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Effective Theoretical Approach to Backreaction of the Dynamical Casimir Effect in 1+1 Dimensions

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Abstract

Backreaction of the dynamical Casimir effect is investigated by a scalar effective theory in $1+1$ dimensions. The adiabatic approximation allows us to treat the size of the space as a dynamical variable described by the Robertson-Walker type metric. The effective action for the size of the space is obtained with the background field method in the path-integral formalism. This effective action includes the contributions from the static and the dynamical Casimir effects. From our effective action we have derived the dynamical Casimir force as the backreaction. This force depends on the mirror velocity, and becomes always stronger than the static Casimir force.

1 INTRODUCTION

The Casimir effect originally suggested in 1948 has been generally regarded as the contribution of a non-trivial geometry on the vacuum fluctuations of quantum electromagnetic fields. The corresponding change in the vacuum fluctuations appears as a shift of the vacuum energy and a resulting vacuum pressure. For a standard example, when we insert two perfectly conducting parallel plates into the free space R^3, the plates are attracted towards each other [1], although being uncharged. This attractive force is experimentally confirmed by Sparnaay in 1958 [2] and recently more precise measurements have been provided [3].

The dynamical Casimir effect suggests that the non-uniform accelerative motion of the boundaries (perfect conducting plates or mirrors) excites the electromagnetic field and promotes virtual photons from the vacuum into real photons [5, 6]. The works

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on the dynamical Casimir effect are pioneered by Moore [5] and progressed by many authors [6, 7, 8]. Although the phenomena is interesting, this effect has not yet been experimentally confirmed. This process of particle production also has been predicted in a variety of general-relativistic situations, e.g., black holes [9, 10], domain wall activity in cosmology, and the high-speed collision of atomic nuclei [11].

Our interest in the dynamical Casimir effect is as follows: If moving mirrors create radiation, the mirrors should experience a radiation-reaction force. A number of authors have discussed this subject, and the forces such as negative friction have been shown in $(1+1)$-dimensional cavity [7]. In this paper we propose an effective theory for the backreaction of the dynamical Casimir effect in $1+1$-dimensions. Our effective theory is constructed by the background field method in the path-integral formalism. The motion of the mirrors described by our effective action gives us *the dynamical Casimir force* as the backreaction, which contains the negative frictional forces. For a massless scalar field in $1+1$ dimensions both the static and the dynamical Casimir effects exist. The discussion on the dynamical Casimir effect within the adiabatic approximation in $1+1$ dimensions is the same as that in $3+1$ dimensions.

The principal steps for constructing our model are as follows:

(1) We consider a massless scalar field in one-dimensional finite space with two boundaries (one-dimensional disc D^1), and require the fixed boundary condition, i.e., we consider the massless scalar field between two 'mirrors'. The size of the space $D(t)$, namely the separation of the mirrors, is a dynamical variable depending on the time.

(2) We replace the geometrical configuration from the space D^1 to the space S^1, and change the parameterization $D(t)$ from the separation of the boundaries (the size of D^1) into the size of S^1. We require that the scalar field satisfies the periodic boundary condition rather than the fixed boundary condition. This replacement is justified when the relative velocity between the mirrors is much slower than the velocity of light: $\dot{D}(t) \ll 1$. This condition on the velocity is referred to as the adiabatic condition. In the adiabatic treatment of the dynamical Casimir effect [6], there are no transitions among the oscillation modes of the adiabatic Hamiltonian. Because of this absence of the transitions, the dynamical Casimir effect in the space D^1 is the same as that in the space S^1 except for the overall factor in each adiabatic Hamiltonian. The overall factor changes into twice in this replacement. The equivalence between the original system and the replaced one means that the boundary effect is not essential in the adiabatic approximation, i.e., the dynamical Casimir effect is caused not by the direct contribution of the boundary, but by the change of field configuration in the vacuum.

(3) In order to calculate the motion of $D(t)$ as the backreaction to boundaries (mirrors), we will employ the background field method: The geometrical configuration of the space is treated classically and the scalar field in this space is a quantum object. By integrating out the scalar field, we obtain the effective action for $D(t)$ which describes the motion of the mirrors.

The paper is organized as follows. In Section 2 the general description of our model is given, and the effective action is obtained. In Section 3 the backreaction of the dynamical Casimir effect in our model is investigated, and the dynamical Casimir force is derived. Section 4 is devoted to conclusions and discussions. In Appendix it is shown that the conformal anomaly is induced by means of the Fujikawa method [12].

2 SCALAR MODEL FOR CASIMIR EFFECTS

For the purpose of describing Casimir effects in one-dimensional cavity and the reaction from the Casimir effects received by the moving mirrors, we consider a massless scalar field in the one-dimensional variable-sized space between two boundaries, i.e. D^1 (see Figure 1(a)). Suppose that the size of the space $D(t)$ is a dynamical variable and it receives all backreaction of the Casimir effects. For the adiabatic motion of the mirrors does not mix the oscillation modes with each other we can neglect the boundary effect. Then the adiabatic approximation allows us to replace the space D^1 to the space S^1 (see Figure 1(b)). After this replacement our model is defined on the $R \times S^1$ space-time, where the size of S^1 space is $D(t)$. We require the periodic boundary condition on the scalar field, then the energy levels of the adiabatic oscillation-modes in the original system are two times those in the replaced system.

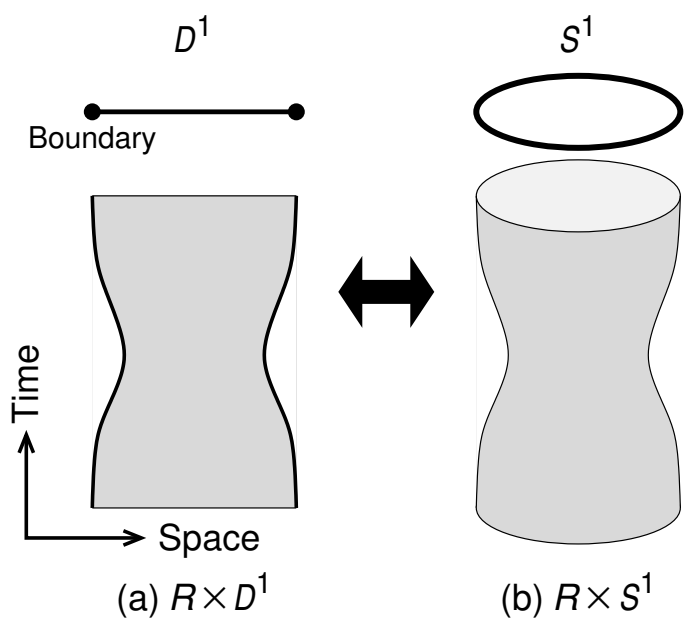

Figure 1: Space-times for the $(1+1)$-dimensional Casimir effects. **(a)** One-dimensional space with two boundaries (one-dimensional disc D^1) as the cavity between two moving 'mirrors'. The scalar field satisfies the fixed boundary condition on the edges. **(b)** S^1 space which is adiabatically equivalent for the scalar field to the geometrical configuration (a). The periodic boundary condition is imposed on the field.

In order to describe the size of S^1 varying with time, we will define the the Robertson-Walker type metric on the space-time $R \times S^1$: $ds^2 = -dt^2 + D(t)^2 dx^2$ $(0 \leq x \leq a)$, where we have redefined $D(t)$ as a dimensionless scale factor, and have introduced a dimensional constant a as a standard size of the space. The classical action of this system is given by the Polyakov action:

$$S[g_{\mu\nu},\, \phi] \;=\; -\frac{1}{2\pi}\int d^2x\,\sqrt{-g}\,\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{1}$$

This action has a conformal symmetry, i.e., this action has an invariance under the general coordinate transformation and the Weyl transformation. The existence of the conformal

symmetry of this action implies that the Casimir effects can not classically occur as follows. Here we can always rewrite down the metric to a conformally flat form by the general coordinate transformation:

$$ds^2 \;=\; -dt^2 + D(t)^2 dx^2 \;=\; -C(\eta)\left(d\eta^2 - dx^2\right) \;=\; g_{\mu\nu}\, dx^\mu dx^\nu, \tag{2}$$

where we have introduced a new coordinate η such that $d\eta \equiv dt/D(t)$ and $C(\eta) \equiv D(t(\eta))^2$. After performing the Weyl transformation $g_{\mu\nu} \to C^{-1}(\eta) g_{\mu\nu}$, we have the $D(t)$-independent flat metric:

$$ds^2 \;=\; -d\eta^2 + dx^2 \;=\; \eta_{\mu\nu}\, dx^\mu dx^\nu. \tag{3}$$

This implies that any deformation of the space-size does not affect the classical action, i.e., the Casimir effects are essentially quantum effects.

For evaluating the contribution of the Casimir effects, we present a path integral formulation. We use background field method in which the metric is treated as classical field and the scalar field is quantized. The effective action for the metric, $\Gamma[g_{\mu\nu}]$, is given by

$$e^{i\Gamma[g_{\mu\nu}]} \;=\; \int \mathcal{D}\phi\; e^{iS[g_{\mu\nu},\phi]}. \tag{4}$$

In order to calculate the effective action for any time evolving metric (2), we perform the conformal transformation on the effective action (4) from any time evolving metric (2) to the flat metric (3): $g_{\mu\nu} \to e^{2\alpha} g_{\mu\nu} = C^{-1}(\eta) g_{\mu\nu} = \eta_{\mu\nu}$. By means of the Fujikawa method [12] this conformal transformation picks up the conformal anomaly as a Jacobian factor from the path-integral measure in the effective action (4):

$$e^{i\Gamma[g_{\mu\nu}]} \;=\; \exp\left[-\frac{i}{2}\int d^2x\, \alpha(x) \sum_n \varphi_n^\dagger(x)\, \varphi_n(x)\right] \;\times\; e^{i\Gamma[\eta_{\mu\nu}]}, \tag{5}$$

where the parameter of the conformal transformation $\alpha(x)$ is chosen as $\alpha(x) = -\frac{1}{2}\ln C(\eta)$, and $\{\varphi_n(x)\}$ is a complete set which consists of the eigenfunctions of the Hamiltonian (see Appendix). Here the first exponential factor in (5) is the conformal anomaly, and the latter factor is the partition function for the free scalar field in the space S^1 with fixed radius.

2.1 CASIMIR ENERGY IN CYLINDRICAL SPACE-TIME

We will find $\Gamma[\eta_{\mu\nu}]$ induces the Casimir energy by evaluating the partition function for the free scalar field. Let us calculate the Euclidean partition function:

$$Z_E \;\equiv\; e^{-\Gamma_E[\eta_{\mu\nu}]} \;=\; \int \mathcal{D}\phi\, \exp\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty} dx^2 \int_0^a dx^1 \frac{1}{2}\,\partial\phi\,\partial\phi\right], \tag{6}$$

where we have defined the imaginary time variable $x^2 \equiv i\eta$, and have used the Euclidean inner product $\partial\phi\partial\phi \equiv \delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Since the free Lagrangian is quadratic in terms of ϕ, this integration can be performed formally, and obtains

$$\ln Z_E \;=\; -\frac{1}{2}\mathrm{Tr}\ln(\partial^2) \;=\; -\frac{1}{2}\int d^2x\, \langle x|\ln\partial^2|x\rangle. \tag{7}$$

In the momentum representation the spatial component of the momentum is discretized in the form $(2\pi n/a)$ for arbitrary integers n due to a compactness of the space.

$$\ln Z_E = -\frac{1}{2}\frac{1}{(2\pi)^2}\int d^2x \int \frac{dk}{2\pi}\frac{1}{a}\sum_n \ln\left[k^2 + (2\pi n/a)^2\right] \equiv -\frac{1}{(2\pi)^2}\int d^2x f_{\rm bare}, \quad (8)$$

where $f_{\rm bare}$ is a bare Euclidean free energy density for the massless field. For the integration over k makes f_{bare} divergent, we introduce mass M of the scalar field to regularize f_{bare} [13], then the integrand is changed as

$$\ln\left[k^2 + (2\pi n/a)^2\right] \to \ln\left[k^2 + (2\pi n/a)^2 + M^2\right]. \quad (9)$$

Employing the indefinite integral of M, we can write

$$f_{\rm bare} = \frac{1}{2}\int \frac{dk}{2\pi}\frac{1}{a}\int dM^2 \sum_n \frac{1}{k^2 + (2\pi n/a)^2 + M^2}. \quad (10)$$

The sum over n can be performed in the expression

$$f_{\rm bare} = \frac{1}{2}\int \frac{dk}{2\pi}\int dM^2 \frac{1}{2\omega_k}\left(1 + \frac{2}{e^{a\omega_k} - 1}\right), \quad (11)$$

where we have defined $\omega_k \equiv \sqrt{k^2 + M^2}$, and have used the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + b^2} = \frac{\pi}{b}\coth \pi b = \frac{\pi}{b}\left[1 + \frac{2}{e^{2\pi b} - 1}\right]. \quad (12)$$

Instead of using k and M we will employ k and ω_k as independent parameters. Substituting $dM^2 = 2\omega_k\, d\omega_k$ gives

$$f_{\rm bare} = \frac{1}{2}\int \frac{dk}{2\pi}\int d\omega_k \left(1 + 2\sum_{n=1}^{\infty} e^{-na\omega_k}\right). \quad (13)$$

For the first term indicates the contribution of infinite space and clearly diverges, we renormalize it as a cosmological term. The second term is relevant for the free energy density, namely, renormalized free energy density:

$$f_{\rm reno} \equiv -\frac{1}{a}\int \frac{dk}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, e^{-na\omega_k}. \quad (14)$$

With the identity

$$e^{-na\omega_k} = \frac{1}{\sqrt{\pi}}\int_0^{\infty} dt\; t^{-1/2}\exp\left[-t - \frac{(na\omega_k)^2}{4t}\right], \quad (15)$$

we perform the integration over k, and obtain

$$f_{\rm reno} = -\frac{M}{\pi a}\sum_{n=1}^{\infty}\frac{1}{n}K_{-1}(naM). \quad (16)$$

Here $K_\nu(z)$ is the modified Bessel function:

$$K_\nu(z) \;=\; \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} \exp\left[-t - \frac{z^2}{4t}\right]. \tag{17}$$

The free energy density for the massless field is obtained by taking the limit $M \to 0$. In this limit we can use the property of the Bessel function, $K_{-1}(z) \approx 1/z$ for small z, and the free energy density (16) becomes

$$f_{\rm reno} \;=\; -\frac{1}{\pi a^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \;=\; -\frac{\pi}{6a^2}. \tag{18}$$

Euclidean partition function is derived by substituting (18) into (8). After performing the spatial integration, and going back to the Minkowski space with $x^2 = i\eta$, we obtain

$$\Gamma[\eta_{\mu\nu}] \;=\; \frac{1}{i}\ln Z \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; \frac{1}{12}\frac{1}{aD(t)}, \tag{19}$$

where we have used the relation $d\eta = dt/D(t)$. It should be noticed that $-1/(12aD)$ is the Casimir energy in 1+1 dimensions.

2.2 CONFORMAL ANOMALY IN SPACE-TIME $R \times S^1$

We will evaluate the conformal anomaly in the space-time $R\times S^1$. It is formally expressed by the first exponent in the right hand side of (5). This anomaly part appears when the size of the space S^1 is varying with time. Then the anomaly part describes the terms of the backreaction of the dynamical Casimir effect. In the Euclidean space-time with the metric $ds^2 = \rho(x^2)\left[(dx^1)^2 + (dx^2)^2\right]$ the Jacobian induced from the conformal transformation $g_{\mu\nu} \to e^{2\alpha} g_{\mu\nu}$ is

$$J_E \;\equiv\; \exp\left[-\frac{1}{2}\int_0^a dx^1 \int_{-\infty}^{\infty} dx^2\, \alpha(x^2) \sum_{n,k} \varphi^\dagger_{n,k}(x)\, \varphi_{n,k}(x)\right], \tag{20}$$

where $\{\varphi_{n,k}(x)\}$ is a complete set of the eigenfunctions of the Hamiltonian operator:

$$\hat{H} \;=\; -\frac{1}{2}\frac{1}{\sqrt{\rho}}\,\partial\partial\,\frac{1}{\sqrt{\rho}}, \qquad \hat{H}\varphi_{n,k}(x) \;=\; \lambda^2_{n,k}\varphi_{n,k}(x). \tag{21}$$

This Jacobian will be evaluated by using the eigenfunctions $\varphi_{n,k}(x)$ which satisfy the periodic boundary condition in the space S^1.

The factor $j(x) \equiv \sum_{n,k} \varphi^\dagger_{n,k}(x)\varphi_{n,k}(x)$ in the Jacobian (20) has a divergence due to the infinite degrees of freedom of the space-time points. In order to regularize this divergence we introduce a cutoff parameter M and insert the cutoff function $\exp(-\lambda_n^2/M^2)$ into $j(x)$:

$$\begin{aligned} j(x) \;&\equiv\; \sum_{n,k} \varphi^\dagger_{n,k}(x)\varphi_{n,k}(x) \\ \to\quad j(x) \;&\equiv\; \lim_{M\to\infty}\sum_{n,k} \varphi^\dagger_{n,k}(x) e^{-\lambda^2_{n,k}/M^2}\varphi_{n,k}(x) = \lim_{M\to\infty}\sum_{n,k} \varphi^\dagger_{n,k}(x) e^{-\hat{H}/M^2}\varphi_{n,k}(x). \end{aligned}$$

Adopted the eigenfunctions $\varphi_{n,k}(x)$ with the form $\varphi_{n,k}(x) = \frac{1}{\sqrt{a}} e^{ikx^2} e^{i(2\pi n/a)x^1}$,

$$j(x) = \lim_{M\to\infty} \frac{1}{a} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[\frac{1}{2M^2}\left(-\frac{k^2+(2\pi n/a)^2}{\rho} + 2\frac{ik}{\sqrt{\rho}}\partial_2\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\rho}}\partial_2^2\frac{1}{\sqrt{\rho}}\right)\right], \quad (22)$$

where we should note that $j(x)$ is independent of x^1. With a rescaled dimensionless parameter $\tilde{k} \equiv Mk$, we can write

$$\begin{aligned} j(x^2) &= \lim_{M\to\infty} \frac{M}{a} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{k}}{2\pi} \\ &\quad \times \exp\left[-\frac{\tilde{k}^2+(2\pi n/Ma)^2}{2\rho} + \frac{i\tilde{k}}{M}\frac{1}{\sqrt{\rho}}\partial_2\frac{1}{\sqrt{\rho}} + \frac{1}{2M^2}\frac{1}{\sqrt{\rho}}\partial_2^2\frac{1}{\sqrt{\rho}}\right]. \end{aligned} \quad (23)$$

The second and the third term in the exponent in (23) are understood as operators, e.g.,

$$\frac{1}{\sqrt{\rho}}\partial\frac{1}{\sqrt{\rho}} = -\frac{\partial\rho}{2\rho^2} + \frac{1}{\rho}\partial. \quad (24)$$

After expanding the integrand in terms of M^{-1}, the order M^2 terms in (23) under integrating over k and summation over n, denoted as $\mathcal{O}(M^2)$, diverge with the limit on M. Notice that $\sum_{n=-\infty}^{\infty} \exp\left[-\frac{(2\pi n/Ma)^2}{2\rho}\right]$ gives the contribution of $\mathcal{O}(M)$. The terms of $\mathcal{O}(M^2)$, however, are renormalizable by adding a bare cosmological term to the starting Lagrangian [12, 14]. In this expansion the terms in (23) including only one operator $\frac{i\tilde{k}}{M}\frac{1}{\sqrt{\rho}}\partial_2\frac{1}{\sqrt{\rho}}$ become $\mathcal{O}(M)$ because of the existence of the dumping factor, $\exp(-\tilde{k}^2/2\rho)$. The part of $\mathcal{O}(M)$ in (23) becomes zero for $\int_{-\infty}^{\infty} d\tilde{k}\tilde{k} = 0$. Then the next reading terms of $\mathcal{O}(M^0)$ in (23) remain under summation over n and the limit on M. The sum of the all $\mathcal{O}(M^0)$ terms consists of two kind of contributions. One comes from the third term of the exponent in (23), $\frac{1}{2M^2}\frac{1}{\sqrt{\rho}}\partial_2^2\frac{1}{\sqrt{\rho}}$, becoming

$$\begin{aligned} &\frac{1}{Ma} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m!}\left[-\frac{k^2+(2\pi n/Ma)^2}{2\rho}\right]^{m-1} \\ &\quad \times \left\{\left(\frac{m^3}{6} + \frac{m^2}{4} - \frac{m}{24}\right)\rho^{-3}(\partial_2\rho)^2 - \frac{m^2}{4}\rho^{-2}\partial_2^2\rho\right\}, \end{aligned}$$

and another comes from the second term, $\frac{i\tilde{k}}{M}\frac{1}{\sqrt{\rho}}\partial_2\frac{1}{\sqrt{\rho}}$, being

$$\begin{aligned} &\frac{1}{Ma} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{m=1}^{\infty} \frac{(ik)^2}{m!}\left[-\frac{k^2+(2\pi n/Ma)^2}{2\rho}\right]^{m-2} \\ &\quad \times \left\{\left(\frac{m^4}{8} - \frac{m^2}{4} + \frac{m}{8}\right)\rho^{-4}(\partial_2\rho)^2 - \left(\frac{m^3}{6} - \frac{m^2}{4} + \frac{m}{12}\right)\rho^{-3}\partial_2^2\rho\right\}. \end{aligned}$$

After performing the integration over k, $j(x^2)$ becomes

$$j(x^2) = \lim_{M\to\infty}\left[F(\rho)\,(\partial_2\rho)^2 + G(\rho)\partial_2^2\rho\right], \quad (25)$$

where $F(\rho)$ and $G(\rho)$ are given by

$$F(\rho) \equiv \frac{\rho^{-5/2}}{\sqrt{2\pi}Ma}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{2\pi n}{Ma\sqrt{\rho}}\right)^2\right]\left\{\frac{5}{32}-\frac{5}{48}\left(\frac{2\pi n}{Ma\sqrt{\rho}}\right)^2+\frac{1}{96}\left(\frac{2\pi n}{Ma\sqrt{\rho}}\right)^4\right\},$$

$$G(\rho) \equiv \frac{\rho^{-3/2}}{\sqrt{2\pi}Ma}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{2\pi n}{Ma\sqrt{\rho}}\right)^2\right]\left\{-\frac{1}{8}+\frac{1}{24}\left(\frac{2\pi n}{Ma\sqrt{\rho}}\right)^2\right\}.$$

Under the limit on M we obtain

$$\lim_{M\to\infty} F(\rho) = \frac{\rho^{-2}}{2\pi}\frac{1}{12}, \qquad \lim_{M\to\infty} G(\rho) = -\frac{\rho^{-1}}{2\pi}\frac{1}{12}, \tag{26}$$

with the help of the definition of the Jacobi θ function and its property:

$$\theta(u,\tau) \equiv \sum_{l=-\infty}^{+\infty}\exp\left(2\pi i u l + i\pi\tau l^2\right), \qquad \theta(0,i\tau) = \frac{1}{\sqrt{\tau}}\,\theta\left(0,\,\frac{i}{\tau}\right).$$

In order to evaluate the effective action (5) with the Euclidean metric $ds^2 = \rho(x^2)\left[(dx^1)^2+(dx^2)^2\right]$, we have to choose the parameter of the conformal transformation as $\alpha(x^2) = -\frac{1}{2}\ln\rho(x^2)$. Then the Jacobian factor (20) becomes

$$\ln J_E[\rho] = \frac{1}{96\pi}\int_0^a dx^1\int_{-\infty}^{\infty} dx^2\,\rho^{-2}(\partial_2\rho)^2. \tag{27}$$

Now we continue back to the Minkowski Jacobian with time evolving metric (2):

$$\frac{1}{i}\ln J[D] = -\frac{1}{2\pi}\int dt\,\frac{a}{12}\frac{\dot{D}^2}{D}. \tag{28}$$

where we have used the relations between the Euclidean parameters and the Minkowski ones: $x^2 = i\eta$, $\rho(x^2) = C(\eta)$, and we note that $d\eta = dt/D(t), C(\eta) = D(t)^2, \int dx = a$.

On the other hand the well-known Polyakov-Liouville action [14], which is the conformal anomaly in the space-time R^2, brings the same result as (28). The Polyakov-Liouville action with the metric $ds^2 = -C(\eta)\,(d\eta^2 - dx^2)$ is given by

$$S_{PL} = -\frac{1}{2\pi}\int d^2x\frac{1}{48}\ln C\,\Box\ln C. \tag{29}$$

Substituted the relation $d\eta = dt/D(t)$, and $C(\eta) = D(t)^2$ into (29), we obtain (28).

Finally, combining the partition function (19) and the Jacobian factor (28) gives the effective action for the size of the space $D(t)$ as

$$\Gamma[D] \equiv \Gamma[g_{\mu\nu}] = \frac{1}{i}\ln J + \frac{1}{i}\ln Z = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\left(-\frac{1}{12}\frac{\dot{D}^2}{D}+\frac{1}{12}\frac{1}{D}\right), \tag{30}$$

where we have redefined $aD \to D$.

3 BACKREACTION OF THE DYNAMICAL CASIMIR EFFECT

We will study the backreaction of the dynamical Casimir effect by considering the relative motion of the "mirrors" (boundaries). Assume that the mirrors receive the force derived by the static and the dynamical Casimir effects. The system of the mirrors is supposed to have finite reduced mass m for the mirror-separation $D(t)$, the dynamical variable in this system. When the mirrors are free, the system obeys the classical equation of motion. Then we describe the system by the classical kinetic term $\frac{m}{2}\dot{D}^2(t)$ and the quantum backreaction terms in the one-dimensional finite space with two boundaries D^1. This semi-classical treatment of the mirrors is relevant for $D(t) \gg 1/m$.

These backreaction terms in the space D^1 equal to those in the space S^1 shown in (30) except for the overall factor 2. The overall factor is caused by the difference of the energy levels of the oscillating modes as mentioned in Section 1 and Section 2. This equivalence is justified when the motion of the space-size $D(t)$ is adiabatic, i.e., $\dot{D}(t) \ll 1$. Then the semi-classical effective action for the motion of the boundaries is obtained as

$$S_{\text{semicl}} = -\frac{1}{2\pi}\int dt \left(\frac{m}{2}\dot{D}^2 - \frac{\kappa}{24}\frac{\dot{D}^2}{D} + \frac{\kappa}{24}\frac{1}{D}\right), \tag{31}$$

where κ is the number of species of scalar fields. In this action the second term is the backreaction term of the dynamical Casimir effect, and the third term is the static Casimir energy term. This action leads to the equation of motion given by

$$\left(m - \frac{\kappa}{12}\frac{1}{D}\right)\ddot{D} = -\frac{\kappa}{24}\left(\frac{\dot{D}}{D}\right)^2 - \frac{\kappa}{12}\frac{1}{D^2}. \tag{32}$$

This equation is integrable, and the resulting relation is given by

$$\left(\frac{m}{2} - \frac{\kappa}{24}\frac{1}{D}\right)\dot{D}^2 - \frac{\kappa}{24}\frac{1}{D} = E, \tag{33}$$

where E is an integral constant. The left hand side is the Hamiltonian of this system, thus E is the energy of this system. Here it should be noticed that the semi-classical condition $m \gg 1/D(t)$ and the adiabatic condition $\dot{D}(t) \ll 1$ lead to the validity condition $|E| \ll m$. Combining the equation of the motion (32) and the description of the energy (33), we obtain the mutual dynamical force between the mirrors (boundaries), namely *the dynamical Casimir force*:

$$F_{\text{dyn}} \equiv m\ddot{D} = -\frac{\kappa}{24}\frac{1}{D^2}\frac{1+\dot{D}^2}{1-\frac{\kappa}{12}\frac{1}{mD}} = -\frac{\kappa}{24}\frac{1}{D^2}\frac{1+\frac{2E}{m}}{\left(1-\frac{\kappa}{12}\frac{1}{mD}\right)^2}. \tag{34}$$

When the reduced mass m is much larger than the scales E and $1/D$, the dynamical Casimir force (34) is approximately equals to the static one:

$$F_{\text{static}} \equiv -\frac{\partial}{\partial D}\left(-\frac{\kappa}{24}\frac{1}{D}\right) = -\frac{\kappa}{24}\frac{1}{D^2}. \tag{35}$$

The ratio of the dynamical force $F_{\rm dyn}$ to the static one $F_{\rm static}$ is given by

$$F_{\rm dyn}/F_{\rm static} \;=\; \frac{1+\dot{D}^2}{1-\frac{\kappa}{12}\frac{1}{mD}} \;=\; 1 \;+\; \frac{\kappa}{12}\frac{1}{mD} \;+\; \dot{D}^2 \;+\; \cdots . \tag{36}$$

Here the $\dot{D}^2$ term in the expansion is known as the negative-frictional-like-force [7]. Since $\dot{D}^2 \geq 0$, we conclude that the dynamical force $F_{\rm dyn}$ is always stronger than the static one $F_{\rm static}$ for $D > \frac{\kappa}{12}\frac{1}{m}$.

4 CONCLUSION AND DISCUSSION

We have investigated the effective theory for the backreaction of the dynamical Casimir effect in the adiabatic approximation. The resultant *dynamical Casimir force* from the static and the dynamical Casimir effects becomes attractive and always stronger than the static Casimir force.

The adiabatic approximation allows us to neglect the boundary effect under deforming the separation of the mirrors $D(t)$. Then we have considered the scalar effective theory in the space-time $R \times S^1$ with the Robertson-Walker type metric rather than in the space-time $R \times D^1$. Our effective action (31) consists of the static Casimir energy term, the backreaction term of the dynamical Casimir effect and the classical kinetic term of the mirrors. The static Casimir energy arises from the compactness of the space. The backreaction term is described by conformal anomaly. The classical kinetic term is added by hand.

The backreaction discussed in this paper is comparable to the backreaction of the Hawking radiation from the two-dimensional dilaton black hole. In the two-dimensional dilaton black hole (CGHS model [16]) the Hawking radiation is represented by the conformal anomaly [16], and the backreaction appears as decreasing the black-hole mass [17].

In our model the contribution from the dynamical Casimir effect to the mirrors has generated a negative definite kinetic term, which also appeared in the analysis of CGHS model [17]. Although this looks strange, it should be noticed that the following point: for we have assumed that the mass scale of the mirrors m is much greater than the scale of the Casimir energy $\sim D^{-1}$, the contribution of the negative definite term is only perturbative to the main kinetic one. Then our result is reliable.

When the mirror separation $D(t)$ is smaller than the inverse of the mirror mass $\frac{\kappa}{12}\frac{1}{m}$, the dynamical Casimir force $F_{\rm dyn}$ seems to become repulsive in our result (36). While the semi-classical treatment becomes inappropriate at that time, this repulsive force might be realized, if one considers that the motion of the mirrors obeys the quantum mechanics.

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APPENDIX

In this appendix we briefly explain the derivation of the effective action (5) from the definition of the effective action (4). This derivation is based on the evaluation of the conformal anomaly by the Fujikawa method [12]. In order to perform the path integration of (4), we make a Wick rotation by introducing an imaginary time variable $x^2 \equiv i\eta$. Then the Euclidean metric corresponding to the Minkowski one (2) becomes

$$ds^2 \;=\; \rho(x^2)\,\left[\left(dx^1\right)^2 + \left(dx^2\right)^2\right]. \tag{37}$$

The Euclidean effective action is

$$e^{-\Gamma_E[g_{\mu\nu}]} \;=\; \int \mathcal{D}\phi\;\exp\left[-\frac{1}{2\pi}\int d^2x\,\sqrt{-g}\,\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\right]. \tag{38}$$

By introducing $\tilde{\phi} \equiv \sqrt[4]{-g}\,\phi$ and changing the measure $\mathcal{D}\phi$ into the invariant form under the general coordinate transformation $\mathcal{D}\tilde{\phi}$, equation (38) becomes

$$e^{-\Gamma_E[g_{\mu\nu}]} \;=\; \int \mathcal{D}\tilde{\phi}\;\exp\left[-\frac{1}{2\pi}\int d^2x\,\frac{1}{2}\partial\left(\frac{\tilde{\phi}}{\sqrt{\rho}}\right)\partial\left(\frac{\tilde{\phi}}{\sqrt{\rho}}\right)\right]. \tag{39}$$

Here we have used a notation $\partial\phi\,\partial\phi \equiv \partial_1\phi\,\partial_1\phi + \partial_2\phi\,\partial_2\phi$. We perform a mode expansion of the field $\tilde{\phi}(x)$ in terms of a complete set $\{\varphi_n(x)\}$:

$$\tilde{\phi}(x) \;=\; \sum_n a_n\varphi_n(x) \;\equiv\; \sum_n \langle x|n\rangle\,a_n, \tag{40}$$

where we have chosen $\varphi_n(x)$ as an eigenfunction of the Hamiltonian operator:

$$\hat{H} \;=\; -\frac{1}{2}\frac{1}{\sqrt{\rho}}\,\partial\partial\,\frac{1}{\sqrt{\rho}}, \qquad \hat{H}\varphi_n(x) \;=\; \lambda_n^2\,\varphi_n(x). \tag{41}$$

Here $\varphi_n(x)$ satisfies the normalization $\int d^2x\,\varphi_m^\dagger(x)\,\varphi_n(x) \;=\; \delta_{mn}$. Now we note that the measure $\mathcal{D}\tilde{\phi}$ is expressed by the mode coefficients a_n as

$$\mathcal{D}\tilde{\phi} \;=\; \prod_x \mathcal{D}\tilde{\phi}(x) \;=\; [\,\det\langle x|n\rangle\,]\prod_n da_n \;=\; \prod_n da_n. \tag{42}$$

Under Weyl transformation $g_{\mu\nu} \to e^{2\alpha(x)}g_{\mu\nu}$ the mode-coefficients of the field $\tilde{\phi}(x)$, a_n, are transformed as an infinitesimal form:

$$\begin{aligned}
\tilde{\phi}(x) \;&\to\; \tilde{\phi}'(x) \;\equiv\; \sum_n a'_n\,\varphi_n(x),\\
a'_n \;&=\; a_n \;+\; \sum_m \frac{1}{2}\int d^2x\,\alpha(x)\varphi_n^\dagger(x)\,\varphi_m(x)a_m \;\equiv\; \sum_m C_{nm}\,a_m.
\end{aligned} \tag{43}$$

Then the measure is transformed as

$$\begin{aligned}
\mathcal{D}\tilde{\phi}' \;&=\; \prod_n da'_n \;=\; [\det(C_{nm})]\prod_l da_l\\
&=\; \exp\left[\mathrm{Tr}\ln\left(\delta_{nm} \;+\; \frac{1}{2}\int d^2x\,\alpha(x)\varphi_n^\dagger(x)\,\varphi_m(x)\right)\right]\prod_l da_l\\
&=\; \exp\left[+\frac{1}{2}\int d^2x\,\alpha(x)\sum_n \varphi_n^\dagger(x)\,\varphi_n(x)\right]\;\mathcal{D}\tilde{\phi}.
\end{aligned} \tag{44}$$

This gives the Jacobian of the conformal transformation. By the Weyl transformation chosen $\alpha(x) = -\frac{1}{2}\ln\rho(x)$ for $\tilde{\phi} \to \tilde{\phi}' = \tilde{\phi}/\sqrt{\rho}$, the effective action (39) becomes

$$\begin{aligned} e^{-\Gamma_E[g_{\mu\nu}]} &= \exp\left[-\frac{1}{2}\int d^2x\,\alpha(x)\sum_n \varphi_n^\dagger(x)\,\varphi_n(x)\right] \\ &\quad \times \int \mathcal{D}\tilde{\phi}'\,\exp\left[-\frac{1}{2\pi}\int d^2x\,\frac{1}{2}\,\partial\tilde{\phi}'\,\partial\tilde{\phi}'\right], \end{aligned} \tag{45}$$

where the second factor equals to the partition function of the free scalar field in the flat space-time. Finally, we can arrive at our destination (5) from the description (45) by the inverse of the Wick rotation.

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