline{ij}}_\pm$. The ordinary Cartan generators t_i and $H_{\underline{i}}$ above are related by $t_i=O_i{}^{\underline{j}}H_{\underline{j}}$.

For concreteness, we consider the prepotential

$$\mathcal{F}=\sum_{\ell=0}\frac{C_\ell}{\ell!}\mathrm{Tr}\Phi^\ell\ . \quad (4\cdot5)$$

Non-vanishing $\langle\mathcal{F}_{ab}\rangle$ and $\langle\mathcal{F}_{abc}\rangle$ are

$$\langle\mathcal{F}_{\underline{ii}}\rangle\ ,\quad\langle\mathcal{F}_{\pm\underline{ij},\pm\underline{ij}}\rangle\equiv\langle\partial^2\mathcal{F}/\partial\Phi^{\underline{ij}}_\pm\partial\Phi^{\underline{ij}}_\pm\rangle\ ,\quad\langle\mathcal{F}_{\underline{iii}}\rangle\ ,\quad\langle\mathcal{F}_{\underline{i},\pm\underline{ij},\pm\underline{ij}}\rangle\ ,\quad\langle\mathcal{F}_{\underline{j},\pm\underline{ij},\pm\underline{ij}}\rangle\ , \quad (4\cdot6)$$

and so the metric $\langle g_{ab}\rangle$ *is diagonal.* The vacuum condition is reduced to

$$0=\langle\mathcal{F}_{\underline{iii}}\boldsymbol{D}^{\underline{i}}\cdot\boldsymbol{D}^{\underline{i}}\rangle\qquad\forall\underline{i} \quad (4\cdot7)$$

because $\langle\boldsymbol{D}^r\rangle\sim\langle g^{rs}(\boldsymbol{\mathcal{E}}\delta^0_s+\mathcal{M}\mathcal{F}^*_{0s})\rangle=0$. The points specified by $\langle\mathcal{F}_{\underline{iii}}\rangle=0$ are not stable vacua because $\langle\partial_{\underline{i}}\partial_{\underline{i}^*}\mathcal{V}\rangle=0$. At the stable vacua, we have

$$\langle\boldsymbol{D}_{\underline{i}}\cdot\boldsymbol{D}_{\underline{i}}\rangle=0\quad\text{where}\quad\langle\boldsymbol{D}_{\underline{i}}\rangle=O_{\underline{i}}{}^j\langle\boldsymbol{D}_j\rangle=\frac{2}{\sqrt{N}}\left(0,\,e+\frac{1}{2}m\langle\mathcal{F}^*_{\underline{ii}}\rangle,\,-\xi\right)\ . \quad (4\cdot8)$$

We have determined the vacuum expectation values $\langle\!\langle\ \rangle\!\rangle$

$$\langle\!\langle\mathcal{F}_{\underline{ii}}\rangle\!\rangle=-2\left(\frac{e}{m}\pm i\frac{\xi}{m}\right) \quad (4\cdot9)$$

and thus

$$\langle\!\langle g_{\underline{ii}}\rangle\!\rangle = \mp 2\frac{\xi}{m} \ , \qquad \langle\!\langle \boldsymbol{D}^{\underline{i}}\rangle\!\rangle = \frac{m}{\sqrt{N}}(0,-i,\pm 1) \ . \tag{4·10}$$

The positivity of the metric implies $\mp\frac{\xi}{m} > 0$, so that on the vacua, we have

$$\langle\!\langle \mathcal{V}\rangle\!\rangle = \mp 2m\xi = 2|m\xi|. \tag{4·11}$$

4.1. *Gauge symmetry breaking*

Let $\langle\!\langle A\rangle\!\rangle$ be

$$\langle\!\langle A\rangle\!\rangle = \mathrm{diag}(\overbrace{\lambda^{(1)},\cdots,\lambda^{(1)}}^{N_1},\overbrace{\lambda^{(2)},\cdots,\lambda^{(2)}}^{N_2},\cdots) \ , \qquad \sum_i N_i = N \tag{4·12}$$

where $\lambda^{(k)}$ are complex roots of (4·9), then $U(N)$ is broken to $\prod_i U(N_i)$, because

$$[E^{\pm}_{\underline{jk}}, \langle A\rangle] = \mp i(\lambda^{\underline{j}} - \lambda^{\underline{k}})E^{\mp}_{\underline{jk}} \ . \tag{4·13}$$

Unbroken $\Pi_i U(N_i)$ is generated by $t_\alpha \in \{t_a|[t_a,\langle\!\langle A\rangle\!\rangle] = 0\}$, while broken ones by $t_\mu \in \{t_a|[t_a,\langle\!\langle A\rangle\!\rangle] \neq 0\}$. As will be seen in the next section, the mass spectrum is expressed in terms of unbroken t_α and broken t_μ, only. For later use, we note that for a given t_μ, there exists a unique $t_{\tilde{\mu}}$ such that $[t_\mu, \langle A\rangle] \sim t_{\tilde{\mu}}$, which implies that $f_{\mu\underline{i}}^{\tilde{\mu}}\lambda^{\underline{i}} = -f_{\tilde{\mu}\underline{i}}^{\mu}\lambda^{\underline{i}}$.

4.2. *Partial supersymmetry breaking*

The supersymmetry transformation of fermions is reduced on the vacua to

$$\langle\!\langle \boldsymbol{\delta\lambda}_I{}^{\underline{i}}\rangle\!\rangle = i\langle\!\langle(\boldsymbol{\tau}\cdot\boldsymbol{D}^{\underline{i}})_I{}^J\rangle\!\rangle\boldsymbol{\eta}_J \qquad \text{while} \qquad \langle\!\langle \boldsymbol{\delta\lambda}_I{}^r\rangle\!\rangle = 0 \tag{4·14}$$

because $\langle\!\langle \boldsymbol{D}^r\rangle\!\rangle = 0$. Note that $\langle\!\langle \det(\boldsymbol{\tau}\cdot\boldsymbol{D}^{\underline{i}})\rangle\!\rangle = -\langle\!\langle \boldsymbol{D}^{\underline{i}}\cdot\boldsymbol{D}^{\underline{i}}\rangle\!\rangle = 0$, thus supersymmetry is partially broken on the vacua. In fact

$$\langle\!\langle \frac{1}{\sqrt{2}}\boldsymbol{\delta}(\lambda^{\underline{i}} \pm \psi^{\underline{i}})\rangle\!\rangle = \pm im\sqrt{\frac{2}{N}}(\eta_1 \mp \eta_2) \ , \qquad \langle\!\langle \frac{1}{\sqrt{2}}\boldsymbol{\delta}(\lambda^{\underline{i}} \mp \psi^{\underline{i}})\rangle\!\rangle = 0 \ . \tag{4·15}$$

The former implies that $\langle\!\langle \frac{1}{\sqrt{2}}\boldsymbol{\delta}(\lambda^i \pm \psi^i)\rangle\!\rangle = \pm 2im\delta^i_0(\eta_1 \mp \eta_2)$. As will be seen soon, $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ is massless and thus is the Nambu-Goldstone (NG) fermion for the partial breaking of $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$. We note that partial supersymmetry breaking is possible only if $\mathrm{Im}\,\langle\!\langle \boldsymbol{D}^{\boldsymbol{a}}\rangle\!\rangle \neq 0$.

§5. Mass spectrum

5.1. *fermion mass spectrum*

We find that the fermion mass term is reduced to

$$\langle\!\langle \mathcal{L}_{\mathrm{Yukawa}}\rangle\!\rangle = \frac{1}{2}\boldsymbol{\lambda}^{\alpha I}(M_{\alpha\alpha})_I{}^J\boldsymbol{\lambda}^{\alpha}_J + \frac{1}{2}\boldsymbol{\lambda}^{\mu I}(M_{\mu\nu})_I{}^J\boldsymbol{\lambda}^{\nu}_J \quad + c.c. \tag{5·1}$$

$$M_{\alpha\alpha} = \frac{\sqrt{N}}{2}\langle\!\langle \mathcal{F}_{0\alpha\alpha}\rangle\!\rangle(\boldsymbol{\tau}\cdot\langle\!\langle \boldsymbol{D}^\alpha\rangle\!\rangle) = \frac{m}{2}\langle\!\langle \mathcal{F}_{0\alpha\alpha}\rangle\!\rangle \begin{pmatrix} \pm 1 & -1 \\ 1 & \mp 1 \end{pmatrix} , \tag{5·2}$$

$$M_{\mu\tilde{\mu}} = -M_{\tilde{\mu}\mu} = \frac{1}{\sqrt{2}}\langle\!\langle g_{\mu\mu}\rangle\!\rangle f^{\mu}_{\tilde{\mu}\underline{k}}\lambda^{*\underline{k}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5·3}$$

Because $\det M_{\alpha\alpha} = 0$, the fermions $\boldsymbol{\lambda}^\alpha_J$ contain massless modes, while all of the fermions $\boldsymbol{\lambda}^\mu_J$ are massive. Taking the normalization of the kinetic terms into account, we find fermion masses on the vacua

field	mass	label	# of polarization states
$\frac{1}{\sqrt{2}}(\lambda^\alpha \pm \psi^\alpha)$	0	A	$2d_u$
$\frac{1}{\sqrt{2}}(\lambda^\alpha \mp \psi^\alpha)$	$\lvert m\langle\!\langle g^{\alpha\alpha}\mathcal{F}_{0\alpha\alpha}\rangle\!\rangle\rvert$	B	$2d_u$
$\boldsymbol{\lambda}^\mu_I$	$\frac{1}{\sqrt{2}}\lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{*\underline{i}}\rvert$	C	$4(N^2 - d_u)$

(5·4)

where $d_u \equiv \dim\prod_i U(N_i)$. We obtain the NG fermion $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ associated with the overall $U(1)$ part.

5.2. *boson mass spectrum*

Gauge boson mass term emerges from the kinetic term

$$-\langle\!\langle \mathcal{L}_{\rm kin}\rangle\!\rangle = \frac{1}{4}\langle\!\langle g_{aa'}\rangle\!\rangle f^a_{bc} v^b_m \lambda^c f^{a'}_{b'c'} v^{b'}_m \lambda^{*c'} = \frac{1}{4}\lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert^2 v^\mu_m v^{m\mu} , \tag{5·5}$$

which implies that v^μ_m are massive while v^α_m massless. The scalar mass term is extracted by substituting $A^a = \langle\!\langle A^a\rangle\!\rangle + \delta A^a$ into $\mathcal{V}$

$$\langle\!\langle \partial_a\partial_{b^*}\mathcal{V}\rangle\!\rangle \delta A^a \delta A^{*b} + \frac{1}{2}\langle\!\langle \partial_a\partial_b\mathcal{V}\rangle\!\rangle \delta A^a \delta A^b + \frac{1}{2}\langle\!\langle \partial_{a^*}\partial_{b^*}\mathcal{V}\rangle\!\rangle \delta A^{*a}\delta A^{*b} \equiv \frac{1}{2}\overrightarrow{\delta A^a}^\dagger M_{ab}\overrightarrow{\delta A^b} \tag{5·6}$$

where $\overrightarrow{\delta A^a} \equiv \binom{\delta A^a}{\delta A^{*a}}$ and

$$M_{\alpha\alpha} = m^2\langle\!\langle g^{\alpha\alpha}\lvert\mathcal{F}_{0\alpha\alpha}\rvert^2\rangle\!\rangle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{5·7}$$

$$M_{\mu\mu} = \frac{1}{2}\langle\!\langle g_{\tilde{\mu}\tilde{\mu}}\rangle\!\rangle \begin{pmatrix} \lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert^2 & -(f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}})^2 \\ -(f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{*\underline{i}})^2 & \lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert^2 \end{pmatrix} = \frac{1}{2}\langle\!\langle g_{\tilde{\mu}\tilde{\mu}}\rangle\!\rangle T^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 2\lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert^2 \end{pmatrix} T . \tag{5·8}$$

The massless mode $(T\overrightarrow{\delta A^\mu})_1$ is absorbed into v^μ_m as the longitudinal mode to form massive vector fields. The resulting boson mass spectrum is summarized as

field	mass	label	# of polarization states
v^α_m	0	A	$2d_u$
A^α	$\lvert m\langle\!\langle g^{\alpha\alpha}\mathcal{F}_{0\alpha\alpha}\rangle\!\rangle\rvert$	B	$2d_u$
v^μ_m	$\frac{1}{\sqrt{2}}\lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert$	C	$3(N^2 - d_u)$
$(T\overrightarrow{A^\mu})_2$	$\frac{1}{\sqrt{2}}\lvert f^{\tilde{\mu}}_{\mu\underline{i}}\lambda^{\underline{i}}\rvert$	C	$N^2 - d_u$

(5·9)

Due to the $\mathcal{N}=1$ supersymmetry on the vacua, the modes in (5·4) and (5·9) form $\mathcal{N}=1$ multiplets as follows. First, fields labelled by A, $(\frac{1}{\sqrt{2}}(\lambda^\alpha \pm \psi^\alpha), v_m^\alpha)$, form massless $\mathcal{N}=1$ vector multiples of spin$(\frac{1}{2},1)$. Secondly, those labelled by B, $(A^\alpha, \frac{1}{\sqrt{2}}(\lambda^\alpha \mp \psi^\alpha))$, form massive $\mathcal{N}=1$ chiral multiplets of spin$(0,\frac{1}{2})$. Finally those labelled by C, $((T\overrightarrow{A^\mu})_2, \boldsymbol{\lambda}_I^\mu, v_m^\mu)$, form two massive $\mathcal{N}=1$ vector multiplets of spin$(0,\frac{1}{2},1)$. The masses for these multiplets are depicted in Figure 1.

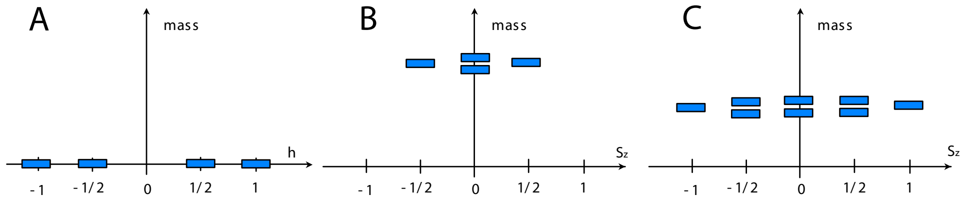

Fig. 1. $\mathcal{N}=1$ supermultiplets

§6. Supercurrents and "central charge"

$U(1)_R$ transformation is given by

$$R: \ \Phi(x,\theta,\bar\theta) \rightarrow e^{i\alpha}\Phi(x, e^{-i\frac{\alpha}{2}}\theta, \bar\theta), \quad \mathcal{W}_\alpha(x,\theta,\bar\theta) \rightarrow \mathcal{W}_\alpha(x, e^{-i\frac{\alpha}{2}}\theta, \bar\theta). \qquad (6\cdot1)$$

If $\mathcal{F}$ transforms as weight two, $R: \mathcal{F} \rightarrow e^{2i\alpha}\mathcal{F}$, then our action is invariant under R and the associated $U(1)_R$ current J^m is

$$\theta\sigma_m\bar\theta \ J^m \equiv \theta J \bar\theta = (\tau_2)_{ab}\left(\bar\theta\bar{\boldsymbol{\lambda}}^{Ia} \ \theta\boldsymbol{\lambda}_I{}^b + iA^{*a}\theta \overleftrightarrow{\mathcal{D}}_m \ \sigma^m\bar\theta A^b\right) \equiv (\tau_2)_{ab}\left(\bar\theta j^{ab}\theta\right) \ .(6\cdot2)$$

Acting a supersymmetry transformation on this we obtain $\mathcal{N}=2$ supercurrents[21,22]

$$\boldsymbol{\eta}_J\boldsymbol{S}^{(J)m} + \bar{\boldsymbol{\eta}}_J\bar{\boldsymbol{S}}^{(J)m} = -\frac{1}{2}(\tau_2)_{ab} \ \mathrm{tr}(\bar\sigma^m \, \boldsymbol{\delta} j^{ab}) \qquad (6\cdot3)$$

where tr is for spinor indices.

Though the R-current is not conserved for general $\mathcal{F}$, we can construct a conserved $\mathcal{N}=2$ supercurrent as a broken $\mathcal{N}=2$ supermultiplet of currents. We write $\mathcal{F}=\sum_n h_n C^{(n)}(A^a)$ where $C^{(n)}(A^a)$ are n-th order $U(N)$ invariant polynomials in A^a and h_n are their coefficients. First, we assign weight $(2-n)$ to h_n. Then the weight of $\mathcal{F}$ can be regarded as two. The local $U(1)_R$ variation of $\mathcal{L}$ implies

$$\partial_m\left(-\frac{1}{2}\mathrm{tr}\bar\sigma^m J\right) = i\left(\sum_n (n-2)\frac{\partial}{\partial h_n}\right)\mathcal{L} \equiv \varDelta_h\mathcal{L} \ . \qquad (6\cdot4)$$

Acting the supersymmetry transformation on it, and noting that $\boldsymbol{\delta}\mathcal{L}=\partial_m X^m$ with some X^m and that $\varDelta_h\partial_m X^m = \partial_m\varDelta_h X^m$, we obtain a general construction of the conserved $\mathcal{N}=2$ supercurrents of our model;

$$\boldsymbol{\eta}_J\boldsymbol{\mathcal{S}}^{(J)m} + \bar{\boldsymbol{\eta}}^J\bar{\boldsymbol{\mathcal{S}}}_{(J)}^{\ m} \equiv -\frac{1}{2}\mathrm{tr}(\bar\sigma^m\boldsymbol{\delta}J) - \varDelta_h X^m. \qquad (6\cdot5)$$

The term $\Delta_h X^m$ is not universal but depends on the concrete form of $\mathcal{F}$. It should be difficult to find the universal coupling to supergravity.

Further action of the supersymmetry transformation generates

$$\theta\boldsymbol{\delta}\boldsymbol{\delta}J\bar{\theta} = 8m\xi\ \bar{\theta}\bar{\boldsymbol{\eta}}\,\boldsymbol{\tau}_1\,\boldsymbol{\eta}\theta\ +\ \cdots , \tag{6·6}$$

from which we can read off the constant matrix $C_I{}^J$ in (1·1) as $C_I{}^J = +4m\xi(\boldsymbol{\tau}_1)_I{}^J$.

§7. $\mathcal{N}=2$ $U(N)$ gauge model coupled with $\mathcal{N}=2$ hypermultiplets

In this section we provide a manifestly $\mathcal{N}=2$ supersymmetric formulation of the $\mathcal{N}=2$ $U(N)$ gauge model coupled with $\mathcal{N}=2$ hypermultiplets.[10)] For this we work in harmonic superspace[11)] $\mathbb{R}^{4|8}\times S^2$ parametrized by

$$(x^m_A, \theta^\pm, \bar{\theta}^\pm, u_I^\pm) = (x^m - 2i\theta^I\sigma^m\bar{\theta}^J u^+_{(I}u^-_{J)}, \theta^I u_I^\pm, \bar{\theta}^I u_I^\pm, u_I^\pm) \tag{7·1}$$

in the analytic basis. $u_I^\pm$ are harmonic variables parametrizing $\mathrm{S}^2 = SU(2)/U(1)$

$$(u_I^+, u_I^-) \in SU(2)\ ,\quad u^{+I}u_I^- = 1\ ,\quad \overline{u^{+I}} = u_I^-\ . \tag{7·2}$$

We introduce an $\mathcal{N}=2$ vector multiplet $V^{++}(x^m,\theta^+,\bar{\theta}^+) = V^{++a}t_a$ transforming as adjoint under $U(N)$. V^{++} is composed of a complex scalar A, a vector v_m, an $SU(2)$ doublet Weyl spinor λ^i_α and an auxiliary field D^{IJ}. $D^I{}_J = \varepsilon_{JK}D^{IK} = iD^A(\tau_A)^I{}_J$ is an $SU(2)$ matrix and D^A is a real three-vector $\overline{D^A} = D^A$. By using the field strength W of V^{++} the action for V^{++} is constructed as

$$S_V = -\frac{i}{4}\int d^4x\left[(D)^4\mathcal{F}(W) - (\bar{D})^4\bar{\mathcal{F}}(\bar{W})\right]\ , \tag{7·3}$$

where $(D)^4 = \frac{1}{16}(D^+)^2(D^-)^2$ and $D^\pm$ are the spinor harmonic derivatives.[11)] A's parametrize the special Kähler geometry with the Kähler metric $g_{ab} = \mathrm{Im}\,(\mathcal{F}_{ab}|)$ where $\mathcal{F}_{ab\cdots}|$ means $\mathcal{F}_{ab\cdots}$ evaluated at $\theta^\pm = \bar{\theta}^\pm = 0$. The metric admits $U(N)$ isometry generated by Killing vectors with the Killing potential $\mathcal{P}^a = -if^a_{bc}\bar{A}^bA^c$.

The $\mathcal{N}=2$ hypermultiplet q^+ is composed of an $SU(2)$ doublet complex scalar f^I, a pair of $SU(2)$ singlet spinors and infinitely many auxiliary fields. We introduce two sets of $\mathcal{N}=2$ hypermultiplets, N_f hypermultiplets q^{+u} $(u=1,\cdots,N)$ and N_a hypermultiplets q^{+a} $(a=0,1,\cdots,N^2-1)$ which transform as fundamental representation and adjoint representation under $U(N)$, respectively. We suppress flavor indices below. The $U(N)$ gauged action is given by (ω-hypermultiplets are also included in ref.10))

$$S_q^{\mathrm{gauged}} = -\int du d\zeta^{(-4)}\left[\tilde{q}^+_u\mathcal{D}^{++}q^{+u} + \tilde{q}^+_a\mathcal{D}^{++}q^{+a}\right] \tag{7·4}$$

where the tilde denotes the analyticity preserving conjugation.[11)] The covariant derivative is defined as $\mathcal{D}^{++}q^{+\mu} = D^{++}q^{+\mu} + iV^{++a}(T_a)^\mu{}_\nu q^{+\nu}$ where D^{++} is the harmonic derivative[11)] and

$$(T_a)^\mu{}_\nu = \begin{cases}(t_a)^u{}_v & \text{for fundamental } q^+\\ \mathrm{ad}(t_a)^b{}_c = if^b_{ac} & \text{for adjoint } q^+\end{cases}\ . \tag{7·5}$$

The $U(N)$ isometry gauged above is generated by Killing vectors with Killing potentials

$$\left\{ \begin{array}{l} \hat{\mathcal{Q}}_a^{IJ} = \mathcal{Q}_a^{IJ}|_{T_a=t_a} \\ \check{\mathcal{Q}}_a^{IJ} = \mathcal{Q}_a^{IJ}|_{T_a=\mathrm{ad}(t_a)} \end{array} \right. \quad \text{where} \quad \mathcal{Q}_a^{IJ} = i\bar{f}_\mu^{(I}(T_a)^\mu{}_\nu f^{J)\nu} \; . \tag{7·6}$$

Next we introduce the electric and magnetic FI terms. The electric FI term is given by

$$S_e = \int du d\zeta^{(-4)} \mathrm{tr}(\Xi^{++}V^{++}) + c.c. = \int d^4x \xi^{IJ} D_{IJ}^0 + c.c. \tag{7·7}$$

where $\Xi^{++} = \xi^{IJ}u_I^+ u_J^+$ is the electric FI parameter. The effect of this term is to shift the dual auxiliary field D_D^{aIJ} in $W_D^a \equiv \mathcal{F}_a$ by an imaginary constant, $D_D^{aIJ} \to D_D^{aIJ} + 8i\xi_D^{IJ}\delta_0^a$. We introduce the magnetic FI term so as to shift the auxiliary field D^{aIJ} in W^a by an imaginary constant, $D^{aIJ} \to \boldsymbol{D}^{aIJ} = D^{aIJ} + 4i\xi^{IJ}\delta_0^a$. By this, the $\mathcal{N}=2$ supersymmetry transformation law $\delta_\eta \lambda^{aI} = (D^a)^I{}_J \eta^J + \cdots$ changes to $\delta_\eta \lambda^{aI} = (\boldsymbol{D}^a)^I{}_J \eta^J + \cdots$, under which the total action

$$S = S_V + S_q^{\mathrm{gauged}} + S_e + S_m \tag{7·8}$$

is invariant. It is straightforward to see that the magnetic FI term of the form

$$S_m = \int d^4x \left[(D)^4 \xi_D^{IJ} \theta_I \theta_J \left(\mathcal{F}_0 + 2i\mathcal{F}_{00} \xi_D^{KL} \theta_K \theta_L \right) + 2i\hat{\mathcal{Q}}_0^{IJ} \xi_{DIJ} \right] + c.c. \tag{7·9}$$

causes the imaginary constant shift of the auxiliary field

$$S_V + S_q^{\mathrm{gauged}} + S_m = \left(S_V + S_q^{\mathrm{gauged}} \right)\Big|_{D\to\boldsymbol{D}} \tag{7·10}$$

where $|_{D\to\boldsymbol{D}}$ means the replacement $D^{aIJ} \to \boldsymbol{D}^{aIJ}$ ($D^{aIJ} \to \bar{\boldsymbol{D}}^{aIJ}$). We find that in the presence of hypermultiplets in the fundamental representation of the gauge group $U(N)$, the magnetic FI term develops an additional term which overcomes the difficulty in coupling fundamental hypermultiplets with the APT model. The adjoint scalars do not appear in (7·9) because $\mathrm{ad}(t_0) = 0$.

It is straightforward to eliminate infinitely many auxiliary fields in q^+ and the auxiliary field D in V^{++} and obtain the scalar potential

$$\begin{aligned} \mathcal{V} = & \frac{1}{4} g_{ab} \boldsymbol{D}^{aIJ} | \bar{\boldsymbol{D}}_{IJ}^b | + g_{ab}\mathcal{P}^a\mathcal{P}^b + 2i(\xi^{IJ} + \bar{\xi}^{IJ})(\xi_{DIJ} - \bar{\xi}_{DIJ}) \\ & + \overline{f^I}_u (\bar{A}A + A\bar{A})^u{}_v f^{Iv} + \overline{f^I}_a (\bar{A}A + A\bar{A})^a{}_b f^{Ib} \end{aligned} \tag{7·11}$$

where

$$\boldsymbol{D}^{aIJ}| = -2g^{ab} \left[(\xi^{IJ} + \bar{\xi}^{IJ})\delta_b^0 + (\xi_D^{IJ} + \bar{\xi}_D^{IJ})\bar{\mathcal{F}}_{0b}| + \hat{\mathcal{Q}}_b^{IJ} + \check{\mathcal{Q}}_b^{IJ} \right] \; .$$

The vacua are determined by $\mathcal{V}$ and exhibit various phases. On the Coulomb phase $\langle\langle A^{\underline{i}} \rangle\rangle \neq 0$, $\langle\langle A^r \rangle\rangle = \langle\langle f_u^I \rangle\rangle = \langle\langle f_r^I \rangle\rangle = 0$ and thus $\langle\langle \hat{\mathcal{Q}}_b^{IJ} \rangle\rangle = \langle\langle \check{\mathcal{Q}}_b^{IJ} \rangle\rangle = 0$. In this way we have arrived at the vacuum condition for $\mathcal{N}=2$ $U(N)$ gauge model without hypermultiplets, $\langle\!\langle \partial_{A^a}\mathcal{V}\rangle\!\rangle = \frac{i}{4}\langle\!\langle \mathcal{F}_{abc}|\boldsymbol{D}^{bA}\boldsymbol{D}^{cA}\rangle\!\rangle = 0$. Let us examine the case with $\mathcal{F} = \sum_n \frac{c_n}{n!}\mathrm{tr}W^n$ for concreteness. Then it is further reduced to

$$\sum_A \langle\!\langle \boldsymbol{D}^{\underline{i}A}\boldsymbol{D}^{\underline{i}A}\rangle\!\rangle = 0\ , \quad \underline{i} = \underline{1}, \cdots, \underline{N}\ . \tag{7·12}$$

It is easy to show that by fixing $SU(2)$ appropriately $\langle\!\langle \mathcal{F}_{\underline{ii}}\rangle\!\rangle = -2(\frac{e}{m} \pm i\frac{\xi}{m})$ in (4·9) can be reproduced. On the vacua the supersymmetry transformations of fermions are found to be trivial except for

$$\langle\!\langle \delta\lambda^{\underline{i}I}\rangle\!\rangle = i\langle\!\langle \boldsymbol{D}^{\underline{i}A}\rangle\!\rangle(\tau_A)^I{}_J\eta^J\ . \tag{7·13}$$

On the other hand the mass term of $\lambda^{\underline{i}I}$ is

$$-\frac{i}{4}\langle\!\langle \mathcal{F}_{\underline{iii}}\boldsymbol{D}^{\underline{i}A}\rangle\!\rangle\lambda^{\underline{i}I}(\tau_2\tau_A)_{IJ}\lambda^{\underline{i}J}\ . \tag{7·14}$$

Because (7·12) implies that $\det \boldsymbol{D}^{\underline{i}I}{}_J = 0$, a half of the fermions $\lambda^{\underline{i}I}$, $I = 1, 2$, say $U^1{}_J\lambda^{\underline{i}J}$ with a constant matrix $U^I{}_J$, is massless but has a nontrivial supersymmetry transformation. In the ordinary basis spanned by matrices t_a, this means that $U^1{}_J\lambda^{0J}$ is the NG fermion for partial supersymmetry breaking.

Acknowledgements

This work is supported in part by the Grant-in-Aid for Scientific Research (No.16540262, No.17540262 and No.17540091) from the Ministry of Education, Science and Culture, Japan. Support from the 21 century COE program "Constitution of wide-angle mathematical basis focused on knots" is gratefully appreciated.

References

1) J. T. Lopuszanski, "The Spontaneously Broken Supersymmetry In Quantum Field Theory," Rept. Math. Phys. **13** (1978) 37.
2) I. Antoniadis, H. Partouche and T. R. Taylor, "Spontaneous Breaking of $\mathcal{N} = 2$ Global Supersymmetry," Phys. Lett. B **372** (1996) 83 [arXiv:hep-th/9512006].
3) S. Ferrara, L. Girardello and M. Porrati, "Spontaneous Breaking of N=2 to N=1 in Rigid and Local Supersymmetric Theories," Phys. Lett. B **376** (1996) 275 [arXiv:hep-th/9512180].
4) H. Partouche and B. Pioline, "Partial spontaneous breaking of global supersymmetry," Nucl. Phys. Proc. Suppl. **56B** (1997) 322 [arXiv:hep-th/9702115].
5) E. A. Ivanov and B. M. Zupnik, "Modified N = 2 supersymmetry and Fayet-Iliopoulos terms," Phys. Atom. Nucl. **62** (1999) 1043 [Yad. Fiz. **62** (1999) 1110] [arXiv:hep-th/9710236].
6) R. Grimm, M. Sohnius and J. Wess, "Extended Supersymmetry And Gauge Theories," Nucl. Phys. B **133** (1978) 275; M. F. Sohnius, "Bianchi Identities For Supersymmetric Gauge Theories," Nucl. Phys. B **136** (1978) 461; M. de Roo, J. W. van Holten, B. de Wit and A. Van Proeyen, "Chiral Superfields In N=2 Supergravity," Nucl. Phys. B **173** (1980) 175.
7) K. Fujiwara, H. Itoyama and M. Sakaguchi, "Supersymmetric $U(N)$ gauge model and partial breaking of $\mathcal{N} = 2$ supersymmetry," Prog. Theor. Phys. **133** (2005) 429 [arXiv:hep-th/0409060].
8) K. Fujiwara, H. Itoyama and M. Sakaguchi, "$U(N)$ gauge model and partial breaking of $\mathcal{N} = 2$ supersymmetry," in the proceedings of SUSY2004 [arXiv:hep-th/0410132].

9) K. Fujiwara, H. Itoyama and M. Sakaguchi, "Partial breaking of $\mathcal{N}=2$ supersymmetry and of gauge symmetry in the $U(N)$ gauge model," Nucl. Phys. B **723** (2005) 33 [arXiv:hep-th/0503113].
10) K. Fujiwara, H. Itoyama and M. Sakaguchi, "Partial Supersymmetry Breaking And $\mathcal{N}=2$ $U(N_c)$ Gauge Model With Hypermultiplets In Harmonic Superspace," Nucl. Phys. B **740** (2006) 58 [arXiv:hep-th/0510255].
11) A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, "Harmonic superspace," Cambridge University Press, 2001.
12) N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N = 2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485] [arXiv:hep-th/9407087].
13) T. R. Taylor and C. Vafa, "RR flux on Calabi-Yau and partial supersymmetry breaking," Phys. Lett. B **474** (2000) 130 [arXiv:hep-th/9912152]; P. Mayr, "On supersymmetry breaking in string theory and its realization in brane worlds," Nucl. Phys. B **593** (2001) 99 [arXiv:hep-th/0003198]; C. Vafa, "Superstrings and topological strings at large N," J. Math. Phys. **42** (2001) 2798 [arXiv:hep-th/0008142]; F. Cachazo, K. A. Intriligator and C. Vafa, "A large N duality via a geometric transition," Nucl. Phys. B **603** (2001) 3 [arXiv:hep-th/0103067]; F. Cachazo, S. Katz and C. Vafa, "Geometric transitions and N = 1 quiver theories," arXiv:hep-th/0108120; F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," Nucl. Phys. B **628** (2002) 3 [arXiv:hep-th/0110028]; J. Louis and A. Micu, "Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes," Nucl. Phys. B **635** (2002) 395 [arXiv:hep-th/0202168]; F. Cachazo and C. Vafa, "N = 1 and N = 2 geometry from fluxes," arXiv:hep-th/0206017.
14) R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644** (2002) 3 [arXiv:hep-th/0206255]; R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644** (2002) 21 [arXiv:hep-th/0207106]; R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048; H. Itoyama and A. Morozov, "The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory," Nucl. Phys. B **657** (2003) 53 [arXiv:hep-th/0211245].
15) F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP **0212**, 071 (2002) [arXiv:hep-th/0211170].
16) P. Kaste and H. Partouche, "On the equivalence of N = 1 brane worlds and geometric singularities with flux," JHEP **0411** (2004) 033 [arXiv:hep-th/0409303].
17) A. Morozov, "Challenges of matrix models," arXiv:hep-th/0502010.
18) J. Bagger and E. Witten, "The Gauge Invariant Supersymmetric Nonlinear Sigma Model," Phys. Lett. B **118**, 103 (1982).
19) C. M. Hull, A. Karlhede, U. Lindstrom and M. Rocek, "Nonlinear Sigma Models And Their Gauging In And Out Of Superspace," Nucl. Phys. B **266**, 1 (1986).
20) P. Fayet and J. Iliopoulos, "Spontaneously Broken Supergauge Symmetries And Goldstone Spinors," Phys. Lett. B **51** (1974) 461; P. Fayet, "Fermi-Bose Hypersymmetry," Nucl. Phys. B **113** (1976) 135.
21) S. Ferrara and B. Zumino, "Transformation Properties Of The Supercurrent," Nucl. Phys. B **87**, 207 (1975).
22) H. Itoyama, M. Koike and H. Takashino, "N = 2 supermultiplet of currents and anomalous transformations in supersymmetric gauge theory," Mod. Phys. Lett. A **13**, 1063 (1998) [arXiv:hep-th/9610228].